SciPost Phys. 12, 142 (2022)

# Fractons in curved space

**Akash Jain[1,2⋆] and Kristan Jensen[3†]**

**1** Institute for Theoretical Physics, University of Amsterdam,
1090 GL Amsterdam, The Netherlands
**2** Dutch Institute for Emergent Phenomena, 1090 GL Amsterdam, The Netherlands
**3** Department of Physics and Astronomy, University of Victoria,
Victoria, BC V8W 3P6, Canada

⋆ a.jain@uva.nl, † kristanj@uvic.ca

## Abstract

**We consistently couple simple continuum field theories with fracton excitations to curved spacetime backgrounds. We consider homogeneous and isotropic fracton field theories, with a conserved $U(1)$ charge and dipole moment. Coupling to background fields allows us to consistently define a stress-energy tensor for these theories and obtain the respective Ward identities. Along the way, we find evidence for a mixed gauge-gravitational anomaly in the symmetric tensor gauge theory which naturally couples to conserved dipoles. Our results generalise to systems with arbitrarily higher conserved moments, in particular, a conserved quadrupole moment.**



Received 02-02-2022
Accepted 11-04-2022
Published 29-04-2022
Check for updates
doi:10.21468/SciPostPhys.12.4.142

## Contents

# 1 Introduction

We are interested in quantum mechanical models with fractons [1–5]. These models describe exotic, at this time purely hypothetical, phases of quantum matter displaying features that challenge our usual notions of quantum field theory in the continuum limit. Perhaps the most striking of these is the existence of finite-energy excitations with restricted mobility: fractons are excitations which are "pinned" to a point, lineons are able to move along a one-dimensional sublattice of a lattice model, etc. In some instances, like the X-cube model [6], they exhibit an ultraviolet-sensitive but non-extensive ground state degeneracy. A priori, it is unclear how to describe these phenomena with textbook quantum field theory and, indeed, much recent attention (see [7, 8] and follow-up work) has been devoted to answering the question of how to carefully coarse-grain these models and what are the rules of the game for their continuum limits.

Fortunately, the strange behaviour of these phases of matter is intimately tied with their symmetries, which we are well-positioned to study in field theory. Models of fracton order have exotic spacetime symmetries, like a conserved dipole moment, or subsystem symmetry as in e.g. [6, 9, 10]. It is intuitively simple to understand how a conserved dipole moment leads to "fracton" excitations. Namely, individual charges can carry finite energy, but an isolated charge cannot move without changing the dipole moment. The ultraviolet sensitive ground state degeneracies are also tied to these symmetries. Take the X-cube model of [6], a theory of $\mathbb{Z}_2$ spins on a hypercubic lattice. For each plane of the lattice, the Hamiltonian has a $\mathbb{Z}_2$ subsystem symmetry that flips all of the spins on that plane. The ground state of that model is not invariant under this subsystem symmetry, so that there is a large space of vacua generated by acting with the symmetry. The ground state degeneracy is parametrically the volume of the symmetry group, which is sensitive to the number of lattice sites in each direction.

The focus of this work is to better understand the spacetime symmetries of simple, continuum models of fractons. It is the first in a series of works whose broad goal is to study the role of these symmetries, as well as their spontaneous breaking, in interacting models of fractons and the ensuing implications for transport. Ultimately, we endeavour to find new, interacting, and soluble models of fractons, whose low-energy symmetry breaking pattern and careful quantisation thereby inform us as to what we might expect in the low-energy physics of these exotic theories. A simple but useful future application is to construct theories of transport, both at zero and at finite temperature, i.e. the hydrodynamic description of fracton models, which is strongly constrained by symmetry. (See [11–14] for earlier work on the hydrodynamics of fracton field theories.) We hope that the resulting theory of transport may prove useful in giving predictions which, perhaps, lead to the experimental discovery of models with fracton order.

Here we take the very first steps. As in standard Lorentz-invariant field theory, we would like to couple the symmetry currents of these models to external sources. That is, we would like to study fractons in curved space. This task is non-trivial on account of exotic spacetime symmetries, which lead to a somewhat intricate coupling to a background spacetime. We focus on what could be described as the "most symmetric" fracton models, which are isotropic and contain a conserved $U(1)$ charge and dipole moment. Our methods naturally extend to field theories with conserved multipole moments, including those with conserved dipole moment and quadrupole trace. This particular symmetry pattern is perhaps the most experimentally relevant one on the market, given the arguments that these symmetries approximately govern real-world systems including vortices in superfluid helium [15,16], defects in 2+1-dimensional elastic media [17], and the lowest Landau level of a quantum Hall state [18]. See also [19,20] for other proposals to realize models with conserved dipole moment.

The output of our analysis is a systematic description of the symmetry currents of these models, the algebra of local symmetries, and their Ward identities, which are stable under radiative corrections in the absence of anomalies. Indeed, while it seems unlikely to us that simple fracton models (like the scalar theory of Pretko we review in Section 2) possess anomalies, we do note that the coupling to external fields allows for the future classification and computation of perturbative anomalies, assuming they can exist.

As a byproduct of our work, we find that the symmetric tensor gauge theory with local dipole symmetry considered in [21] cannot be consistently coupled to curved space in a covariant way, on account of the fact that it does not possess a conserved and gauge-invariant stress tensor.[1] This is reminiscent of a gauge-gravitational mixed anomaly, in the sense that there is an obstruction to simultaneously maintaining gauge invariance and covariance. It remains to be seen if the tensor gauge theory can be redefined in a way so as to maintain gauge-invariance while at the expense of covariance, as for mixed gauge-gravitational anomalies in relativistic field theory.

This work is a stepping stone. The next step is to pose, and solve, interacting large $N$ fracton models. This will be done in [24], mostly using the imaginary time formalism at finite temperature. At least in those models, one can deduce the low-energy symmetry breaking pattern and accompanying Goldstone effective theory. This effective description brings us most of the way to a theory of transport. From there, it is straightforward to generalise the methods of [25–29] (as well as e.g. [30–34] on constructing theories of transport in the non-relativistic setting) to obtain the dissipative hydrodynamic description of these models, either from the point of view of constitutive relations and conservation equations, or from the point of view of a Keldysh effective field theory.

The remainder of this paper is organized as follows. In Section 2 we briefly review field theories with conserved dipole moment. The heart of the paper is Section 3, where we couple the symmetry currents of these models to background fields. This coupling leads to a notion of general covariance, and an infinite-dimensional algebra of charges generated by diffeomorphisms and gauge transformations. We deduce this algebra in Section 4, which, for technical reasons, is easiest to do when working in an analogue of the first-order formulation of the background spacetime. We discuss the extension of our results to even more exotic models with conserved multipole moments in Section 5, and wrap up with a discussion in Section 6.

[1] The tensor gauge theory was "minimally coupled" to a time-independent spatial metric in [22,23], and those authors found that the resulting model was gauge-non-invariant unless the spatial metric is one of constant curvature. Our result goes beyond those, insofar as we find that there is no curved space definition of the tensor gauge theory which is simultaneously gauge- and diffeomorphism-invariant.

We relegate a few technical computations to the Appendix.

*Note*: While this paper was nearing completion, we became aware of [35], whose authors also study the problem of putting models with conserved dipole moment into curved spacetime.

## 2 Field theories with a conserved dipole moment

Consider a rotationally and translationally invariant field theory with a conserved $U(1)$ charge and dipole moment. We denote the Hamiltonian as H, momenta as $\mathrm{P}_a$, angular momenta as $\mathrm{M}_{ab}$, $U(1)$ charge as Q, and dipole moments as $\mathrm{D}_a$. The symmetry algebra of charges is

$$\begin{aligned} [\mathrm{P}_a, \mathrm{D}_b] &= i\delta_{ab}\mathrm{Q}\,, \\ [\mathrm{M}_{ab}, \mathrm{D}_c] &= i(\delta_{ac}\mathrm{D}_b - \delta_{bc}\mathrm{D}_a)\,, \\ [\mathrm{M}_{ab}, \mathrm{P}_c] &= i(\delta_{ac}\mathrm{P}_b - \delta_{bc}\mathrm{P}_a)\,, \\ [\mathrm{M}_{ab}, \mathrm{M}_{cd}] &= i(\delta_{ac}\mathrm{M}_{bd} - \delta_{bc}\mathrm{M}_{ad} - \delta_{ad}\mathrm{M}_{bc} + \delta_{bd}\mathrm{M}_{ac})\,, \end{aligned} \tag{2.1}$$

with all other commutators vanishing. The $U(1)$ charge Q appears as a central extension.

The work of [36] prescribed a systematic procedure to writing down the action for a charged scalar field $\Phi$ that is invariant under these symmetries. We review this construction in this Section. Take, for example, a scalar field theory described by the action

$$S = \int \mathrm{d}t\mathrm{d}^d x\Big(i\Phi^*\partial_t\Phi + \lambda\,\mathrm{D}_{ij}(\Phi^*,\Phi^*)\mathrm{D}^{ij}(\Phi,\Phi) - V(\Phi^*\Phi)\Big)\,, \tag{2.2}$$

where $\mathrm{D}_{ij}(\Phi,\Phi) = \Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi$. It is easy to see that this theory is invariant under constant $U(1)$ rotations of the complex scalar field $\Phi \to e^{i\Lambda}\Phi$. In fact, this theory has another invariance under spatially linear $U(1)$ rotations $\Phi \to e^{i\psi_i x^i}\Phi$. This latter symmetry leads to the conserved dipole moment. To wit, we can compute the conserved charge density and flux associated with the global $U(1)$ symmetry of the theory to be

$$J^t = \Phi^*\Phi\,, \qquad J^i = \partial_j\Big(i\lambda\,\mathrm{D}^{ij}(\Phi^*,\Phi^*)\Phi^2 - i\lambda\,(\Phi^*)^2\mathrm{D}^{ij}(\Phi,\Phi)\Big)\,, \tag{2.3}$$

satisfying $\partial_t J^t + \partial_i J^i = 0$ on the solutions of the equations of motion. It is easy to see that the total charge defined as

$$Q = \int \mathrm{d}^d x\, J^t\,, \tag{2.4}$$

is conserved. However, the dipole moment defined as

$$D^i = \int \mathrm{d}^d x\, x^i J^t\,, \tag{2.5}$$

is also conserved. This is precisely the Noether current associated with the linear $U(1)$ rotations. This conservation implies that the $U(1)$ flux $J^i$ in eq. (2.3) can be expressed as the divergence of a dipole flux $J^{ij}$ as $J^i = \partial_j J^{ij}$. The conservation of $U(1)$ charge and dipole moment are then simultaneously encoded in the Ward identity

$$\partial_\mu J^\mu = \partial_t J^t + \partial_i\partial_j J^{ij} = 0\,. \tag{2.6}$$

Note that only the symmetric part of $J^{ij}$ appears here. For the purposes of better understanding the symmetries of the problem, we can then regard $J^{ij}$ as symmetric.[2]

[2] More precisely, since the antisymmetric part of $J^{ij}$ drops out of the conservation equation, this antisymmetric part represents an ambiguity in the definition of the dipole current. In the language of high energy physics, we may consider an "improved" version of the dipole current whereby we redefine it to be symmetric. We can do this as long as the antisymmetric part of the original dipole current is a gauge-invariant operator.

There are infinitely more terms that can be included into the action (2.2) consistent with the symmetries, like $\partial_t\Phi^*\partial_t\Phi$, $(\phi^*)^2 f(\Phi^*\Phi)\mathrm{D}^i{}_i(\Phi,\Phi) + (\text{c.c.})$, etc. The crucial point is that terms with spatial derivatives are strongly constrained by the conserved dipole moment. In particular, the standard rotationally invariant term $\partial_i\Phi^*\partial^i\Phi$ is forbidden.

The particular set of allowed terms can be understood by coupling to background fields. One introduces a field $A_t$ which couples to the charge density $J^t$, and a field $a_{ij}$ (with $a_{ij} = a_{ji}$) which couples to the dipole current $J^{ij}$, with

$$\delta S = \int \mathrm{d}t\mathrm{d}^d x \left( J^t \delta A_t + \frac{1}{2} J^{ij} \delta a_{ij} \right). \tag{2.7}$$

The action is now a functional of the quantum field $\Phi$ and the background fields $A_t$ and $a_{ij}$. The peculiar form of current conservation (2.6) then arises if we impose a symmetry under

$$\Phi \to e^{i\Lambda(t,\vec{x})}\Phi\,, \qquad A_t \to A_t + \partial_t\Lambda(t,\vec{x})\,, \qquad a_{ij} \to a_{ij} - 2\partial_i\partial_j\Lambda(t,\vec{x})\,. \tag{2.8}$$

The absence of a vector gauge field $A_i$ implies that while there is a covariant derivative with respect to time, i.e.

$$\mathrm{D}_t\Phi = \partial_t\Phi - iA_t\Phi\,, \tag{2.9}$$

but no covariant derivative in spatial directions. Instead, the simplest covariant object that acts with spatial derivatives on charged fields includes two fields and two derivatives,

$$\mathrm{D}_{ij}(\Phi,\Phi) = \Phi\partial_i\partial_j\Phi - \partial_i\Phi\partial_j\Phi + \frac{i}{2}a_{ij}\Phi^2\,, \tag{2.10}$$

which one can readily verify transforms covariantly under (2.8), with $\mathrm{D}_{ij}(\Phi,\Phi) \to e^{2i\Lambda}\mathrm{D}_{ij}(\Phi,\Phi)$. Acting on two fields $\Phi_1$ and $\Phi_2$ with charges $q_1$ and $q_2$, there is a more general expression

$$\mathrm{D}_{ij}(\Phi_1,\Phi_2) = \frac{1}{2}\left( \frac{q_1}{q_2}\Phi_1\partial_i\partial_j\Phi_2 + \frac{q_2}{q_1}\Phi_2\partial_i\partial_j\Phi_1 - \partial_i\Phi_1\partial_j\Phi_2 - \partial_j\Phi_1\partial_i\Phi_2 \right) + i\frac{q_1+q_2}{4}a_{ij}\Phi_1\Phi_2\,, \tag{2.11}$$

which transforms as

$$\mathrm{D}_{ij}(\Phi_1,\Phi_2) \to e^{i(q_1+q_2)\Lambda}\mathrm{D}_{ij}(\Phi_1,\Phi_2)\,. \tag{2.12}$$

For the quantum field theory of $\Phi$, it then follows that the simplest terms in the effective action with spatial derivatives involve at least four powers of $\Phi$. For this reason, one expects the model with spatial kinetic terms to be strongly correlated, and indeed, in [24] we find that this is the case in soluble large $N$ generalizations of these theories.

There is also a simple free field theory of dynamical fields $A_t$ and $a_{ij}$, a symmetric tensor gauge theory analogous to pure electromagnetism, first written down in [21]. In this theory one identifies $(A_t, a_{ij})$ modulo the gauge symmetry. The gauge-invariant analogues of the electric and (the Hodge dual of) magnetic fields in this theory are

$$E_{ij} = -\partial_t a_{ij} - 2\partial_i\partial_j A_t\,, \qquad F_{ijk} = \partial_i a_{jk} - \partial_j a_{ik}\,, \tag{2.13}$$

and so it is easy to write down gauge-invariant actions, like

$$S = \int \mathrm{d}t\mathrm{d}^d x \left( \frac{\epsilon_0}{2}E_{ij}E^{ij} - \frac{1}{4\mu_0}F_{ijk}F^{ijk} \right). \tag{2.14}$$

As we will see, there is an obstruction to placing this theory in a general curved space.

# 3 Coupling to curved space

In the previous Section, we studied how to write down simple continuum field theories of a charged scalar field $\Phi$ with a conserved dipole moment. The goal of the present section is to write such a theory covariantly, which will allow us to place such a model into "curved spacetime," coupling it to the analogue of an external metric.

