Entanglement, partial set of measurements,
and diagonality of the density matrix in the parton model

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Abstract

To study quantum properties of the hadron wavefunction at small x, we derived the reduced density matrix for soft gluons in the CGC framework. We explicitly showed that the reduced density matrix is not diagonal in the particle number basis. The off-diagonal components are usually ignored in the conventional parton model. We thus defined the density matrix of ignorance by keeping only the part of the reduced density matrix which can be probed in a limited set of experimental measurements. We calculated Von Neumann entropy for both the reduced and ignorance density matrices. The entropy of ignorance is always greater than the entanglement entropy (computed drom the reduced density matrix) of gluons. Finally, we showed that the CGC reduced density matrix for soft gluons can be diagonalized in a thermal basis with Boltzmann weights suggesting thermalization of new quasi-particle states which we dubbed entangolons.

1 Introduction

This work was motivated by the paradox raised in Ref. [1]. On one hand the proton as a quantum object is in a pure state and is described by a completely coherent wavefunction with zero entropy. On the other hand in high energy experiments when probed by a small external probe, it behaves like an incoherent ensemble of (quasi-free) partons. Such an ensemble carries a nonvanishing entropy. Reference [1] suggested that the origin of this entropy is entanglement between the degrees of freedom one observes in DIS (partons in the small spatial region of the proton) and the rest of the proton wavefunction which are not measured in the final state and therefore play the role of an "environment". Formally, this can be formulated in terms of the reduced density matrix

$$\hat{\rho}_r = \text{Tr}_{\text{unobs}}\left[\, |P\rangle\langle P| \,\right], \tag{1}$$

where $|P\rangle$ is the proton wave function and the partial trace is taken over the unobserved degrees of freedom. The entropy of the parton model is then identified with the von Neumann entropy of the reduced density matrix according to

$$S_{\text{PM}} = -\text{Tr}\Big[\hat{\rho}_r \ln \hat{\rho}_r\Big]. \tag{2}$$

Most of the quantities measured DIS are related to the average number of particles or multi-parton momentum distributions $\langle a^\dagger(\underline{k}_1)a(\underline{k}_1)...a^\dagger(\underline{k}_n)a(\underline{k}_n)\rangle$. These quantities do not provide an access to the off-diagonal elements of the density matrix in the number operator basis. This leads to the fundamental problem that one cannot reconstruct the full density matrix from the available experimental data.

This lack of knowledge of the actual density matrix of the system can be characterized by an entropy. We will dub this entropy "*the entropy of ignorance*". Among the family of all density matrices that reproduce the same experimental observable, we defined the matrix that have zeros on all off-diagonal elements as the density matrix of ignorance (the operation of dropping the off-diagonal elements of the density matrix can be considered as an example of a quantum-to-classical channel). The associated entropy

$$S_I = -\text{Tr}\Big[\hat{\rho}_I \ln \hat{\rho}_I\Big]. \tag{3}$$

It has the following property $S_I \geq -\text{Tr}\Big[\hat{\rho}_r \ln \hat{\rho}_r\Big]$ (see e.g. Refs. [2,3]). The entropy of ignorance is exactly equal to the Boltzmann entropy of the classical ensemble of partons with the probability distribution defined by the diagonal matrix elements

$$S_I = S_B = -\sum_n p_n \ln p_n; \qquad p_n = \langle n|\hat{\rho}_r|n\rangle. \tag{4}$$

2 The CGC density matrices

The Color Glass Condensate (CGC) effective theory describes scattering at high energy (see Refs. [4–8] for reviews). According to the CGC, the valence ("hard") partons can be treated as static classical sources of the small-x "soft" gluons. This partial separation of degrees of freedom can be formalized by

$$|\psi\rangle = |s\rangle \otimes |v\rangle, \tag{5}$$

where $|v\rangle$ is the state vector characterizing the valence degrees of freedom and $|s\rangle$ is the soft fields in the presence of the valence source. Despite appearances, the state is not of a direct product form since the soft vacuum depends on the valence degrees of freedom. In the leading perturbative order, the CGC soft gluon state is

$$|s\rangle = \mathcal{C}|0\rangle \tag{6}$$

with the coherent operator

$$\mathcal{C} = \exp\left\{2i\text{tr}\int \frac{d^2k}{(2\pi)^2} b^i(\underline{k})\phi_i(\underline{k})\right\} = \exp\left\{2i\text{tr}\int_{\underline{k}} b^i(\underline{k})\phi_i(\underline{k})\right\}, \tag{7}$$

where $\phi_i(\underline{k}) \equiv a_i^+(\underline{k}) + a_i(-\underline{k})$, and the trace operation (tr) is over all colors. The background field b_a^i is determined by the valence color charge density ρ via:

$$b_a^i(\underline{k}) = g\rho_a(\underline{k})\frac{ik_i}{k^2} + \mathcal{O}(\rho^2). \tag{8}$$

For simplicity we restrict our consideration only to the leading order contribution in the charge density ρ which captures only the longitudinal polarization of the soft gluons. We do not expect our results to change if higher order corrections are included.