## 3.1 Aristotelian background sources

We are interested in physical systems that are invariant under spacetime translations and spatial rotations, but with no boost symmetry – Galilean or Lorentz. Such systems naturally couple to the so-called Aristotelian background[3] sources [32–34].[4] The sources consist of a clock-form $n_\mu$ and a degenerate symmetric spatial metric tensor $h_{\mu\nu}$. Together, $n_\mu$ and $h_{\mu\nu}$ can be thought of as the analogue of the spacetime metric $g_{\mu\nu}$, but when no Lorentz boost symmetry has been imposed to combine the space and time components into a single object. Physically, $n_\mu$ couples to the energy density $\epsilon^t$ and energy flux $\epsilon^i$ of the system, while $h_{\mu\nu}$ couples to the momentum density $\pi^i$ and stress tensor $\tau^{ij}$ respectively. Note that one of the components of $h_{\mu\nu}$ is not independent due to the degeneracy condition. One typically also includes a gauge field $A_\mu$ that couples to some conserved $U(1)$ particle-number/charge density $J^t$ and the associated flux $J^i$ in the theory.

We denote the zero eigenvector of $h_{\mu\nu}$ by $v^\mu$, normalised as $v^\mu n_\mu = 1$, such that $v^\mu h_{\mu\nu} = 0$. This is to be understood as the velocity of the preferred reference frame that is observing the physical system under consideration. Using this, we can also define an inverse spatial metric $h^{\mu\nu}$ satisfying $h^{\mu\nu}n_\nu = 0$ and $h^{\mu\rho}h_{\nu\rho} = h^\mu_\nu = \delta^\mu_\nu - v^\mu n_\nu$.

An important aspect of Aristotelian spacetimes is that they come equipped with a covariant derivative. Just like in general relativity, where the covariant derivative is defined so that the metric tensor $g_{\mu\nu}$ is covariantly constant, we can define an Aristotelian covariant derivative via the connection

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2}h^{\lambda\rho}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right). \tag{3.1}$$

The connection satisfies

$$\begin{gathered}\nabla_\lambda n_\nu = \nabla_\lambda h^{\mu\nu} = 0\,, \qquad \nabla_\lambda h_{\mu\nu} = -n_{(\mu}\pounds_\nu h_{\nu)\lambda}\,, \qquad h_{\nu\mu}\nabla_\lambda v^\mu = \frac{1}{2}\pounds_v h_{\nu\lambda}\,, \\ \Gamma^\mu_{\mu\nu} + F^n_{\nu\mu}v^\mu = \frac{1}{\sqrt{\gamma}}\partial_\nu\sqrt{\gamma}\,, \qquad T^\lambda{}_{\mu\nu} \equiv 2\Gamma^\lambda{}_{[\mu\nu]} = v^\lambda F^n_{\mu\nu}\,,\end{gathered} \tag{3.2}$$

where $\gamma = \det(n_\mu n_\nu + h_{\mu\nu})$, $F^n_{\mu\nu} = \partial_\mu n_\nu - \partial_\nu n_\mu$, and $\pounds_X$ is the Lie derivative along $X^\mu$. Round and square brackets indicate symmetrisation and anti-symmetrisation over indices, with $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$ and $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$. A curious contrast compared to Riemannian geometry is that the connection is torsional, with torsion $T^\lambda{}_{\mu\nu}$; it is not possible to define a torsionless connection that annihilates $n_\mu$. Note that this is purely a matter of definition and has no physical relevance. One could also add a more general form of torsion to this connection or use a different connection that annihilates $n_\mu$ and $h^{\mu\nu}$, but we shall refrain from

[3] It is amusing to contrast the invocation of Aristotle with the more common naming convention in theoretical physics, where a result is named for the last author to discover it.

[4] These are a generalisation of the Newton-Cartan background sources that show up when coupling to Galilean (non-relativistic) field theories [31, 37–39], but when no Galilean or Milne boost symmetry is imposed.

delving into these possibilities here.[5] We can define the curvature tensor as

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\lambda}{}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}{}_{\mu\rho} + \Gamma^{\lambda}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} - \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\rho} \,. \tag{3.4}$$

Note that $n_\lambda R^{\lambda}{}_{\rho\mu\nu} = 0$.

By introducing an Aristotelian background, we can now take a non-relativistic field theory and render it generally covariant, by coupling it to sources in such a way as to be invariant under diffeomorphisms and gauge transformations. Parametrising these symmetry transformations by $\mathcal{X} = (\chi^\mu, \Lambda)$, their infinitesimal action on the background fields is given as

$$\begin{aligned} \delta_{\mathcal{X}} n_\mu = \$_\chi n_\mu \qquad &= \nabla_\mu (n_\lambda \chi^\lambda) + \chi^\lambda F^n_{\lambda\mu} \,, \\ \delta_{\mathcal{X}} h_{\mu\nu} = \$_\chi h_{\mu\nu} \qquad &= \chi^\lambda \nabla_\lambda h_{\mu\nu} + 2 h_{\lambda(\mu} \nabla_{\nu)} \chi^\lambda \,, \\ \delta_{\mathcal{X}} A_\mu = \$_\chi A_\mu + \partial_\mu \Lambda &= \nabla_\mu (\Lambda + A_\mu \chi^\mu) + \chi^\lambda F_{\lambda\mu} \,. \end{aligned} \tag{3.5}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the $U(1)$ strength. The action of the symmetry transformations on the derived fields $v^\mu$ and $h^{\mu\nu}$ can also be obtained accordingly $\delta_{\mathcal{X}} v^\mu = \$_\chi v^\mu$ and $\delta_{\mathcal{X}} h^{\mu\nu} = \$_\chi h^{\mu\nu}$.

This coupling to background is rather trivial for an "ordinary" non-relativistic theory. All we have done is make precise what needs to be done in order to write a translationally and rotationally invariant field theory in a general set of coordinates. However, more structure is required for theories that have a conserved dipole moment, where there is a richer interplay between internal and spacetime symmetries.

### 3.2 The dipole shift symmetry

Before we adapt the Aristotelian backgrounds to account for physical systems with conserved dipole moment, we need to pay a closer attention to the dipole symmetry. In Section 2, we reviewed how, when coupling the density $J^t$ and dipole flux $J^{ij}$ to the background fields $A_t$ and $a_{ij}$ respectively, the dipole moment conservation can be understood as the invariance of the theory under a $U(1)$ transformation of the background fields: $A_t \to A_t + \partial_t \Lambda$, $a_{ij} \to a_{ij} - 2\partial_i \partial_j \Lambda$. Note that the symmetry acts on $a_{ij}$ with a non-Leibniz differential operator. In order to discuss the algebra of charges generated by gauge transformations and diffeomorphisms, it is convenient to instead realise this symmetry with linear differential operators. This approach is also useful when coupling to a background.

To this end, we introduce a vector gauge field $A_i$ coupled to the flux $J^i$ and impose the usual invariance under

$$A_t \to A_t + \partial_t \Lambda \,, \qquad A_i \to A_i + \partial_i \Lambda \,, \tag{3.6}$$

leading to the regular $U(1)$ conservation law $\partial_\mu J^\mu = 0$. We supplement it with an additional "dipole shift symmetry" given as

$$A_i \to A_i + \psi_i \,, \qquad a_{ij} \to a_{ij} + \partial_i \psi_j + \partial_j \psi_i \,, \tag{3.7}$$

which imposes the constraint $J^i = \partial_j J^{ij}$. Together these relations lead to the desired conservation equation $\partial_t J^t + \partial_i \partial_j J^{ij} = 0$. Of course, we can entirely "gauge fix" the dipole shift

[5] For completeness, we note that the most general form of the Aristotelian connection with the properties $\nabla_\lambda n_\mu = \nabla_\lambda h^{\mu\nu} = 0$ is given as

$$\Gamma^{\lambda}{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2} h^{\lambda\rho} \left( \partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \right) + n_{(\mu} \Omega_{\nu)\rho} h^{\lambda\rho} + \frac{1}{2} \left( T^{\lambda}{}_{\mu\nu} - 2 h_{\rho(\nu} T^{\rho}{}_{\mu)\sigma} h^{\sigma\lambda} \right) , \tag{3.3}$$

where $T^\lambda_{\mu\nu} = h^\lambda_\rho \Gamma^\rho_{\mu\nu}$ (satisfying $n_\lambda T^\lambda_{\mu\nu}$) is the spatial torsion tensor and $\Omega_{\mu\nu} = 2 h_{\lambda[\nu} \nabla_{\mu]} v^\lambda$ is the background frame vorticity tensor.

symmetry by choosing $A_i = 0$, which forces us to set $\psi_i = -\partial_i \Lambda$ and gives back our original $U(1)$ symmetry.

We already have a gauge field $A_\mu$ corresponding to a conserved $U(1)$ current in the Aristotelian framework. To account for the conserved dipole moment, we also need to introduce a degenerate symmetric spatial dipole gauge field $a_{\mu\nu}$, constrained as $v^\mu a_{\mu\nu} = 0$. In the reference frame of the background observer, when $v^\mu = \delta^\mu_t$, this reduces to the dipole gauge field $a_{ij}$ discussed previously. The dipole shift transformations can now be stated as

$$A_\mu \to A_\mu + \psi_\mu\,, \qquad a_{\mu\nu} \to a_{\mu\nu} + h^\rho_\mu h^\sigma_\nu \left(\nabla_\rho \psi_\sigma + \nabla_\sigma \psi_\rho\right), \tag{3.8}$$

for some dipole shift parameter $\psi_\mu$ obeying $v^\mu \psi_\mu = 0$. We will study the consistency of this symmetry as a Lie algebra in the next Subsection.

Note that the dipole source $a_{\mu\nu}$ cannot be used as a connection to define "dipole-covariant derivatives," because it is only sensitive to the symmetric spatial derivative of $\psi_\mu$. However, we can define an object that is nearly a dipole connection by combining $F_{\mu\nu}$ and $a_{\mu\nu}$ as

$$A^\lambda{}_\mu = n_\mu v^\rho F_{\rho\sigma} h^{\sigma\lambda} + \frac{1}{2}\left(h^\rho_\mu F_{\rho\sigma} h^{\sigma\lambda} + a_{\mu\sigma} h^{\sigma\lambda}\right). \tag{3.9}$$

Note that $n_\lambda A^\lambda{}_\mu = 0$. It can be checked that this object transforms as

$$A^\lambda{}_\mu \to A^\lambda{}_\mu + \nabla_\mu \psi^\lambda + n_\mu \psi^\nu \nabla_\nu v^\lambda\,, \tag{3.10}$$

where $\psi^\mu = h^{\mu\nu}\psi_\nu$. We can also define the "dipole field strength"

$$F^\lambda{}_{\mu\nu} = \nabla_\mu A^\lambda{}_\nu - \nabla_\nu A^\lambda{}_\mu + F^n_{\mu\nu} v^\rho A^\lambda{}_\rho + 2 n_{[\mu} A^\rho{}_{\nu]} \nabla_\rho v^\lambda\,, \tag{3.11}$$

which transforms as

$$F^\lambda{}_{\mu\nu} \to F^\lambda{}_{\mu\nu} + \left(R^\lambda{}_{\rho\mu\nu} + F^n_{\mu\nu}\nabla_\rho v^\lambda - 2 n_{[\mu}\nabla_{\nu]}\nabla_\rho v^\lambda\right)\psi^\rho\,. \tag{3.12}$$

The "dipole field strength" $F^\lambda{}_{\mu\nu}$ is nothing more than the curved space version of the dipole electric/magnetic fields discussed in (2.13). Note $F^\lambda{}_{\mu\nu}$ is not dipole-invariant in a general background. As a result there is no way to define curved space versions of the dipole electric and magnetic fields $E_{ij}$ and $B_{ijk}$ while preserving the dipole symmetry. This presents an obstruction to coupling the symmetric tensor gauge theory in (2.14) to a curved spacetime, which we discuss briefly at the end of this Section.

These definitions will be useful later when we compute the algebra of charges. There, $A^\lambda{}_\mu$ and $F^\lambda{}_{\mu\nu}$ appear as the dipole versions of $A_\mu$ and $F_{\mu\nu}$ respectively.

In is interesting to note that a priori $A^\lambda{}_\mu$ has $d(d+1)$ independent components on account of the condition $n_\lambda A^\lambda{}_\mu = 0$, as we would expect for a connection for a $d$-parameter symmetry transformation. However, in our case, it only has $d(d+1)/2$ independent components in the form of $a_{\mu\nu} = 2h_{\lambda(\mu} h^\rho_{\nu)} A^\lambda{}_\rho$. The remaining $d(d+1)/2$ components are fixed in terms of the $U(1)$ field strength $F_{\mu\nu}$ via the relation $A^\lambda{}_\mu h_{\nu\lambda} - A^\lambda{}_\nu h_{\mu\lambda} = F_{\mu\nu}$. This may seem like a technical aside, but there is some potentially interesting physics here. In Section 2 we considered field theories with a conserved dipole moment. In these models the elementary, bare, excitations are $U(1)$ charges. Charges can acquire dipole moments through quantum corrections, and perhaps there might be bound charge-anticharge states with nonzero dipole moment. For such models the dipole density is not an independent operator from the charge density. However one can envision effective low-energy field theories with elementary *dipoles* with an "internal" dipole density $\mathcal{D}^i$, so that the total dipole moment is of the form

$$D^i = \int d^d x \left(J^t x^i + \mathcal{D}^i\right). \tag{3.13}$$

In such a model the dipole Ward identity would be modified to become $J^i = \partial_t \mathcal{D}^i + \partial_j J^{ij}$, where the dipole current $J^{ij}$ need no longer be symmetric. The "internal" dipole density $\mathcal{D}^i$ and antisymmetric part of $J^{ij}$ have just the right number of independent components to couple to the remaining $d(d+1)/2$ components of $A^\lambda{}_\mu$. Indeed, it is easy to see that this is the correct interpretation for the remaining components of $A^\lambda{}_\mu$, i.e. if they were present, they would couple to precisely such an "internal" dipole density and antisymmetric dipole current.

### 3.3 Conserved currents and Ward identities

We define the symmetry currents through the variation of the generating functional $W = -i \ln Z$ with respect to background fields:[6]

$$\delta W = \int \mathrm{d}^{d+1}x \sqrt{\gamma} \left( -\epsilon^\mu \delta n_\mu + \left( v^{(\mu} \pi^{\nu)} + \frac{1}{2} \tau^{\mu\nu} \right) \delta h_{\mu\nu} + J^\mu \delta A_\mu + J^\mu{}_\lambda \delta A^\lambda{}_\mu \right), \tag{3.14}$$

where $\gamma = \det(n_\mu n_\nu + h_{\mu\nu})$. Here $\epsilon^\mu$ is the energy current, $\pi^\mu$ the momentum current satisfying $\pi^\mu n_\mu = 0$, $\tau^{\mu\nu}$ the spatial stress tensor satisfying $\tau^{\mu\nu} n_\nu = 0$, $J^\mu$ the $U(1)$ current, and $J^\mu{}_\lambda$ the dipole current. This expression should be understood as the "covariant definition" of the conserved currents in curved spacetime. Explicitly picking coordinates $x^\mu = (t, x^i)$, where we only mandate $v^t \neq 0$, the various components of the currents read

$$\begin{gathered}\epsilon^\mu = \begin{pmatrix} \epsilon \\ \epsilon^i \end{pmatrix}, \qquad \pi_\mu = \begin{pmatrix} -v^k \pi_k / v^t \\ \pi_i \end{pmatrix}, \qquad \tau^{\mu\nu} = \begin{pmatrix} n_k n_l \tau^{kl}/n_t^2 & -n_k \tau^{kj}/n_t \\ -n_k \tau^{ik}/n_t & \tau^{ij} \end{pmatrix}, \\ J^\mu = \begin{pmatrix} J^t \\ J^i \end{pmatrix}, \qquad J^{\mu\nu} = \begin{pmatrix} n_k n_l J^{kl}/n_t^2 & -n_k J^{kj}/n_t \\ -n_k J^{ik}/n_t & J^{ij} \end{pmatrix},\end{gathered} \tag{3.15}$$

where indices have been raised and lowered using the spatial metric $h_{\mu\nu}$ and its "inverse" $h^{\mu\nu}$. These expressions satisfy the various identities $\pi_\mu v^\mu = \tau^{\mu\nu} n_\nu = J^{\mu\nu} n_\nu = 0$, $\tau^{[\mu\nu]} = 0$, and $J^{[\mu\nu]} = 0$.

We have in mind a dipole-symmetric field theory coupled to a background spacetime in such a way as to be invariant under diffeomorphisms, $U(1)$ gauge transformations, and dipole transformations, acting on both the quantum and background fields. More precisely, in the absence of anomalies, the generating functional $W$ of the theory is required to be invariant under these transformation, leading to the Ward identities. Let $\chi^\mu$ denote an infinitesimal diffeomorphism, $\Lambda$ an infinitesimal gauge transformation, and $\psi_\mu$ an infinitesimal dipole transformation. Collectively denoting them as $\hat{\mathcal{X}} = (\chi^\mu, \Lambda, \psi_\mu)$, the action of these transformations on the background fields is given by

$$\begin{aligned} \delta_{\hat{\mathcal{X}}} n_\mu &= \$_\chi n_\mu , \\ \delta_{\hat{\mathcal{X}}} h_{\mu\nu} &= \$_\chi h_{\mu\nu} , \\ \delta_{\hat{\mathcal{X}}} A_\mu &= \$_\chi A_\mu + \partial_\mu \Lambda + \psi_\mu , \\ \delta_{\hat{\mathcal{X}}} A^\lambda{}_\mu &= \$_\chi A^\lambda{}_\mu + \nabla_\mu \psi^\lambda + n_\mu \psi^\nu \nabla_\nu v^\lambda . \end{aligned} \tag{3.16}$$

By assumption the generating functional is invariant under these transformations, i.e. $\delta_{\hat{\mathcal{X}}} W = 0$. This variation is given by plugging the symmetry variation in eq. (3.16) into

[6] We have decided to couple $J^{\mu\nu}$ to the full connection $A^\lambda{}_\mu$ instead of the spatial connection $a_{\mu\nu}$. This is purely a matter of definition because $J^{\mu\nu} h_{\nu\lambda} \delta A^\lambda{}_\mu = J^{\mu\nu} v^\rho F_{\rho\nu} \delta n_\mu - J^{\mu\rho} A^\nu{}_\rho \delta h_{\mu\nu} + \frac{1}{2} J^{\mu\nu} \delta a_{\mu\nu}$. This should be thought of as a convenient definition of other conserved currents in the presence of a dipole symmetry, so that they have "nice" transformations properties under the dipole shift symmetry.

$\delta W$ in eq. (3.14). Imposing that this symmetry variation vanishes and integrating by parts, we find the following Ward identities

$$\begin{aligned} \nabla'_\mu \epsilon^\mu &= -v^\mu f_\mu - \left(\tau^{\mu\nu} + \tau_{\mathrm{d}}^{\mu\nu}\right) h_{\lambda\nu} \nabla_\mu v^\lambda \,, \\ \nabla'_\mu \left(v^\mu \pi^\nu + \tau^{\mu\nu} + \tau_{\mathrm{d}}^{\mu\nu}\right) &= h^{\nu\mu} f_\mu - \pi_\mu h^{\nu\lambda} \nabla_\lambda v^\mu \,, \\ \nabla'_\mu J^\mu &= 0 \,, \\ \nabla'_\mu J^{\mu\nu} &= h^\nu_\mu J^\mu \,, \end{aligned} \tag{3.17}$$

where $\nabla'_\mu = \nabla_\mu + F^n_{\mu\nu} v^\nu$ and

$$\begin{aligned} f_\mu &= -F^n_{\mu\nu}\epsilon^\nu - h_{\mu\lambda}A^\lambda{}_\nu J^\nu + F^\lambda{}_{\mu\nu} h_{\rho\lambda} J^{\nu\rho} - n_\mu A^\lambda{}_\rho J^\rho{}_\nu \nabla_\lambda v^\nu \,, \\ \tau_{\mathrm{d}}^{\mu\nu} &= -A^\mu{}_\rho J^{\rho\nu} \,. \end{aligned} \tag{3.18}$$

The $f_\mu$ contributions to the right-hand-side of the energy and momentum conservation Ward identities correspond to the power-force density due to the background fields sourcing energy and momentum, analogous to the familiar Joule heating term in $F_{\mu\nu}J^\nu$ in the stress tensor Ward identity of a relativistic field theory. The terms involving $\nabla_\mu v^\nu$, on the other hand, can be thought of as pseudo-power and pseudo-force contributions due to the background observed not being inertial. On the other hand, $\tau_{\mathrm{d}}^{\mu\nu}$ is an effective contribution to the stress tensor due to dipole sources.

The Ward identities are manifestly covariant under diffeomorphisms, with the background fields and currents transforming as tensors, and are invariant under gauge transformations. However, the behaviour of various fields under dipole transformations is not so obvious. To derive the transformation laws of the currents, we use the dipole symmetry to equate the generating functional $W$ with its value evaluated on a dipole-transformed background

$$W[n_\mu, h_{\nu\rho}; A_\mu, A^\lambda{}_\mu] = W[n_\mu, h_{\nu\rho};\, A_\mu + \psi_\mu \,,\, A^\lambda{}_\mu + \nabla_\mu \psi^\lambda + n_\mu \psi^\nu \nabla_\nu v^\lambda \,] \,, \tag{3.19}$$

and then take a variation of this identity with respect to the various background fields. The variations of the left-hand-side give the currents, and those of the right-hand-side can be expressed in terms of the dipole-transformed currents, which follow from the variations of with respect to the transformed background fields. For infinitesimal $\psi_\mu$, this amounts to setting the second variation $\delta(\delta_\psi W)$ to vanish. This second variation reads

$$\begin{aligned} \delta(\delta_\psi W) = \int \mathrm{d}^{d+1}x \sqrt{\gamma}\Big( &-\delta_\psi \epsilon^\mu \delta n_\mu + \left(v^\mu \delta_\psi \pi^\nu + \frac{1}{2}\delta_\psi \tau^{\mu\nu}\right)\delta h_{\mu\nu} + \delta_\psi J^\mu \delta A_\mu + \delta_\psi J^\mu{}_\lambda \delta A^\lambda_\mu \\ &+ J^\mu \psi^\sigma \delta h_{\sigma\mu} + \left(2J^{\mu(\rho}\psi^{\sigma)} - J^{\rho\sigma}\psi^\mu\right)\frac{1}{2}\$_v h_{\rho\sigma}\delta n_\mu \\ &- J^{\mu\sigma} v^\rho F^n_{\mu\nu}\psi^\nu \delta h_{\sigma\rho} + \frac{1}{2} J^{\mu\nu}\psi^\lambda \nabla_\lambda \delta h_{\mu\nu}\Big) \,, \end{aligned} \tag{3.20}$$

where we have used the dipole Ward identity. In deriving this we have used the variation of the connection

$$\begin{aligned} \delta\Gamma^\lambda{}_{\mu\nu} = &\, v^\lambda \nabla_\mu \delta n_\nu + \frac{1}{2} h^{\lambda\rho}\left(\nabla_\mu \delta h_{\nu\rho} + \nabla_\nu \delta h_{\mu\rho} - \nabla_\rho \delta h_{\mu\nu}\right) \\ &+ \frac{1}{2} h^{\lambda\rho}(\$_v h_{\mu\nu})\delta n_\rho - \frac{1}{2} h^{\lambda\sigma}\left(v^\rho F^n_{\mu\nu}\delta h_{\sigma\rho} + 2v^\rho F^n_{\sigma(\mu}\delta h_{\nu)\rho}\right) . \end{aligned} \tag{3.21}$$

Reading off the contributions coming from individual background field variations, we infer that the currents shift under dipole transformations as

$$\begin{aligned}\epsilon^\mu &\to \epsilon^\mu + \left(2J^{\mu(\rho}\psi^{\sigma)} - J^{\rho\sigma}\psi^\mu\right)\frac{1}{2}\$_\nu h_{\rho\sigma}\,,\\ \pi^\mu &\to \pi^\mu - (J^\nu n_\nu)\psi^\mu + J^{\rho\mu}F^n_{\rho\sigma}\psi^\sigma\,,\\ \tau^{\mu\nu} &\to \tau^{\mu\nu} - 2J^\lambda h^{(\mu}_\lambda\psi^{\nu)} + \nabla'_\lambda(\psi^\lambda J^{\mu\nu})\,,\end{aligned} \tag{3.22}$$

while $J^\mu$ and $J^{\mu\nu}$ are invariant. One can explicitly see that this is a symmetry of the Ward identities.

It is interesting to note that the energy density and flux contained in $\epsilon^\mu$ are invariant under dipole shifts in flat space, but the same is not true for the momentum density $\pi^\mu$ or the stress tensor $\tau^{\mu\nu}$. One physical implication of this fact is that a relativistic field theory cannot have a conserved dipole moment, because momentum density is equated to energy flux by Lorentz boost symmetry. The same is true for a Galilean theory, where the Galilean boost symmetry equates momentum density to charge (particle number) flux. These transformation properties of conserved currents can also be used to diagnose the consistency of seemingly dipole-symmetric field theories written out in flat space. We shall illustrate this in the next subsection for the symmetric tensor gauge theory given in Eq. (2.14).

### 3.4 Dipole-symmetric field theory in curved space

We can use the Aristotelian background sources to write down the covariant version of the field theories mentioned in Section 2. Firstly, we note that we can define a gauge covariant derivative for the scalar field $\Phi$ with charge $q$ as

$$\mathrm{D}_\mu\Phi = \partial_\mu\Phi - iqA_\mu\Phi\,, \tag{3.23}$$

however, it is not invariant under the dipole shift symmetry: $\mathrm{D}_\mu\Phi \to \mathrm{D}_\mu\Phi - iq\psi_\mu\Phi$. The "time-component" of the covariant derivative $v^\mu\mathrm{D}_\mu\Phi$ is still dipole-invariant, but for the spatial part we instead need to define a double derivative operator

$$\mathrm{D}_{\mu\nu}(\Phi,\Phi) = h^\rho_{(\mu}h^\sigma_{\nu)}\left(\Phi\,\mathrm{D}_\rho\mathrm{D}_\sigma\Phi - \mathrm{D}_\rho\Phi\,\mathrm{D}_\sigma\Phi\right) + \frac{iq}{2}a_{\mu\nu}\Phi^2\,. \tag{3.24}$$

With this, the original scalar field theory may be written covariantly as

$$S = \int \mathrm{d}^{d+1}x\,\sqrt{\gamma}\Big(i\Phi^* v^\mu\mathrm{D}_\mu\Phi + \lambda h^{\mu\rho}h^{\nu\sigma}\mathrm{D}_{\mu\nu}(\Phi^*,\Phi^*)\mathrm{D}_{\rho\sigma}(\Phi,\Phi) - V(\Phi^*\Phi)\Big)\,. \tag{3.25}$$

Now that we have a theory coupled to background sources, we can read off all the conserved currents by varying with respect to these. We find

$$\begin{aligned}\epsilon^\mu &= -v^\mu\Big(\lambda h^{\lambda\rho}h^{\tau\sigma}\mathrm{D}_{\lambda\tau}(\Phi^*,\Phi^*)\mathrm{D}_{\rho\sigma}(\Phi,\Phi) - V(\Phi^*\Phi)\Big)\,,\\ &\qquad + 2\lambda\Big(\mathrm{D}^{\mu\rho}(\Phi^*,\Phi^*)\left(\Phi\, v^\nu\mathrm{D}_{(\nu}\mathrm{D}_{\rho)}\Phi - v^\nu\mathrm{D}_\nu\Phi\,\mathrm{D}_\rho\Phi\right) + \text{c.c.}\Big)\\ &\qquad\quad + \Big(iq\lambda\mathrm{D}^{\mu\nu}(\Phi^*,\Phi^*)\Phi^2 - iq\lambda\Phi^2\mathrm{D}^{\mu\nu}(\Phi,\Phi)\Big)v^\rho F_{\rho\nu}\,,\\ \pi^\mu &= -ih^{\mu\nu}\Phi^*\mathrm{D}_\nu\Phi\,,\\ \tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}} &= h^{\mu\nu}\Big(i\Phi^* v^\nu\mathrm{D}_\nu\Phi + \lambda h^{\lambda\rho}h^{\tau\sigma}\mathrm{D}_{\lambda\tau}(\Phi^*,\Phi^*)\mathrm{D}_{\rho\sigma}(\Phi,\Phi) - V(\Phi^*\Phi)\Big)\,,\\ &\qquad - 4\lambda h_{\rho\sigma}\mathrm{D}^{\rho(\mu}(\Phi^*,\Phi^*)\mathrm{D}^{\nu)\sigma}(\Phi,\Phi)\\ &\qquad\quad + \Big(iq\lambda\mathrm{D}^{\mu\rho}(\Phi^*,\Phi^*)\Phi^2 - iq\lambda\Phi^2\mathrm{D}^{\mu\rho}(\Phi,\Phi)\Big)A^\nu_{\ \rho}\,,\\ J^\mu &= q\Phi^*\Phi v^\mu + \mathrm{D}_\nu\Big(iq\lambda\mathrm{D}^{\mu\nu}(\Phi^*,\Phi^*)\Phi^2 - iq\lambda\Phi^2\mathrm{D}^{\mu\nu}(\Phi,\Phi)\Big)\,,\\ J^{\mu\nu} &= iq\lambda\mathrm{D}^{\mu\nu}(\Phi^*,\Phi^*)\Phi^2 - iq\lambda\Phi^2\mathrm{D}^{\mu\nu}(\Phi,\Phi)\,.\end{aligned} \tag{3.26}$$

What of the symmetric tensor gauge theory (2.14)? The curved space versions of the electric field $E_{ij} = 2F^{i}{}_{jt}$ and (the Hodge dual of) magnetic fields $F_{ijk} = 2F^{k}{}_{ij}$ are encoded in the same object $F^{\lambda}{}_{\mu\nu}$ defined in Eq. (3.11). However, as we saw in Eq. (3.12), this object is not dipole-invariant in a general spacetime. While we cannot conclusively rule out the possibility that there is another dipole- and gauge-invariant curved space tensor that reduces to $E_{ij}$ and $F_{ijk}$ in flat space, we have worked rather hard to find such an object without success. As such, we are inclined to take this non-invariance seriously and arrive at the conclusion that there is an obstruction to defining a dipole-symmetric version of the electric and magnetic fields in a general curved space.[7] Because the dipole symmetry is really a part of the gauge symmetry of this model, rather than a global symmetry, this renders the symmetric tensor gauge theory inconsistent in a general background.

This obstruction has a simple explanation in the flat space theory. Consider gauge fixing the dipole shift symmetry by setting $A_i = 0$, which fuses the dipole transformation to $U(1)$ gauge transformations as $\psi_i = -\partial_i \Lambda$. Then, using the transformation rules in Eq. (3.22), we infer that the energy current is gauge-invariant, but the momentum density $\pi^i$ and spatial stress tensor $\tau^{ij}$ are no longer gauge invariant. In fact, in flat space, they ought to transform as $\pi^i \to \pi^i + J^t \partial^i \Lambda$ and $\tau^{ij} \to \tau^{ij} + 2\partial_k J^{k(i)} \partial^{j}\Lambda - \partial_k(\partial^k \Lambda J^{ij})$. However, the equations of motion set $J^t$ and $J^{ij}$ to zero. Consequently, the full stress tensor ought to be dipole-invariant on-shell. Relatedly, the spatial stress tensor $\tau^{ij}$ ought to be symmetric on-shell. Interestingly, we find that this is not the case for the symmetric tensor gauge theory given by the action in Eq. (2.14): the momentum current and spatial stress tensor are not gauge-invariant, nor is the spatial stress tensor symmetric.