The valence wave function $|v\rangle$ is customarily modeled following the McLerran-Venugopalan (MV) model [9, 10]

$$\langle \rho | v \rangle \langle v | \rho \rangle = \mathcal{N} e^{-\int_{\underline{k}} \frac{1}{2\mu^2} \rho_a(\underline{k}) \rho_a^*(\underline{k})} , \tag{9}$$

where $\mathcal{N}$ is the normalization factor and the parameter μ^2 determines the average color charge density. Now we can formally define the hadron density matrix:

$$\hat{\rho} = |v\rangle \otimes |s\rangle \langle s| \otimes \langle v| . \tag{10}$$

Integrating out the valence degrees of freedom, we obtain the reduced density matrix for the soft gluons:

$$\hat{\rho}_r = \mathcal{N} \int D\rho \; e^{-\int_{\underline{k}} \frac{1}{2\mu^2} \rho_a(\underline{k}) \rho_a^*(\underline{k})} \mathcal{C}(\rho_b, \phi_b^i) |0\rangle \langle 0| \mathcal{C}^\dagger(\rho_c, \phi_c^j) . \tag{11}$$

To extract the ignorance density matrix, $\hat{\rho}_I$, in gluon number basis, we need the basis vectors

$$\prod_c \prod_k |n_c(\underline{k}), m_c(-\underline{k})\rangle \propto \prod_c \prod_k \left(\frac{[a_c^\dagger(\underline{k})]^{n_c}}{\sqrt{n_c!}} \right) \left(\frac{[a_c^\dagger(-\underline{k})]^{m_c}}{\sqrt{m_c!}} \right) |0\rangle , \tag{12}$$

where c is the color indices, and we only consider longitudinal polarization. Since a mode with momentum $\underline{k}$ mixes only with the mode with momentum $-\underline{k}$ due to the fact that $\rho_a^*(\underline{k}) = \rho_a(-\underline{k})$, we introduce $N_c = n_c + m_c$. Furthermore, $\hat{\rho}_r$ is a direct product of density matrices in a fixed transverse momentum sector due to transnational in variance, see [11] for details.

The matrix elements of the density matrix for a given value of momentum q and given color c are

$$\langle n_c(\underline{q}), m_c(-\underline{q}) | \hat{\rho}_r(\underline{q}) | \alpha_c(\underline{q}), \beta_c(-\underline{q}) \rangle = (1-R) \frac{(n+\beta)!}{\sqrt{n!m!\alpha!\beta!}} \left(\frac{R}{2} \right)^{n+\beta} \delta_{(n+\beta),(m+\alpha)}, \tag{13}$$

$$R = \left(1 + \frac{q^2}{2g^2\mu^2} \right)^{-1} . \tag{14}$$

We thus explicitly demonstrated that the off-diagonal components are non zero! This is one of the main results of this paper. If we ignore the off-diagonal elements we obtain the ignorance density matrix in the number of gluons representation

$$\langle n_c(\underline{q}), m_c(-\underline{q}) | \hat{\rho}_I(\underline{q}) | n_c(\underline{q}), m_c(-\underline{q}) \rangle = (1-R) \frac{(n+m)!}{n!m!} \left(\frac{R}{2} \right)^{n+m} . \tag{15}$$

3 Quantum Entropies

Using the reduced density matrix, the entropy of entanglement can be readily calculated. It reproduces the known result obtained in Ref. [12]:

$$S_E = \frac{(N_c^2-1)S_\perp}{2} \int \frac{d^2q}{(2\pi)^2} \left[\ln\left(\frac{g^2\mu^2}{q^2} \right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \right.$$
$$\left. \times \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2} \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \right) \right]. \tag{16}$$
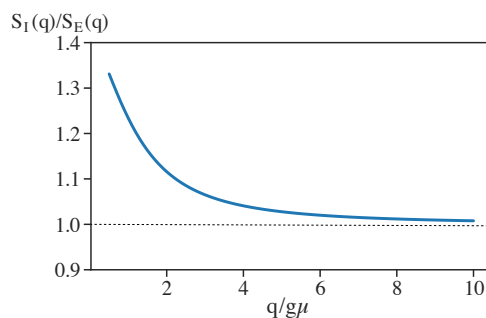
Figure 1: Ratios of entropy densities at a given magnitude of the transverse momentum $q/g\mu$. $S_I(q^2)$ is the von Neumann entropy density of ignorance and $S_E(q^2)$ is the corresponding entanglement entropy density.

There is no analytic expression for the entropy of ignorance. Hence, we compare the results numerically in Fig. 1. According to the ratio S_I/S_E, the dominant contribution to their difference is at small q where states with more than one particle cannot be ignored. While at large q, only the vacuum state survives and the two density matrix are approximately the same hence the ratio approaching 1.

4 Thermal quasi-particle basis

It is amusing that the entanglement entropy can be rewritten in the form

$$S_E = (N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2}\left[(1+f)\ln(1+f) - f \ln f\right] \tag{17}$$

with $f = (\exp(\beta\omega) - 1)^{-1}$ and $\omega\beta = 2\ln\left(\frac{q}{2g\mu} + \sqrt{1 + \left(\frac{q}{2g\mu}\right)^2}\right)$. This suggests that there exists a quasi-particle basis in the reduced density matrix is diagonal and thermal (i.e. all eigenvalues of the density matrix are powers of the same number.). We named this quasi-particles entangalons.

At small momentum, $\omega\beta \approx \frac{q}{g\mu}$, we thus can identify the effective temperature of entangolons with $g\mu$.

At large momentum, the quasi-particle distribution function coincides with the Weizsacker-Williams distribution of gluons $f \approx g^2\mu^2/q^2$.

5 Conclusions

Our explicit calculation confirms that, within the CGC framework, the reduced density matrix for soft gluons is not diagonal in parton/particle number basis. Neglecting the off-diagonal elements results in the loss of information and thus in larger entropy. We introduced a new form of entropy associated with this loss and called it entropy of ignorance. It is greater than the entanglement entropy which is supposedly related to the entropy of final state particles. In addition, by analysing the entanglement entropy, we hypothesized that the reduced density matrix directly is diagonal in some quasi-particle basis with the eigenvalues having the form of Boltzmann weights.

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