Note that the canonical momentum density, the Noether density that generates spatial translations, is already gauge-non-invariant in ordinary Maxwell theory. But one can still define an improved gauge-invariant momentum density as the generator of a spatial translation together with an appropriate gauge transformation. Consider a symmetry transformation involving a spatial translation along a constant translation parameter $\chi^i$ and a gauge transformation $\Lambda = -\chi^i \lambda_i$, for some quantity $\lambda_i$ that depends on the gauge fields. The improved momentum density of Maxwell theory is

$$\pi_i = \frac{1}{\mu_0}\left(\partial_i A_j - \partial_j \lambda_i\right) E^j \,, \tag{3.27}$$

which can be rendered gauge-invariant by setting $\lambda_i = A_i$, leading to the standard expression for momentum density as $\pi_i = \frac{1}{\mu_0} F_{ij} E^j$. Applying the same procedure to the symmetric tensor gauge theory in Eq. (2.14), we get the momentum density

$$\pi_i = \epsilon_0 \left(\partial_i a_{jk} + 2\partial_j \partial_k \lambda_i\right) E^{jk} \,. \tag{3.28}$$

This expression cannot be made gauge-invariant for any choice of $\lambda_i$, so we may as well set $\lambda_i = 0$. We have similarly computed the energy current and spatial stress tensor for the theory (2.14). There is a gauge-invariant improved energy current, but the spatial stress tensor

[7] There can, of course, be special curved backgrounds to which the symmetric tensor gauge theory can be coupled. A notable example comes from the work of [22, 23]. Consider a background spacetime where time is absolute and the spatial metric is time-independent, i.e. $\nabla_\mu v^\nu = \frac{1}{2} h^{\nu\rho} h^{\sigma}_{\mu} \$_v h_{\rho\sigma} = 0$, and whose spatial part is an Einstein manifold, i.e. $R^{\lambda}{}_{\rho\mu\nu} = \frac{1}{L^2}(h^{\lambda}_{\mu} h_{\rho\nu} - h^{\lambda}_{\nu} h_{\rho\mu})$. For such background spacetimes, we find $F^{\lambda}{}_{\mu\nu} \to F^{\lambda}{}_{\mu\nu} + \frac{1}{L^2}(h^{\lambda}_{\mu}\psi_\nu - h^{\lambda}_{\nu}\psi_\mu)$. Consequently, the curved space electric field $F^{\lambda}{}_{\nu\rho} v^\rho$ and the "symmetric" magnetic field $F^{\lambda}{}_{\mu\nu} + \frac{2}{d-1} F^{\rho}{}_{\rho[\mu} h^{\lambda}_{\nu]}$ are dipole-symmetric, and can be used to define a covariant version of the symmetric tensor gauge theory in Eq. (2.14). This conclusion is consistent with that of [23].

is not gauge-invariant either. We find

$$
\begin{aligned}
\tau^{ij} = & -\epsilon_0 E^{ik}E^j{}_k + \frac{1}{\mu_0}F^{ikl}F^j{}_{kl} + \frac{1}{2}\delta^{ij}\left(\epsilon_0 E_{kl}E^{kl} - \frac{1}{2\mu_0}F_{klm}F^{klm}\right) \\
& - \epsilon_0 E^{ik}\partial_t\left(a^j{}_k + 2\partial_k\lambda^j\right) - 2\epsilon_0\partial_k E^{ki}\left(\partial^j A_t - \partial_t\lambda^j\right) + \frac{1}{\mu_0}F^{ikl}\partial_k\left(a^j{}_l + 2\partial_l\lambda^j\right),
\end{aligned} \tag{3.29}
$$

which is neither gauge-invariant nor symmetric for any choice of $\lambda_i$.

Suppose, now, that we couple the flat space theory to linearised perturbations of the background spacetime. On account of the non-symmetric spatial stress tensor, in order to maintain diffeomorphism invariance, this can be done only by coupling to a background in the first-order formalism, where we decompose the spatial metric into a spatial vielbein as $h_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu$, with $a = 1, 2, \ldots, d$. This is done with a perturbation in the action,

$$
\delta S = \int \mathrm{d}t\mathrm{d}^d x\left(-\epsilon^t\delta n_t - \epsilon^i\delta n_i + \pi^i\delta e^i_t + \tau^{ij}\delta e^i_j\right). \tag{3.30}
$$

This linearised coupling is diffeomorphism invariant on account of the conservation equations

$$
\partial_\mu\epsilon^\mu = 0, \qquad \partial_t\pi^i + \partial_j\tau^{ij} = 0, \tag{3.31}
$$

however it is not gauge-invariant, rendering the model inconsistent. We therefore conclude that the symmetric tensor gauge theory cannot be consistently coupled to curved spacetime while preserving diffeomorphism invariance. This is reminiscent of a mixed gauge-gravitational anomaly. We however hesitate to use that term just yet. In relativistic field theories with mixed flavor-gravitational anomalies, one can redefine the theory in such a way as to be either non-invariant under flavor transformations or under diffeomorphisms. Here, it is not yet clear if one can redefine the tensor gauge theory in such a way as to be gauge-invariant, but non-covariant in curved space.

However, if charged matter coupled to the gauge fields condenses so that the tensor gauge theory is in a Higgs phase, then the massive gauge theory may be placed in curved spacetime. Consider the following invariant action where we also introduce a charged scalar $\Phi$ with action

$$
S = \int \mathrm{d}^{d+1}x\, h_{\lambda\tau}h^{\nu\sigma}\mathcal{F}^\lambda{}_{\mu\nu}\mathcal{F}^\tau{}_{\rho\sigma}\left(2\epsilon_0 v^\mu v^\rho - \frac{1}{\mu_0}h^{\mu\rho}\right). \tag{3.32}
$$

We have defined the dipole-invariant combination of the dipole field strength $F^\lambda{}_{\mu\nu}$ and the charged scalar field $\Phi$ via

$$
\mathcal{F}^\lambda{}_{\mu\nu} = \Phi^*\Phi F^\lambda{}_{\mu\nu} - \frac{i}{q}\Phi^*\mathrm{D}_\rho\Phi\, h^{\rho\sigma}\left(R^\lambda{}_{\sigma\mu\nu} + F^n_{\mu\nu}\nabla_\sigma v^\lambda - 2n_{[\mu}\nabla_{\nu]}\nabla_\sigma v^\lambda\right). \tag{3.33}
$$

Of course, we can also add the usual dipole-invariant $\Phi$ terms to the action from the theory in Eq. (3.25). Unlike the theory in Eq. (2.14), however, in the flat space limit the $E_{ij}E^{ij}$ and $F_{ijk}F^{ijk}$ terms in this theory come coupled to a factor of $\Phi^*\Phi$. Assuming that the dynamics allow $\Phi$ to condense, this model has a Higgs phase whose low-energy description ought to be a massive version of the symmetric gauge theory (2.14) coupled to the phase of $\Phi$.

## 4 First order formulation and dipole symmetry algebra

In the previous Section, we have worked with the second-order formulation of Aristotelian geometries, where the curved spacetime background is captured by a clock-form $n_\mu$ and a spatial metric $h_{\mu\nu}$. While this formulation is sufficient to couple to field theories with conserved dipole moment, to better appreciate the structure of the dipole shift symmetry, it is convenient to pass to a first-order formulation. Here the spatial metric $h_{\mu\nu}$ is exchanged for a spatial vielbein $e^a_\mu$ and a local $SO(d)$ symmetry which rotates the spatial one-forms $e^a_\mu$ into each other. The dipole symmetry is then naturally defined as acting in the tangent bundle.

### 4.1 First order formulation of Aristotelian geometries

The first-order formulation of Aristotelian backgrounds has the clock-form $n_\mu$ and a spatial vielbein $e^a_\mu$, where $a = 1, \ldots, d$ is an index enumerating this basis of spatial one-forms. There must be a zero linear combination $v^\mu e^a_\mu = 0$ for some vector field $v^\mu$ normalised as $v^\mu n_\mu = 1$. We can also define the inverse vielbein $e^\mu_a$ using the completeness relations $v^\mu n_\nu + e^\mu_a e^a_\nu = \delta^\mu_\nu$ and $e^\mu_a e^b_\mu = \delta^b_a$. The spatial metric $h_{\mu\nu}$ and cometric $h^{\mu\nu}$ are related in terms of these as

$$h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu \,, \qquad h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b \,. \tag{4.1}$$

The raising and lowering of $a, b, \ldots$ indices is done using $\delta^{ab}$ and $\delta_{ab}$. Note that the spatial metric $h_{\mu\nu}$ has $(d+1)(d+2)/2-1$ independent components, while the spatial vielbein $e^a_\mu$ has $d(d+1)$ independent components. This additional $d(d-1)/2$ components in $e^a_\mu$ arise from a redundancy under local $SO(d)$ rotations that arises from the decomposition of $h_{\mu\nu}$ into $e^a_\mu$. This local $SO(d)$ symmetry acts as $e^a_\mu \to R^a{}_b e^b_\mu$ for $R$ a rotation matrix.

We also introduce a spin connection $\omega^a{}_{b\mu}$ associated with the $SO(d)$ symmetry,

$$\omega^a{}_{b\mu} = e^a_\lambda \partial_\mu e^\lambda_b + e^a_\lambda \Gamma^\lambda{}_{\mu\nu} e^\nu_b \,, \tag{4.2}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the connection defined in Eq. (3.1). It can be explicitly checked that $\omega^{ab}{}_\mu = -\omega^{ba}{}_\mu$ and the associated covariant derivative, which acts both on spacetime indices $\mu$, $\nu$ and flat spatial indices $a, b$, satisfies the properties

$$\begin{aligned} \nabla_\mu \delta_{ab} &= -\delta_{cb}\omega^c{}_{a\mu} - \delta_{ac}\omega^c{}_{b\mu} = 0 \,, \\ \nabla_\mu e^\nu_a &= \partial_\mu e^\nu_a + \Gamma^\nu{}_{\mu\rho} e^\rho_a - e^\nu_b \omega^b{}_{a\mu} = 0 \,, \\ \nabla_\mu e^a_\nu &= \partial_\mu e^a_\nu - \Gamma^\rho{}_{\mu\nu} e^a_\rho + e^b_\nu \omega^a{}_{b\mu} = -\frac{1}{2} n_\nu e^{a\rho} \$_v h_{\mu\rho} \,. \end{aligned} \tag{4.3}$$

If we were to introduce torsion into the Aristotelian connection (3.1) (see Footnote 5), it would correspondingly introduce a torsion into the spin connection. We can define the associated spatial torsion and curvature tensors as

$$\begin{aligned} T^a{}_{\mu\nu} &= \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a{}_{b\mu} e^b_\nu - \omega^a{}_{b\nu} e^b_\mu \,, \\ R^a{}_{b\mu\nu} &= \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} \,. \end{aligned} \tag{4.4}$$

These are related to the full Aristotelian torsion and curvature tensors as

$$\begin{aligned} T^\lambda{}_{\mu\nu} &= e^\lambda_a T^a{}_{\mu\nu} - 2 n_{[\mu} \nabla_{\nu]} v^\lambda + v^\lambda F^n_{\mu\nu} \,, \\ R^\lambda{}_{\sigma\mu\nu} &= e^\lambda_a e^b_\sigma R^a{}_{b\mu\nu} + 2 n_\sigma \nabla_{[\mu} \nabla_{\nu]} v^\lambda \,. \end{aligned} \tag{4.5}$$

For our choice of connection, the Aristotelian torsion is simply $T^\lambda{}_{\mu\nu} = v^\lambda F^n_{\mu\nu}$ and correspondingly $T^a{}_{\mu\nu} = e_{a\lambda} n_{[\mu} \nabla_{\nu]} v^\lambda$.

The $U(1)$ connection $A_\mu$ is borrowed directly from the second order formulation, whereas the spatial dipole gauge field is instead taken to be $a_{ab}$. The covariant spatial dipole gauge field $a_{\mu\nu}$ is defined as

$$a_{\mu\nu} = a_{ab} e^a_\mu e^b_\nu \,, \tag{4.6}$$

and automatically satisfies the constraint $v^\mu a_{\mu\nu} = 0$. We can define the dipole "connection" and "field strength" (the analogues of $A^\lambda{}_\mu$ and $F^\lambda{}_{\mu\nu}$) in the first-order formulation as

$$\begin{aligned} A^a{}_\mu &= n_\mu v^\rho F_{\rho\sigma} e^{a\sigma} + \frac{1}{2}\left(h^\rho_\mu F_{\rho\sigma} e^{a\sigma} + a^{ab} e_{b\mu}\right), \\ F^a{}_{\mu\nu} &= \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + \omega^a{}_{b\mu} A^b{}_\nu - \omega^a{}_{b\nu} A^b{}_\mu + n_{[\mu} A^b{}_{\nu]} e^{a\rho} e^\sigma_b \$_v h_{\rho\sigma} . \end{aligned} \tag{4.7}$$

Now, when coupling a field theory to this first-order formulation of the background, we impose invariance under diffeomorphisms, local rotations, $U(1)$ gauge transformations, and dipole transformations. At the infinitesimal level, these act on the background fields in the following way. Let $\chi^\mu$ be an infinitesimal diffeomorphism, $\Omega^a{}_b$ an infinitesimal rotation (with $\Omega_{(ab)} = 0$), $\Lambda$ a $U(1)$ gauge transformation, and $\psi_a$ an infinitesimal dipole transformation. Collectively denoting the transformation as $\hat{\mathfrak{X}} = (\xi^\mu, \Omega^a{}_b, \Lambda, \psi_a)$, the corresponding generator $\delta_{\hat{\mathfrak{X}}}$ of the transformation acts on the background fields as

$$\begin{aligned} \delta_{\hat{\mathfrak{X}}} n_\mu &= \$_\chi n_\mu , \\ \delta_{\hat{\mathfrak{X}}} e^a_\mu &= \$_\chi e^a_\mu - \Omega^a{}_b e^b_\mu , \\ \delta_{\hat{\mathfrak{X}}} A_\mu &= \$_\chi A_\mu + \partial_\mu \Lambda + e^a_\mu \psi_a , \\ \delta_{\hat{\mathfrak{X}}} a_{ab} &= \$_\chi a_{ab} + \Omega^c{}_a a_{cb} + \Omega^c{}_b a_{ac} + e^\mu_a \nabla_\mu \psi_b + e^\mu_b \nabla_\mu \psi_a . \end{aligned} \tag{4.8}$$

So defined, the spin and "dipole" connections transform as

$$\begin{aligned} \delta_{\hat{\mathfrak{X}}} \omega^a{}_{b\mu} &= \$_\chi \omega^a{}_{b\mu} + \nabla_\mu \Omega^a{}_b , \\ \delta_{\hat{\mathfrak{X}}} A^a_\mu &= \$_\chi A^a{}_\mu - \Omega^a{}_b A^b{}_\mu + \nabla_\mu \psi^a + \frac{1}{2} n_\mu \psi^b e^{a\rho} e^\sigma_b \$_v h_{\rho\sigma} , \end{aligned} \tag{4.9}$$

so that the spin connection is indeed a connection under local rotations, and $A^a{}_\mu$ is nearly a connection under dipole transformations. It can be checked that these symmetry variations lead to the second-order symmetry variations given in eq. (3.16). A lengthy computation also shows that the symmetry generators form an algebra, with

$$[\delta_{\hat{\mathfrak{X}}}, \delta_{\hat{\mathfrak{X}}'}] = \delta_{[\hat{\mathfrak{X}}, \hat{\mathfrak{X}}']} , \tag{4.10}$$

where the commutator transformation is defined as

$$\begin{aligned} \chi^\mu_{[\hat{\mathfrak{X}}', \hat{\mathfrak{X}}]} &= \delta_{\hat{\mathfrak{X}}'} \chi^\mu &&= \$_{\chi'} \chi^\mu , \\ (\Omega_{[\hat{\mathfrak{X}}', \hat{\mathfrak{X}}]})^a{}_b &= \delta_{\hat{\mathfrak{X}}'} \Omega^a{}_b &&= \$_{\chi'} \Omega^a{}_b - \$_\chi \Omega'^a{}_b + \Omega^a{}_c \Omega'^c{}_b - \Omega'^a{}_c \Omega^c{}_b , \\ \Lambda_{[\hat{\mathfrak{X}}', \hat{\mathfrak{X}}]} &= \delta_{\hat{\mathfrak{X}}'} \Lambda &&= \$_{\chi'} \Lambda - \$_\chi \Lambda' , \\ (\psi_{[\hat{\mathfrak{X}}', \hat{\mathfrak{X}}]})_a &= \delta_{\hat{\mathfrak{X}}'} \psi_a &&= \$_{\chi'} \psi_a - \$_\chi \psi'_a + \psi_b \Omega'^b{}_a - \psi'_b \Omega^b{}_a . \end{aligned} \tag{4.11}$$

More details can be found in Appendix (A.1). Applying the identity (4.10) to the generating functional leads to Wess-Zumino consistency conditions, just as for relativistic field theories.

## 4.2 Currents and conservation

The coupling of conserved currents to background sources follows analogous to our discussion in Subsection 3.3. Directly plugging in the definition of second-order sources in terms of the first-order ones into the generating function variation in Eq. (3.14), we can obtain

$$\delta W = \int \mathrm{d}^{d+1} x \sqrt{\gamma} \left( -\epsilon^\mu \delta n_\mu + \tau^\mu_a \delta e^a_\mu + J^\mu \delta A_\mu + J^\mu_a \delta A^a{}_\mu \right) , \tag{4.12}$$

where the full momentum current is defined as $\tau^{\mu}{}_{a} = (v^{\mu}\pi^{\nu} + \tau^{\mu\nu} + \tau_{\rm d}^{\mu\nu})e_{a\nu}$, including the contributions from momentum density $\pi^{\mu}$, symmetric stress tensor $\tau^{\mu\nu}$, and stress contributions from the multipole currents $\tau_{\rm d}^{\mu\nu} = -A^{\mu}_{\rho}J^{\rho\nu}$. The dipole current is defined as $J^{\mu}_{a} = J^{\mu}{}_{\lambda}e^{\lambda}_{a}$. This coupling structure, of course, looks more natural from a field theoretic perspective.

Invariance of the action under $SO(d)$ rotations leads to the relation $\tau^{\mu}_{[a}e_{b]\mu} = -J^{\mu}_{[a}A_{b]\mu}$ that is identically satisfied. On the other hand, demanding invariance under the remaining symmetry variations in (4.8) lead to the Ward identities in the first-order form

$$\begin{aligned}
\nabla'_{\mu}\epsilon^{\mu} &= -v^{\mu}f_{\mu} - \tau^{\mu}{}_{a}e^{a}_{\nu}\nabla_{\mu}v^{\nu}\,,\\
\nabla'_{\mu}\tau^{\mu}{}_{a} &= e^{\mu}_{a}f_{\mu} - n_{\mu}\tau^{\mu}{}_{b}e^{b}_{\nu}e^{\rho}_{a}\nabla_{\rho}v^{\nu}\,,\\
\nabla'_{\mu}J^{\mu} &= 0\,,\\
\nabla'_{\mu}J^{\mu}_{a} &= e_{a\mu}J^{\mu}\,.
\end{aligned} \tag{4.13}$$

The power-force density $f_{\mu}$ has already been defined in Eq. (3.18).

Finally, the dipole transformation properties of various currents can directly be derived from Eq. (3.22). We find

$$\begin{aligned}
\epsilon^{\mu} &\to \epsilon^{\mu} + \left(2J^{\mu(\rho}\psi^{\sigma)} - J^{\rho\sigma}\psi^{\mu}\right)\frac{1}{2}\$_{v}h_{\rho\sigma}\,,\\
\tau^{\mu}{}_{a} &\to \tau^{\mu}{}_{a} - J^{\mu}\psi_{a} + \nabla'_{\nu}\left(\psi^{\nu}J^{\mu}_{a} - J^{\nu}_{a}\psi^{\mu}\right) + v^{\mu}J^{\rho}_{a}F^{n}_{\rho\sigma}\psi^{\sigma}\,,
\end{aligned} \tag{4.14}$$

while $J^{\mu}$ and $J^{\mu}_{a}$ are invariant.

### 4.3 Dipole algebra in curved space

We have verified the consistency of the symmetry algebra as a Lie algebra. But one might still wonder what this algebra has to do with the original dipole symmetry algebra of the flat space theory in Eq. (2.1). To make this connection, let us follow the analysis of [40] and consider a symmetry variation $\delta_{\hat{\mathfrak{X}}}$, given in terms of the parameters $\hat{\mathfrak{X}} = (\chi^{\mu}, \Omega^{a}{}_{b}, \Lambda, \psi_{a})$, and formally decompose it in a basis as

$$\delta_{\hat{\mathfrak{X}}} = i\chi^{\mu}n_{\mu}\mathrm{H} - i\chi^{\mu}e^{a}_{\mu}\mathrm{P}_{a} + \frac{i}{2}(\Omega^{ab} + \chi^{\mu}\omega^{ab}{}_{\mu})\mathrm{M}_{ab} - i(\Lambda + \chi^{\mu}A_{\mu})\mathrm{Q} - i(\psi^{a} + \chi^{\mu}A^{a}{}_{\mu})\mathrm{D}_{a}\,. \tag{4.15}$$

The Hamiltonian H, momenta $\mathrm{P}_a$, angular momenta $\mathrm{M}_{ab}$, charge Q, and dipole moment $\mathrm{D}_a$ should be understood as generators of local time translations, spatial translations, rotations, $U(1)$ transformations, and dipole transformations in a general curved background. Inserting this decomposition into the infinite-dimensional symmetry algebra, (4.10), we find that it can be expressed in terms of a finite set of commutators involving H, $\mathrm{P}_a$, etc.:

$$\begin{aligned}
[\mathrm{H},\mathrm{P}_a] &= iv^{\mu}e^{\nu}_{a}\mathcal{C}_{\mu\nu}\,, & [\mathrm{P}_a,\mathrm{P}_b] &= -ie^{\mu}_{a}e^{\nu}_{b}\mathcal{C}_{\mu\nu}\,,\\
[\mathrm{H},\mathrm{D}_a] &= -\frac{i}{2}e^{\rho}_{a}e^{b\sigma}\$_{v}h_{\rho\sigma}\mathrm{D}_b\,, & [\mathrm{P}_a,\mathrm{D}_b] &= i\delta_{ab}\mathrm{Q}\,,\\
[\mathrm{M}_{ab},\mathrm{D}_c] &= i(\delta_{ac}\mathrm{D}_b - \delta_{bc}\mathrm{D}_a)\,,\\
[\mathrm{M}_{ab},\mathrm{P}_c] &= i(\delta_{ac}\mathrm{P}_b - \delta_{bc}\mathrm{P}_a)\,,\\
[\mathrm{M}_{ab},\mathrm{M}_{cd}] &= i(\delta_{ac}\mathrm{M}_{bd} - \delta_{bc}\mathrm{M}_{ad} - \delta_{ad}\mathrm{M}_{bc} + \delta_{bd}\mathrm{M}_{ac})\,,
\end{aligned} \tag{4.16}$$

and all other commutators zero. We have defined the "curvature" operator

$$\mathcal{C}_{\mu\nu} = -F^{n}_{\mu\nu}\mathrm{H} + 2T^{a}{}_{\mu\nu}\mathrm{P}_a - \frac{1}{2}R^{ab}{}_{\mu\nu}\mathrm{M}_{ab} + F^{a}{}_{\mu\nu}\mathrm{D}_a\,. \tag{4.17}$$

To obtain this we made use of the identities

$$
\begin{aligned}
\delta_{\hat{\chi}'}(\Omega^{ab}+\chi^\mu\omega^{ab}{}_\mu) &= \$_{\chi'}(\Omega^{ab}+\chi^\mu\omega^{ab}{}_\mu)+2\Omega'^{[a}{}_c(\Omega^{b]c}+\chi^\mu\omega^{b]c}{}_\mu)\,, \\
\delta_{\hat{\chi}'}(\Lambda+\chi^\mu A_\mu) &= \$_{\chi'}(\Lambda+\chi^\mu A_\mu)+\chi^\mu e^a_\mu\psi'_a\,, \\
\delta_{\hat{\chi}'}(\psi^a+\chi^\mu A^a_\mu) &= \$_{\chi'}(\psi^a+\chi^\mu A^a_\mu)-\Omega'^a{}_b(\psi^b+\chi^\mu A^b_\mu) \\
&\qquad +(\Omega^a{}_b+\chi^\mu\omega^a{}_{b\mu})\psi'^b+\frac{1}{2}\chi^\mu n_\mu\psi'^b e^{a\rho}e^\sigma_b\$_v h_{\rho\sigma}\,.
\end{aligned}
\tag{4.18}
$$

Further details can be found in Appendix A.3. Note that the $U(1)$ field strength $F_{\mu\nu}$ does not appear in the curvature operator on its own as it does not transform homogeneously under dipole transformations.

Eq. (4.16) is the dipole algebra generalised to curved space. It reduces to the original dipole algebra given in Eq. (2.1) in the flat space limit. On curved space, the algebra is almost the same, except that the mutual commutators of H and $\mathrm{P}_a$ are not zero, but are sourced by the curvature operator $\mathcal{C}_{\mu\nu}$. This should be physically expected because local translations on curved spacetime do not commute. $[\mathrm{H},\mathrm{D}_a]$ is also nonzero in curved space. The source of this commutator is proportional to $\$_v h_{\mu\nu}$, which can be understood as the "time derivative" of the spatial metric in the reference frame of the background observer with velocity $v^\mu$. In other words, the dipole moment is no longer conserved when the system is coupled to a time-dependent spatial metric. This should also be physically expected because a dipole is a non-local degree of freedom and the associated dipole moment is sensitive to the spatial separation between the charges making up the dipole.

## 5 Conserved multipole moments

The techniques that we have developed in this work can easily be adapted to covariantise field theories with higher conserved multipole moments. A system with $2^n$-pole symmetry has a series of conserved multipole charges $\mathrm{Q}^{(r)}_{a_1\ldots a_r}$, for $r=0,\ldots,n$, obeying the "multipole algebra" [41]

$$
\begin{aligned}
[\mathrm{P}_a,\mathrm{Q}^{(r)}_{a_1\ldots a_r}] &= ir\delta_{a(a_1}\mathrm{Q}^{(r-1)}_{a_2\ldots a_r)}\,, \\
[\mathrm{M}_{ab},\mathrm{Q}^{(r)}_{c_1\ldots c_r}] &= ir\left(\delta_{a(c_1}\mathrm{Q}^{(r)}_{c_2\ldots c_r)b}-\delta_{b(c_1}\mathrm{Q}^{(r)}_{c_2\ldots c_r)a}\right), \\
[\mathrm{M}_{ab},\mathrm{P}_c] &= i(\delta_{ac}\mathrm{P}_b-\delta_{bc}\mathrm{P}_a)\,, \\
[\mathrm{M}_{ab},\mathrm{M}_{cd}] &= i(\delta_{ac}\mathrm{M}_{bd}-\delta_{bc}\mathrm{M}_{ad}-\delta_{ad}\mathrm{M}_{bc}+\delta_{bd}\mathrm{M}_{ac})\,,
\end{aligned}
\tag{5.1}
$$

and all other commutators zero. The $U(1)$ monopole charge is $\mathrm{Q}=\mathrm{Q}^{(0)}$, while the dipole moment is $\mathrm{D}_a=\mathrm{Q}^{(1)}_a$. The $r^{\text{th}}$ moment charge (i.e. $2^r$-pole moment) commutes with the Hamiltonian and transforms as an rank-$r$ tensor under rotations. Under translations, on the other hand, it picks up the $(r-1)^{\text{th}}$ moment charge. For example, $[P_a,Q]=0$, $[\mathrm{P}_a,\mathrm{D}_b]=i\delta_{ab}\mathrm{Q}$, and $[\mathrm{P}_a,\mathrm{Q}^{(2)}_{bc}]=i(\delta_{ab}\mathrm{D}_c+\delta_{ac}\mathrm{D}_b)$. Note that the monopole charge commutes with everything, while the dipole moment obeys the same algebra as noted previously.

In a field theory, $2^n$-pole symmetry charges can be realised by a $U(1)$ charge density $J^t$, obeying the conservation equation of the form

$$
\partial_t J^t+\partial_{i_1}\ldots\partial_{i_{n+1}}J^{i_1\ldots i_{n+1}}_{(n)}=0\,,
\tag{5.2}
$$

where $J^{i_1\ldots i_{n+1}}_{(n)}$ is the totally-symmetric $n^{\text{th}}$ pole current. The conserved multipole moments are

$$
\mathrm{Q}^{i_1\ldots i_r}_{(r)}=\int \mathrm{d}^d x\, J^t x^{i_1}\ldots x^{i_r}\,,\qquad \text{for}\quad r=0,1,\ldots n\,.
\tag{5.3}
$$

Similar to our analysis of models with conserved dipole moment, it is convenient to exchange the conservation equation (5.2) for a series of Ward identities

$$\partial_\mu J^\mu = 0\,, \qquad \partial_i J^{ij_1\ldots j_r}_{(r)} = J^{j_1\ldots j_r}_{(r-1)}\,, \quad \text{for} \quad r = 1,\ldots,n\,, \tag{5.4}$$

where $J^i_{(0)} = J^i$. Each successive $r^{\text{th}}$ pole current is given in terms of the divergence of the $(r+1)^{\text{th}}$ pole current, all the way up to $r = n-1$. Expressed in this language, it is straightforward to couple this theory to background fields.

### 5.1 Conserved quadrupole moment

To not get overwhelmed by the number of multipole charges, let us consider first the case of a theory with conserved quadrupole moment. An action for such a model with a single complex quantum field $\Phi$ is given by

$$S = \int \mathrm{d}t\mathrm{d}^d x\Big(i\Phi^*\partial_t\Phi + \mathrm{D}_{ijk}(\Phi^*,\Phi^*,\Phi^*)\mathrm{D}_{ijk}(\Phi,\Phi,\Phi) - V(\Phi^*\Phi)\Big)\,, \tag{5.5}$$

with

$$\mathrm{D}_{ijk}(\Phi,\Phi,\Phi) = \Phi^2\partial_i\partial_j\partial_k\Phi - 3\Phi\partial_{(i}\Phi\partial_j\partial_{k)}\Phi + 2\partial_i\Phi\partial_j\Phi\partial_k\Phi\,. \tag{5.6}$$

Note that the simplest term with spatial derivatives involves six powers of $\Phi$.

Coupling to curved space for this theory follows along the same lines as before. We have the normal Aristotelian background sources: the clock-form $n_\mu$, spatial metric $h_{\mu\nu}$, and $U(1)$ gauge field $A_\mu$. In addition, we have a symmetric spatial dipole gauge field $a^{(1)}_{\mu\nu}$ and analogously a totally-symmetric spatial quadrupole gauge field $a^{(2)}_{\mu\nu\rho}$, satisfying $v^\mu a^{(1)}_{\mu\nu} = v^\mu a^{(2)}_{\mu\nu\rho} = 0$. Under an infinitesimal diffeomorphism $\chi^\mu$, gauge transformation $\Lambda$, dipole transformation $\psi^{(1)}_\mu$ (obeying $v^\mu\psi^{(1)}_\mu = 0$), and quadrupole transformation $\psi^{(2)}_{\mu\nu}$ (symmetric, and obeying $v^\nu\psi^{(2)}_{\mu\nu} = 0$), which we collectively denote as $\hat{\mathfrak{X}} = (\chi^\mu,\Lambda,\psi^{(1)}_\mu,\psi^{(2)}_{\mu\nu})$, the background and quantum fields transform as

$$\begin{aligned}
\delta_{\hat{\mathfrak{X}}}\Phi &= \pounds_\chi\Phi + i\Lambda\Phi\,,\\
\delta_{\hat{\mathfrak{X}}}A_\mu &= \pounds_\chi A_\mu + \partial_\mu\Lambda + \psi^{(1)}_\mu\,,\\
\delta_{\hat{\mathfrak{X}}}a^{(1)}_{\mu\nu} &= \pounds_\chi a^{(1)}_{\mu\nu} + h^\rho_\mu h^\sigma_\nu\left(\nabla_\rho\psi^{(1)}_\sigma + \nabla_\sigma\psi^{(1)}_\rho\right) + 2\psi^{(2)}_{\mu\nu}\,,\\
\delta_{\hat{\mathfrak{X}}}a^{(2)}_{\mu\nu\rho} &= \pounds_\chi a^{(2)}_{\mu\nu\rho} + h^\sigma_\mu h^\tau_\nu h^\lambda_\rho\left(\nabla_\sigma\psi^{(2)}_{\tau\lambda} + \nabla_\tau\psi^{(2)}_{\lambda\sigma} + \nabla_\lambda\psi^{(2)}_{\sigma\tau}\right)\,,
\end{aligned} \tag{5.7}$$

while $n_\mu$ and $h_{\mu\nu}$ transform as before. The numerical factor of 2 in the transformation of $a^{(1)}_{\mu\nu}$ is chosen for convenience. The covariantised version of the quadrupole-symmetric theory (5.5) is then invariant under diffeomorphisms, $U(1)$ gauge transformations, dipole transformations, and quadrupole transformations.

Moving on, we had already defined a dipole "connection" and "field strength" in Eqs. (3.9) and (3.11), which we reprise here,

$$\begin{aligned}
A^\lambda_{(1)\mu} &= n_\mu v^\rho F_{\rho\sigma}h^{\sigma\lambda} + \frac{1}{2}\left(h^\rho_\mu F_{\rho\sigma}h^{\lambda\sigma} + a^{(1)}_{\mu\rho}h^{\rho\lambda}\right)\,,\\
F^\lambda_{(1)\mu\nu} &= \nabla_\mu A^\lambda_{(1)\nu} - \nabla_\nu A^\lambda_{(1)\mu} + F^n_{\mu\nu}v^\rho A^\lambda_{(1)\rho} + 2n_{[\mu}A^\rho_{(1)\nu]}\nabla_\rho v^\lambda\,.
\end{aligned} \tag{5.8}$$

These objects transform "nicely" under the dipole symmetry, but not under the quadrupole symmetry, similar to how $A_\mu$ and $F_{\mu\nu}$ transform nicely under the $U(1)$ monopole symmetry but not under the dipole symmetry. To wit, we have

$$\begin{aligned}
\delta_{\hat{\mathfrak{X}}}A^\lambda_{(1)\mu} &= \pounds_\chi A^\lambda_{(1)\mu} + \nabla_\mu\psi^\lambda_{(1)} + n_\mu\psi^\nu_{(1)}\nabla_\nu v^\lambda + \psi^{\lambda\nu}_{(2)}h_{\mu\nu}\,,\\
\delta_{\hat{\mathfrak{X}}}F^\lambda_{(1)\mu\nu} &= \pounds_\chi F^\lambda_{(1)\mu\nu} + \mathcal{R}^\lambda{}_{\rho\mu\nu}\psi^\rho_{(1)} + h^{\lambda\rho}\nabla_\mu\psi^{(2)}_{\nu\rho} - h^{\lambda\rho}\nabla_\nu\psi^{(2)}_{\mu\rho} + 2n_{[\mu}\psi^\rho_{(2)\nu]}\nabla_\rho v^\lambda\,,
\end{aligned} \tag{5.9}$$

where

$$\mathcal{R}^{\lambda}{}_{\rho\mu\nu} = h^{\sigma}_{\rho}\left(R^{\lambda}{}_{\sigma\mu\nu} + F^{n}_{\mu\nu}\nabla_{\sigma}v^{\lambda} - 2n_{[\mu}\nabla_{\nu]}\nabla_{\sigma}v^{\lambda}\right). \tag{5.10}$$

Similarly, we define the quadrupole "connection"

$$A^{\lambda\tau}_{(2)\mu} = n_{\mu}v^{\rho}F^{(\lambda}{}_{(1)\rho\nu}h^{\tau)\nu} + \frac{1}{3}\left(2h^{\rho}_{\mu}F^{(\lambda}{}_{(1)\rho\sigma}h^{\tau)\sigma} + a^{(2)}_{\mu\rho\sigma}h^{\rho\lambda}h^{\sigma\tau}\right), \tag{5.11}$$

which transforms as

$$\delta_{\hat{\mathfrak{X}}}A^{\lambda\tau}_{(2)\mu} = \$_{\chi}A^{\lambda\tau}_{(2)\mu} + \left(n_{\mu}v^{\rho} + \frac{2}{3}h^{\rho}_{\mu}\right)\mathcal{R}^{(\lambda}{}_{\sigma\rho\nu}h^{\tau)\nu}\psi^{\sigma}_{(1)} + \nabla_{\mu}\psi^{\lambda\tau}_{(2)} + 2n_{\mu}\psi^{\sigma(\lambda}_{(2)}\nabla_{\sigma}v^{\tau)}. \tag{5.12}$$

The quadrupole "field strength," on the other hand, is defined as

$$\begin{aligned} F^{\lambda\tau}_{(2)\mu\nu} &= \nabla_{\mu}A^{\lambda\tau}_{(2)\nu} - \nabla_{\nu}A^{\lambda\tau}_{(2)\mu} - F^{n}_{\mu\nu}v^{\rho}A^{\lambda\tau}_{(2)\rho} \\ &\quad + 2A^{\sigma}_{(1)[\mu}\left(n_{\nu]}v^{\rho} + \frac{2}{3}h^{\rho}_{\nu]}\right)\mathcal{R}^{(\lambda}{}_{\sigma\rho\nu}h^{\tau)\nu} + 4n_{[\mu}A^{\rho(\lambda}_{(2)\ \nu]}\nabla_{\rho}v^{\tau)}. \end{aligned} \tag{5.13}$$

The utility of all this structure is that we can now define symmetry currents as conjugate to external fields via

$$\delta W = \int \mathrm{d}^{d+1}x\sqrt{\gamma}\Big(-\epsilon^{\mu}\delta n_{\mu} + \left(v^{\mu}\pi^{\nu} + \frac{1}{2}\tau^{\mu\nu}\right)\delta h_{\mu\nu} + J^{\mu}\delta A_{\mu} + J^{\mu}_{(1)\lambda}\delta A^{\lambda}_{(1)\mu} + J^{\mu}_{(2)\lambda\tau}\delta A^{\lambda\tau}_{(2)\mu}\Big). \tag{5.14}$$

Invariance under diffeomorphisms, gauge transformations, etc., leads to Ward identities as in our analysis in Subsection 3.3,

$$\begin{aligned} \nabla'_{\mu}\epsilon^{\mu} &= -v^{\mu}f_{\mu} - \left(\tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}}\right)h_{\nu\lambda}\nabla_{\mu}v^{\lambda}, \\ \nabla'_{\mu}\left(v^{\mu}\pi^{\nu} + \tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}}\right) &= h^{\nu\mu}f_{\mu} - \pi_{\mu}h^{\nu\lambda}\nabla_{\lambda}v^{\mu}, \\ \nabla'_{\mu}J^{\mu} &= 0, \\ \nabla'_{\mu}J^{\mu\nu}_{(1)} &= h^{\nu}_{\mu}J^{\mu}, \\ \nabla'_{\mu}J^{\mu\nu\rho}_{(2)} &= J^{\nu\rho}_{(1)}. \end{aligned} \tag{5.15}$$

The derivatives on the left-hand-side are as in Subsection 3.3, $\nabla'_{\mu} = \nabla_{\mu} + F^{n}_{\mu\nu}v^{\nu}$. The power-force density gets modified to

$$\begin{aligned} f_{\mu} = -F^{n}_{\mu\nu}\epsilon^{\nu} - h_{\mu\lambda}A^{\lambda}{}_{\nu}J^{\nu} + J^{\nu}_{(1)\lambda}\left(F^{\lambda}_{(1)\mu\nu} - h_{\nu\rho}A^{\rho\lambda}_{(2)\mu}\right) + F^{\lambda\tau}_{(2)\mu\nu}J^{\nu}_{(2)\lambda\tau} \\ - n_{\mu}\left(A^{\rho}_{(1)\nu}J^{\nu}_{(1)\lambda} + 2A^{\rho\tau}_{(2)\nu}J^{\nu}_{(2)\lambda\tau}\right)\nabla_{\rho}v^{\lambda} \\ - \left(n_{\mu}v^{\sigma}A^{\rho}_{(1)\nu} + 2h^{\sigma}_{[\mu}A^{\rho}_{(1)\nu]}\right)J^{\nu}_{(2)\lambda\tau}\mathcal{R}^{\lambda}{}_{\rho\sigma\beta}h^{\tau\beta}, \end{aligned} \tag{5.16}$$

whereas $\tau^{\mu\nu}_{\mathrm{d}}$ now has contributions from the quadrupole current as well

$$\tau^{\mu\nu}_{\mathrm{d}} = -A^{\mu}_{(1)\rho}J^{\rho\nu}_{(1)} - 2A^{\mu\lambda}_{(2)\rho}J^{\rho\nu}_{(2)\ \lambda}. \tag{5.17}$$

## 5.2 Conserved higher moments

This construction can easily be generalised to systems with higher multipole symmetry. For a system with conserved $n^{\mathrm{th}}$ moments, the background fields comprise of the clock form $n_{\mu}$, spatial metric $h_{\mu\nu}$, gauge field $A_\mu$, and a series of spatial multipole gauge fields $a^{(r)}_{\mu_1\ldots\mu_{r+1}}$ for $r = 1, \ldots, n$. Their transformation laws are

$$\begin{aligned}
\delta_{\hat{\chi}} A_\mu &= \$_\chi A_\mu + \partial_\mu \Lambda + \psi^{(1)}_\mu\,, \\
\delta_{\hat{\chi}} a^{(r)}_{\mu_1\ldots\mu_{r+1}} &= \$_\chi a^{(r)}_{\mu_1\ldots\mu_{r+1}} + (r+1) h^{\rho_1}_{\mu_1} \ldots h^{\rho_{r+1}}_{\mu_{r+1}} \nabla_{(\rho_1} \psi^{(r)}_{\rho_2\ldots\rho_{r+1})} + (r+1)\psi^{(r+1)}_{\mu_1\ldots\mu_{r+1}}\,, \quad \text{for } r < n\,, \\
\delta_{\hat{\chi}} a^{(n)}_{\mu_1\ldots\mu_{n+1}} &= \$_\chi a^{(n)}_{\mu_1\ldots\mu_{n+1}} + (n+1) h^{\rho_1}_{\mu_1} \ldots h^{\rho_{n+1}}_{\mu_{n+1}} \nabla_{(\rho_1} \psi^{(n)}_{\rho_2\ldots\rho_{n+1})}\,,
\end{aligned} \tag{5.18}$$

while $n_\mu$ and $h_{\mu\nu}$ transform as usual. So defined, the symmetry generators obey an algebra

$$[\delta_{\hat{\chi}}, \delta_{\hat{\chi}'}] = \delta_{[\hat{\chi},\hat{\chi}']}\,, \tag{5.19}$$

with a suitably defined commutator, so that one can speak of Wess-Zumino consistency conditions for the currents of these theories. Using these variations, we can define multipole "connections" and "field strengths" as

$$\begin{aligned}
A_{(r)}^{\lambda_1\ldots\lambda_r}{}_\mu &= n_\mu v^\rho F_{(r-1)}^{(\lambda_1\ldots\lambda_{r-1}}{}_{\rho\nu} h^{\lambda_r)\nu} + \frac{1}{r+1}\left( r h^\rho_\mu F_{(r-1)}^{(\lambda_1\ldots\lambda_{r-1}}{}_{\rho\nu} h^{\lambda_r)\nu} + a_{(r)}^{\lambda_1\ldots\lambda_r}{}_\mu \right), \\
F_{(r)}^{\lambda_1\ldots\lambda_r}{}_{\mu\nu} &= 2\nabla_{[\mu} A_{(r)}^{\lambda_1\ldots\lambda_r}{}_{\nu]} + v^\rho F^n_{\mu\nu} A_{(r)}^{\lambda_1\ldots\lambda_r}{}_\rho - 2\sum_{s=1}^{r} A_{(s)}^{\rho_1\ldots\rho_s}{}_{[\mu} \mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\nu]\ \ \rho_1\ldots\rho_s}\,.
\end{aligned} \tag{5.20}$$

Here $\mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\mu\ \ \rho_1\ldots\rho_s}$ are some $(r+s+1)$-rank tensors that are entirely given in terms of the curvature $R^\lambda{}_{\rho\mu\nu}$, frame acceleration $\nabla_\mu v^\nu$, and their derivatives. See Appendix A.2 for more details. The "connections" transform as

$$\delta_{\hat{\chi}} A_{(r)}^{\lambda_1\ldots\lambda_r}{}_\mu = \$_\chi A_{(r)}^{\lambda_1\ldots\lambda_r}{}_\mu + \nabla_\mu \psi_{(r)}^{\lambda_1\ldots\lambda_r} + \sum_{s=1}^{r} \mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\mu\ \ \nu_1\ldots\nu_s} \psi_{(s)}^{\nu_1\ldots\nu_s} + h_{\mu\nu}\psi_{(r+1)}^{\nu\lambda_1\ldots\lambda_r}\,, \tag{5.21}$$

and the variation of the "field strengths" is given in the Appendix. The symmetry currents are then defined by the variation

$$\delta W = \int \mathrm{d}^{d+1}x\,\sqrt{\gamma}\left[ -\epsilon^\mu \delta n_\mu + \left( v^\mu \pi^\nu + \frac{1}{2}\tau^{\mu\nu}\right)\delta h_{\mu\nu} + j^\mu \delta A_\mu + \sum_{r=1}^{n} J^\mu_{(r)\lambda_1\ldots\lambda_r} \delta A_{(r)}^{\lambda_1\ldots\lambda_r}{}_\mu \right]. \tag{5.22}$$

The Ward identities read

$$\begin{aligned}
\nabla'_\mu \epsilon^\mu &= -v^\mu f_\mu - \left(\tau^{\mu\nu} + \tau_{\mathrm{d}}^{\mu\nu}\right) h_{\nu\lambda} \nabla_\mu v^\lambda\,, \\
\nabla'_\mu \left( v^\mu \pi^\nu + \tau^{\mu\nu} + \tau_{\mathrm{d}}^{\mu\nu}\right) &= h^{\nu\mu} f_\mu - \pi_\mu h^{\nu\lambda} \nabla_\lambda v^\mu\,, \\
\nabla'_\mu J^\mu &= 0\,, \\
\nabla'_\mu J_{(r)}^{\mu\nu_1\ldots\nu_r} &= J_{(r-1)}^{\nu_1\ldots\nu_r}\,,
\end{aligned} \tag{5.23}$$

where the power-force density gets modified to

$$\begin{aligned}
f_\mu = &-F^n_{\mu\nu}\epsilon^\nu - h_{\mu\lambda} A_{(1)}^\lambda{}_\nu J^\nu \\
&+ \sum_{r=1}^{n-1} \left( F_{(r)}^{\lambda_1\ldots\lambda_r}{}_{\mu\nu} - h_{\nu\rho} A_{(r+1)}^{\rho\lambda_1\ldots\lambda_r}{}_\mu \right) J_{(r)\lambda_1\ldots\lambda_r}{}^\nu + F_{(n)}^{\lambda_1\ldots\lambda_n}{}_{\mu\nu} J_{(n)\lambda_1\ldots\lambda_n}{}^\nu \\
&- \sum_{r=1}^{n}\sum_{s=1}^{r} \left( \mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\mu\ \ \rho_1\ldots\rho_s} A_{(s)}^{\rho_1\ldots\rho_s}{}_\nu - \mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\nu\ \ \rho_1\ldots\rho_s} A_{(s)}^{\rho_1\ldots\rho_s}{}_\mu \right) J_{(r)\lambda_1\ldots\lambda_r}{}^\nu\,,
\end{aligned} \tag{5.24}$$

while $\tau_{\mathrm{d}}^{\mu\nu}$ gets contributions from all the multipole currents

$$\tau_{\mathrm{d}}^{\mu\nu} = -\sum_{r=1}^{n} r h^{\nu\lambda_r} A_{(r)}^{\mu\lambda_1\ldots\lambda_{r-1}}{}_\rho J_{(r)\lambda_1\ldots\lambda_r}{}^\rho\,. \tag{5.25}$$

### 5.3 Multipole algebras in curved space

As in our discussion of the dipole algebra in curved space, we can understand the commutator of symmetry generators $[\delta_{\hat{\mathfrak{X}}},\delta_{\hat{\mathfrak{X}}'}]=\delta_{[\hat{\mathfrak{X}},\hat{\mathfrak{X}}']}$ in terms of commutators for a finite set of operators, giving the curved space version of the multipole algebra [40]. To do this efficiently, we need to pass to the first-order formulation discussed in Section 4. Without going into even more detail, we note that the first order versions of multipole "connections," "field strengths," and symmetry parameters are defined naturally as $A_{(r)}^{a_1\dots a_r}{}_{\mu}=e^{a_1}_{\lambda_1}\dots e^{a_r}_{\lambda_r}A_{(r)}^{\lambda_1\dots\lambda_r}{}_{\mu}$, $F_{(r)}^{a_1\dots a_r}{}_{\mu\nu}=e^{a_1}_{\lambda_1}\dots e^{a_r}_{\lambda_r}F_{(r)}^{\lambda_1\dots\lambda_r}{}_{\mu\nu}$, and $\psi_{(r)}^{a_1\dots a_r}=e^{a_1}_{\lambda_1}\dots e^{a_r}_{\lambda_r}\psi_{(r)}^{\lambda_1\dots\lambda_r}$ respectively. A symmetry transformation parametrised by $\hat{\mathfrak{X}}=(\chi^\mu,\Omega^a{}_b,\Lambda,\psi_{(r)}^{a_1\dots a_r})$ can be decomposed into a basis of generators according to

$$\begin{aligned}\delta_{\hat{\mathfrak{X}}}&=i\chi^\mu n_\mu \mathrm{H}-i\chi^\mu e^a_\mu \mathrm{P}_a+\frac{i}{2}(\Omega^{ab}+\chi^\mu\omega^{ab}{}_\mu)\mathrm{M}_{ab}-i(\Lambda+\chi^\mu A_\mu)\mathrm{Q}\\&\quad-i\sum_{r=1}^{n}\frac{1}{r!}\left(\psi_{(r)}^{a_1\dots a_r}+\chi^\mu A_{(r)}^{a_1\dots a_r}{}_\mu\right)\mathrm{Q}^{(r)}_{a_1\dots a_r}\,.\end{aligned}\tag{5.26}$$

Plugging this parameterisation into the infinite-dimensional algebra $[\delta_{\hat{\mathfrak{X}}},\delta_{\hat{\mathfrak{X}}'}]=\delta_{[\hat{\mathfrak{X}},\hat{\mathfrak{X}}']}$, we find that the infinite-dimensional algebra is equivalent to a finite set of commutators, in the form of the curved space algebra

$$\begin{aligned}&[\mathrm{H},\mathrm{P}_a]=iv^\mu e^\nu_a\mathcal{C}_{\mu\nu}\,, &&[\mathrm{P}_a,\mathrm{P}_b]=-ie^\mu_a e^\nu_b\mathcal{C}_{\mu\nu}\,,\\&[\mathrm{H},\mathrm{Q}^{(r)}_{a_1\dots a_r}]=-iv^\mu\mathcal{Y}^{(r)}_{\mu a_1\dots a_r}\,, &&[\mathrm{P}_a,\mathrm{Q}^{(r)}_{a_1\dots a_r}]=ir\delta_{a(a_1}\mathrm{Q}^{(r-1)}_{a_2\dots a_r)}+ie^\mu_a\mathcal{Y}^{(r)}_{\mu a_1\dots a_r}\,,\\&[\mathrm{M}_{ab},\mathrm{Q}^{(r)}_{c_1\dots c_r}]=ir\left(\delta_{a(c_1}\mathrm{Q}^{(r)}_{c_2\dots c_r)b}-\delta_{b(c_1}\mathrm{Q}^{(r)}_{c_2\dots c_r)a}\right)\,,\\&[\mathrm{M}_{ab},\mathrm{P}_c]=i(\delta_{ac}\mathrm{P}_b-\delta_{bc}\mathrm{P}_a)\,,\\&[\mathrm{M}_{ab},\mathrm{M}_{cd}]=i(\delta_{ac}\mathrm{M}_{bd}-\delta_{bc}\mathrm{M}_{ad}-\delta_{ad}\mathrm{M}_{bc}+\delta_{bd}\mathrm{M}_{ac})\,,\end{aligned}\tag{5.27}$$

and other commutators zero. The operator $\mathrm{Q}^{(0)}$ should be understood as Q. We have defined the multipole shift parameter

$$\mathcal{Y}^{(r)}_{\mu a_1\dots a_r}=\sum_{s=r}^{n}\frac{r!}{s!}v^\mu\mathcal{Y}_\mu^{(s,r)b_1\dots b_s}{}_{a_1\dots a_r}\mathrm{Q}^{(s)}_{b_1\dots b_s}\,,\tag{5.28}$$

while the definition of the curvature operator is also modified to

$$\mathcal{C}_{\mu\nu}=-F^n_{\mu\nu}\mathrm{H}+2T^a{}_{\mu\nu}\mathrm{P}_a-\frac{1}{2}R^{ab}{}_{\mu\nu}\mathrm{M}_{ab}+\sum_{r=1}^{n}\frac{1}{r!}\left(F_{(r)}^{a_1\dots a_r}{}_{\mu\nu}+2e_{b[\mu}A_{(r+1)}^{ba_1\dots a_r}{}_{\nu]}\right)\mathrm{Q}^{(r)}_{a_1\dots a_r}\,.\tag{5.29}$$

Along the way we have used

$$\begin{aligned}\delta_{\hat{\mathfrak{X}}'}\left(\psi_{(r)}^{a_1\dots a_r}+\chi^\mu A_{(r)}^{a_1\dots a_r}{}_\mu\right)&=\$_{\chi'}\left(\psi_{(r)}^{a_1\dots a_r}+\chi^\mu A_{(r)}^{a_1\dots a_r}{}_\mu\right)-r\Omega'^{(a_1}{}_b\left(\psi_{(r)}^{a_2\dots a_r)b}+\chi^\mu A_{(r)}^{a_2\dots a_r)b}{}_\mu\right)\\&\quad+r\left(\Omega^{(a_1}{}_b+\chi^\mu\omega^{(a_1}{}_{b\mu}\right)\psi'^{a_2\dots a_r)b}_{(r)}\\&\quad+\sum_{s=1}^{r}\chi^\mu\mathcal{Y}_\mu^{(r,s)a_1\dots a_r}{}_{b_1\dots b_s}\psi'^{b_1\dots b_s}_{(s)}+\chi^\mu e_{b\mu}\psi'^{a_1\dots a_r b}_{(r+1)}\,.\end{aligned}\tag{5.30}$$

The curved space algebra (5.27) should be compared with the flat space algebra (5.1). The two agree in flat space, where the generalised curvatures $\mathcal{C}_{\mu\nu}$ and $\mathcal{Y}$'s vanish. As in the dipole algebra, $[\mathrm{H},\mathrm{P}_a]$ and $[\mathrm{P}_a,\mathrm{P}_b]$ are nonzero in curved space. Also like before $[\mathrm{H},\mathrm{Q}^{(r)}_{a_1\dots a_r}]$ is nonzero, affirming that multipole moments are not time-independent when coupled to generic time-dependent backgrounds. A novel feature is that $[\mathrm{P}_a,\mathrm{Q}^{(r)}_{a_1\dots a_r}]$ gets contributions from higher moments $\mathrm{Q}^{p\geq r}_{a_1\dots a_p}$ and not just the preceding moment $\mathrm{Q}^{(r-1)}_{a_1\dots a_{r-1}}$.

## 6 Discussion

In this work we have proposed a coupling of simple continuum field theories with fracton order to curved spacetime backgrounds. The models we considered are, in flat space, invariant under spacetime translations and rotations, with a conserved $U(1)$ charge and a conserved dipole moment. A prototypical example of such a system is the theory of charged matter in [36]. The Noether currents associated with these symmetries naturally couple to (i) a spacetime geometry consisting of a spatial metric $h_{\mu\nu}$ and a clock one-form $n_\mu$, (ii) a $U(1)$ gauge field $A_\mu$, and (iii) a source $a_{ij}$ for the dipole current. The conservation of $U(1)$ charge and dipole moment in flat space is encoded in two Ward identities, namely

$$\partial_\mu J^\mu = 0\,, \qquad J^i = \partial_j J^{ij}\,, \tag{6.1}$$

which when put together yield $\partial_t J^t + \partial_i \partial_j J^{ij} = 0$. The first is the standard Ward identity provided that we couple matter fields to a background gauge field $A_\mu$ in such a way that the total action is invariant under local $U(1)$ transformations. The second is enforced by introducing a local "dipole symmetry," which only acts on the background fields as

$$A_i \to A_i + \psi_i\,, \qquad a_{ij} \to a_{ij} + \partial_i \psi_j + \partial_j \psi_i\,. \tag{6.2}$$

The dipole symmetry could be potentially gauge-fixed by setting $A_i = 0$, which ties the dipole shift parameter to the gauge parameter as $\psi_i = -\partial_i \Lambda$. By introducing the background spacetime $n_\mu$, $h_{\mu\nu}$, one can covariantise the theory of charged matter in [36], and generalise the dipole transformation above to curved space. One can also use the background coupling to derive the relevant Ward identities associated with energy and momentum conservation in these systems.

The procedure of coupling to curved background sources follows as usual. Given a theory with fracton order in flat space, we strive to introduce the background sources in such a way so as to make the full theory, including background and dynamical fields, invariant under spacetime diffeomorphisms, local $U(1)$ transformations, and local dipole transformations. At the infinitesimal level, these symmetry transformations are enacted by first-order linear differential operators, $\delta_{\hat{\mathfrak{X}}}$, where $\hat{\mathfrak{X}}$ collectively labels the infinitesimal diffeomorphism, gauge transformation, and dipole transformation. We showed that $\delta_{\hat{\mathfrak{X}}}$ obey an infinite-dimensional Lie algebra,[8] $[\delta_{\hat{\mathfrak{X}}}, \delta_{\hat{\mathfrak{X}}'}] = \delta_{[\hat{\mathfrak{X}},\hat{\mathfrak{X}}']}$, leading to Wess-Zumino consistency conditions. In the absence of anomalies, radiative corrections are expected to preserve these symmetries, leading to Ward identities for correlation functions. We computed these along with the transformation laws of the various symmetry currents under dipole transformations.

Along the way, we considered the symmetric tensor gauge theory of [21]. This theory of dynamical gauge fields $A_t$, $a_{ij}$ is an analogue of electromagnetism for systems with local dipole-invariance and naturally couples to the scalar matter theory mentioned above. In flat space, one can define the analogues of gauge-invariant "electric" and "magnetic" fields built from $A_t$, $a_{ij}$. However, we found that there is no gauge-invariant notion of an electric or magnetic field in a general curved background, suggesting that a covariant curved space theory does not exist.[9] Indeed, there is an obstruction to placing this theory in curved space while preserving diffeomorphism invariance, visible already in the flat space theory. Namely, there is no gauge-invariant definition of the flat space momentum density and spatial stress tensor. Consequently, the linearized coupling of the flat space theory to metric perturbations breaks the gauge symmetry, rendering the model inconsistent. Interestingly, we did find that the Higgs phase of the gauge theory can be coupled to curved space in a gauge-invariant way.

[8] It turns out that this infinite-dimensional algebra is highly redundant. See Subsection 4.3 for more detail.

[9] Gauge-invariant combinations do exist on spacetimes with absolute time and a time-independent Einstein spatial metric. That result matches the previous work of [22, 23].

In the Introduction, we mentioned that perhaps the most experimentally relevant spacetime symmetry of a fracton model is where $U(1)$ charge, dipole moment, and the trace of the quadrupole moment are all conserved. In such a theory isolated charges are immobile, while dipoles can move in a direction perpendicular to their dipole moment. While we have not discussed this symmetry pattern in the main text, it is easy to generalise our formalism to these systems. Namely, one takes the background fields and symmetries we considered in Section 3, but tunes the couplings of the field theory in question so that the spatial trace of the dipole source $a_{\mu\nu}$ does not appear in the action, i.e. the spatial trace of the dipole current vanishes, $h_{\mu\nu}J^{\mu\nu} = 0$. It would be interesting to visit these theories in greater detail in the future.

So far, several anomalies involving background tensor gauge fields have been discovered for field theories of fracton order [42–44]. In the relativistic setting, we are of course familiar with gravitational and mixed gravitational-flavor anomalies. One might wonder if there are gravitational anomalies, pure or mixed, in the landscape of field theories of fractons. In this work, we found evidence that this is the case for the symmetric tensor gauge theory. We found an the obstruction to coupling the symmetric tensor gauge theory to curved background in a covariant way, which strongly suggests the existence of a mixed gauge-gravitational anomaly signaling the breakdown of gauge symmetry in curved spacetime. It would be interesting to uncover if this particular non-invariance can be cured by sacrificing covariance, or by the inflow mechanism, and the role of gravitational anomalies in this landscape more generally.

Existing field theory models with conserved dipole moment have the feature that dipoles are composite objects. Dipole moments can either be generated by radiative corrections in the presence of an elementary charge, or through a bound state of a charge-anticharge pair. Indeed, one can envision a low-energy limit of a model where there is an energy gap to the production of charges, and the lowest-energy degrees of freedom are electrically neutral dipoles. What would such a field theory look like? A plausible hint comes from our construction of the "dipole connection" $A^{\lambda}{}_{\mu}$, which allows for coupling to field theories where degrees of freedom can carry an "intrinsic dipole moment," much like an independent spin connection $\omega^{a}{}_{b\mu}$ allows for coupling to field theories with intrinsic spin. It will be interesting to pursue this line of thought further.

Finally, thanks to this work, we have developed an appropriate notion of curved spacetime for theories with conserved dipole moment, along with the right curved space symmetries, Ward identities, and the transformation laws for the symmetry currents. With this information at hand, we are nearly ready to construct the hydrodynamic effective description of these models. The last ingredient we require is the low-energy symmetry breaking pattern of interacting fracton models, which, at least in certain soluble large $N$ models, will be presented in [24].

## Acknowledgements

We would like to thank P. Glorioso, A. Gromov, and A. Raz for enlightening discussions. AJ would like to thank the University of Victoria, Canada where part of this project was done. AJ is funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement NonEqbSK No. 101027527. AJ is also supported by the Netherlands Organization for Scientific Research (NWO) and by the Dutch Institute for Emergent Phenomena (DIEP) cluster at the University of Amsterdam. This work is supported in part by the NSERC Discovery Grant program of Canada.

# A Calculational details

In this Appendix we compile some of the calculational details that have been used in this work.

## A.1 Consistency of symmetry algebra

The main goal of this Appendix is to demonstrate that the generators $\delta_{\hat{\mathfrak{X}}}$ of an infinitesimal symmetry transformation furnish a Lie algebra. Doing so requires that we show that the commutator closes, i.e. the commutator of two symmetry variations is itself be a symmetry variation, $[\delta_{\hat{\mathfrak{X}}}, \delta_{\hat{\mathfrak{X}}'}] = \delta_{[\hat{\mathfrak{X}},\hat{\mathfrak{X}}']}$, for some appropriately defined commutator operation $[\hat{\mathfrak{X}}, \hat{\mathfrak{X}}']$, and that the variations $\delta_{\hat{\mathfrak{X}}}$ obey the Jacobi identity. Because the symmetry variations $\delta_{\hat{\mathfrak{X}}}$ are represented by first-order linear differential operators, they automatically satisfy the Jacobi identity, so we need only show that the commutator closes.

In fact, we will find that not only do the $\delta_{\hat{\mathfrak{X}}}$ form an algebra, but so do the infinitesimal transformations $\hat{\mathfrak{X}}$ themselves. We begin by reprising the variation of the symmetry parameters $\hat{\mathfrak{X}} = (\chi^\mu, \Omega^a{}_b, \Lambda, \psi_a)$ given by Eq. (4.11) as

$$\begin{aligned}
\chi^\mu_{[\hat{\mathfrak{X}}',\hat{\mathfrak{X}}]} &= \delta_{\hat{\mathfrak{X}}'}\chi^\mu &&= \$_{\chi'}\chi^\mu\,,\\
(\Omega_{[\hat{\mathfrak{X}}',\hat{\mathfrak{X}}]})^a{}_b &= \delta_{\hat{\mathfrak{X}}'}\Omega^a{}_b &&= \$_{\chi'}\Omega^a{}_b - \$_\chi\Omega'^a{}_b + \Omega^a{}_c\Omega'^c{}_b - \Omega'^a{}_c\Omega^c{}_b\,,\\
\Lambda_{[\hat{\mathfrak{X}}',\hat{\mathfrak{X}}]} &= \delta_{\hat{\mathfrak{X}}'}\Lambda &&= \$_{\chi'}\Lambda - \$_\chi\Lambda'\,,\\
(\psi_{[\hat{\mathfrak{X}}',\hat{\mathfrak{X}}]})_a &= \delta_{\hat{\mathfrak{X}}'}\psi_a &&= \$_{\chi'}\psi_a - \$_\chi\psi'_a + \psi_b\Omega'^b{}_a - \psi'_b\Omega^b{}_a\,.
\end{aligned}\tag{A.1}$$

Let us compute the action of successive variations on the symmetry parameters themselves

$$\begin{aligned}
\delta_{\hat{\mathfrak{X}}''}\delta_{\hat{\mathfrak{X}}'}\chi^\mu &= \$_{\chi''}\$_{\chi'}\chi^\mu\,,\\
\delta_{\hat{\mathfrak{X}}''}\delta_{\hat{\mathfrak{X}}'}\Omega^a{}_b &= \$_{\chi''}\$_{\chi'}\Omega^a{}_b + \$_\chi\left(\$_{\chi'}\Omega''^a{}_b - \Omega'^a{}_c\Omega''^c{}_b\right)\\
&\quad - \Omega^a{}_c\left(\$_{\chi'}\Omega''^c{}_b - \Omega'^c{}_d\Omega''^d{}_b\right) + \left(\$_{\chi'}\Omega''^a{}_c + \Omega''^a{}_d\Omega'^d{}_c\right)\Omega^c{}_b\\
&\quad + \left(\Omega''^a{}_c\$_\chi\Omega'^c{}_b + \Omega'^a{}_c\$_\chi\Omega''^c{}_b\right) - \left(\Omega''^a{}_d\Omega'^c{}_b + \Omega'^a{}_d\Omega''^c{}_b\right)\Omega^d{}_c\\
&\quad + \$_{\chi'}\left(-\$_\chi\Omega''^a{}_b + \Omega^a{}_c\Omega''^c{}_b - \Omega''^a{}_c\Omega^c{}_b\right)\\
&\quad + \$_{\chi''}\left(-\$_\chi\Omega'^a{}_b + \Omega^a{}_c\Omega'^c{}_b - \Omega'^a{}_c\Omega^c{}_b\right)\,,\\
\delta_{\hat{\mathfrak{X}}''}\delta_{\hat{\mathfrak{X}}'}\Lambda &= \$_{\chi''}\$_{\chi'}\Lambda + \$_\chi\$_{\chi'}\Lambda''\\
&\quad - \$_{\chi'}\$_\chi\Lambda'' - \$_{\chi''}\$_\chi\Lambda'\,,\\
\delta_{\hat{\mathfrak{X}}''}\delta_{\hat{\mathfrak{X}}'}\psi_a &= \$_{\chi''}\$_{\chi'}\psi_a + \$_\chi\left(\$_{\chi'}\psi''_a + \psi''_b\Omega'^b{}_a\right)\\
&\quad - \psi_b\left(\$_{\chi'}\Omega''^b{}_a - \Omega'^b{}_c\Omega''^c{}_a\right) + \left(\$_{\chi'}\psi''_b + \psi''_c\Omega'^c{}_b\right)\Omega^b{}_a\\
&\quad - \left(\psi'_b\Omega''^c{}_a + \psi''_b\Omega'^c{}_a\right)\Omega^b{}_c - \left(\$_\chi\psi'_b\Omega''^b{}_a + \$_\chi\psi''_b\Omega'^b{}_a\right)\\
&\quad + \$_{\chi'}\left(-\$_\chi\psi''_a + \psi_b\Omega''^b{}_a - \psi''_b\Omega^b{}_a\right) + \$_{\chi''}\left(-\$_\chi\psi'_a + \psi_b\Omega'^b{}_a - \psi'_b\Omega^b{}_a\right)\,.
\end{aligned}\tag{A.2}$$

The gray terms are symmetric under the exchange of $\hat{\mathfrak{X}}' \leftrightarrow \hat{\mathfrak{X}}''$ and drop out of the commutator. Inspecting the remaining terms, it is trivial to verify that $[\delta_{\hat{\mathfrak{X}}''}, \delta_{\hat{\mathfrak{X}}'}]\hat{\mathfrak{X}} = \delta_{[\hat{\mathfrak{X}}'',\hat{\mathfrak{X}}']}\hat{\mathfrak{X}}$. It is also easy to prove the Jacobi identity from here

$$\begin{aligned}
[\hat{\mathfrak{X}}'', [\hat{\mathfrak{X}}', \hat{\mathfrak{X}}]] = -\delta_{[\hat{\mathfrak{X}}',\hat{\mathfrak{X}}]}\hat{\mathfrak{X}}'' &= -\delta_{\hat{\mathfrak{X}}'}\delta_{\hat{\mathfrak{X}}}\hat{\mathfrak{X}}'' + \delta_{\hat{\mathfrak{X}}}\delta_{\hat{\mathfrak{X}}'}\hat{\mathfrak{X}}''\\
&= -\delta_{\hat{\mathfrak{X}}'}[\hat{\mathfrak{X}}, \hat{\mathfrak{X}}''] - \delta_{\hat{\mathfrak{X}}}[\hat{\mathfrak{X}}'', \hat{\mathfrak{X}}']\\
&= -[\hat{\mathfrak{X}}', [\hat{\mathfrak{X}}, \hat{\mathfrak{X}}'']] - [\hat{\mathfrak{X}}, [\hat{\mathfrak{X}}'', \hat{\mathfrak{X}}']]\,.
\end{aligned}\tag{A.3}$$

This demonstrates that the $\hat{\mathfrak{X}}$'s form an infinite-dimensional Lie algebra.

We now compute the second variation of background fields, finding

$$
\begin{aligned}
\delta_{\hat{\mathcal{X}}'}\delta_{\hat{\mathcal{X}}} n_\mu &= \$_{\chi'}\$_\chi n_\mu \,, \\
\delta_{\hat{\mathcal{X}}'}\delta_{\hat{\mathcal{X}}} e^a_\mu &= \$_{\chi'}\$_\chi e^a_\mu + \left(\$_\chi \Omega'^a{}_b + \Omega'^a{}_c \Omega^c{}_b\right) e^b_\mu \\
&\quad - \$_\chi(\Omega'^a{}_b e^b_\mu) - \$_{\chi'}(\Omega^a{}_b e^b_\mu) \,, \\
\delta_{\hat{\mathcal{X}}'}\delta_{\hat{\mathcal{X}}} A_\mu &= \$_{\chi'}\$_\chi A_\mu - \partial_\mu \$_\chi \Lambda' - e^a_\mu \left(\$_\chi \psi'_a + \psi'_b \Omega^b{}_a\right) \\
&\quad + \$_\chi \partial_\mu \Lambda' + \$_{\chi'} \partial_\mu \Lambda + \$_\chi (e^a_\mu \psi'_a) + \$_{\chi'}(e^a_\mu \psi_a) \,, \\
\delta_{\hat{\mathcal{X}}'}\delta_{\hat{\mathcal{X}}} A_{ab} &= \$_{\chi'}\$_\chi A_{ab} - \left(\$_\chi \Omega'^c{}_a - \Omega^c{}_d \Omega'^d{}_a\right) A_{cb} - \left(\$_\chi \Omega'^c{}_b - \Omega^c{}_d \Omega'^d{}_b\right) A_{ca} \\
&\quad - e^\mu_a \nabla_\mu \left(\$_\chi \psi'_b + \psi'_c \Omega^c{}_b\right) - e^\mu_b \nabla_\mu \left(\$_\chi \psi'_a + \psi'_c \Omega^c{}_a\right) \\
&\quad + e^\mu_c \left(\Omega^c{}_a \nabla_\mu \psi'_b + \Omega'^c{}_a \nabla_\mu \psi_b\right) + e^\mu_b \left(\Omega^c{}_a \nabla_\mu \psi'_c + \Omega'^c{}_a \nabla_\mu \psi_c\right) \\
&\quad + e^\mu_c \left(\Omega^c{}_b \nabla_\mu \psi'_a + \Omega'^c{}_b \nabla_\mu \psi_a\right) + e^\mu_a \left(\Omega^c{}_b \nabla_\mu \psi'_c + \Omega'^c{}_b \nabla_\mu \psi_c\right) \\
&\quad + \left(\Omega'^d{}_b \Omega^c{}_a + \Omega'^d{}_a \Omega^c{}_b\right) A_{cd} + \$_\chi \left(\Omega'^c{}_a A_{cb} + \Omega'^c{}_b A_{ac} + e^\mu_a \nabla_\mu \psi'_b + e^\mu_b \nabla_\mu \psi'_a\right) \\
&\quad + \$_{\chi'} \left(\Omega^c{}_a A_{cb} + \Omega^c{}_b A_{ac} + e^\mu_a \nabla_\mu \psi_b + e^\mu_b \nabla_\mu \psi_a\right) .
\end{aligned}
\tag{A.4}
$$

The gray terms are symmetric under the exchange of $\hat{\mathcal{X}} \leftrightarrow \hat{\mathcal{X}}'$ and drop out of the commutator. With the remaining terms, we can see that $[\delta_{\mathcal{X}'}, \delta_{\mathcal{X}}] = \delta_{[\mathcal{X}',\mathcal{X}]}$ while acting on the background fields. Thus the $\delta_{\hat{\mathcal{X}}}$'s generate a Lie algebra as claimed.

## A.2 Multipole connections and transformation properties

In this Appendix, we derive the explicit form of the objects $\mathcal{Y}^{(r,s)}$ that appear in the definition of multipole "connections" in Section 5. We begin by postulating that there exists an $r^{\text{th}}$ pole connection $A_{(r)}$ that varies under multipole transformations as follows

$$
A^{\lambda_1\ldots\lambda_r}{}_{(r)\mu} \to A^{\lambda_1\ldots\lambda_r}{}_{(r)\mu} + \nabla_\mu \psi^{\lambda_1\ldots\lambda_r}_{(r)} + \sum_{s=1}^{r} \mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\mu\quad\;\; \nu_1\ldots\nu_s} \psi^{\nu_1\ldots\nu_s}_{(s)} + h_{\mu\nu} \psi^{\nu\lambda_1\ldots\lambda_r}_{(r+1)} \,. \tag{A.5}
$$

We can allow $r$ to be zero as well, in which case the connection is just the monopole gauge field $A_\mu = A_{(0)\mu}$, with the $U(1)$ gauge parameter being $\psi_{(0)} = \Lambda$. Using these, we can define the $r$ "field strength" $F_{(r)}$ as in Eq. (5.20), so that it transforms homogeneously under all the $s$-pole transformations for $s \leq r$. Explicitly we find

$$
\begin{aligned}
F^{\lambda_1\ldots\lambda_r}{}_{(r)\mu\nu} &\to F^{\lambda_1\ldots\lambda_r}{}_{(r)\mu\nu} + r R^{(\lambda_1}{}_{\rho\mu\nu} \psi^{\lambda_2\ldots\lambda_r)\rho}_{(r)} + \sum_{s=2}^{r} 2\mathcal{Y}^{(r,s-1)\lambda_1\ldots\lambda_r}_{[\mu\qquad\;\; \rho_2\ldots\rho_s} h_{\nu]\rho_1} \psi^{\rho_1\ldots\rho_s}_{(s)} \\
&\quad + \sum_{s=1}^{r} \left[ \left(v^\rho F^n_{\mu\nu} - 2\delta^\rho_{[\mu} \nabla_{\nu]}\right) \mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\rho\quad\;\; \nu_1\ldots\nu_s} + \sum_{t=s}^{r} 2\mathcal{Y}^{(r,t)\lambda_1\ldots\lambda_r}_{[\mu\quad\;\; \rho_1\ldots\rho_t} \mathcal{Y}^{(t,s)\rho_1\ldots\rho_t}_{\nu]\quad\; \nu_1\ldots\nu_s} \right] \psi^{\nu_1\ldots\nu_s}_{(s)} \\
&\quad - 2h_{\rho[\mu} \nabla_{\nu]} \psi^{\rho\lambda_1\ldots\lambda_r}_{(r+1)} + 2\nabla_{[\mu} h_{\nu]\rho} \psi^{\rho\lambda_1\ldots\lambda_r}_{(r+1)} + 2\mathcal{Y}^{(r,r)\lambda_1\ldots\lambda_r}_{[\mu\quad\;\; \rho_1\ldots\rho_r} h_{\nu]\sigma} \psi^{\sigma\rho_1\ldots\rho_r}_{(r+1)} \,.
\end{aligned}
\tag{A.6}
$$

The important piece is here is the inhomogeneous term in the third line. Using $F_{(r)}$ and the spatial $(r+1)$ pole "connection" $a_{(r+1)}$ given in Eq. (5.7), we can construct an object that transforms like a connection under $\psi_{(r+1)}$ shifts. This is precisely the definition of $A_{(r+1)}$ given in Eq. (5.20). We compute its transformation to be

$$
\begin{aligned}
A^{\lambda_1\ldots\lambda_{r+1}}{}_{(r+1)\mu} \to A^{\lambda_1\ldots\lambda_{r+1}}{}_{(r+1)\mu} &+ \nabla_\mu \psi^{\lambda_1\ldots\lambda_{r+1}}_{(r+1)} + \left(n_\mu v^\sigma + \frac{r+1}{r+2}h^\sigma_\mu\right) rR^{\lambda_1}{}_{\rho\sigma\nu} h^{\lambda_{r+1}\nu}\psi^{\lambda_2\ldots\lambda_r\rho}_{(r)} \\
&+ \sum_{s=2}^{r+1}\left(n_\mu v^\sigma h^{\lambda_1}_{\nu_1} + \frac{r+1}{r+2}\left(h^\sigma_\mu h^{\lambda_1}_{\nu_1} - h_{\mu\nu_1}h^{\lambda_1\sigma}\right)\right)\mathcal{Y}^{(r,s-1)\lambda_2\ldots\lambda_{r+1}}_{\sigma\qquad\nu_2\ldots\nu_s}\psi^{\nu_1\ldots\nu_s}_{(s)} \\
&+ \left(n_\mu v^\sigma + \frac{r+1}{r+2}h^\sigma_\mu\right)\sum_{s=1}^{r}\Bigg[\left(v^\rho F^n_{\sigma\nu} - 2\delta^\rho_{[\sigma}\nabla_{\nu]}\right)\mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\rho\qquad\nu_1\ldots\nu_s}h^{\lambda_{r+1}\nu} \\
&+ \sum_{t=s}^{r} 2\mathcal{Y}^{(r,t)\lambda_1\ldots\lambda_r}_{[\sigma\qquad\rho_1\ldots\rho_t}\mathcal{Y}^{(t,s)\rho_1\ldots\rho_t}_{\nu]\qquad\nu_1\ldots\nu_s}h^{\lambda_{r+1}\nu}\Bigg]\psi^{\nu_1\ldots\nu_s}_{(s)} \\
&+ n_\mu\psi^{\rho\lambda_1\ldots\lambda_r}_{(r+1)}\nabla_\rho v^{\lambda_{r+1}} + \psi^{\lambda_1\ldots\lambda_{r+1}\rho}_{(r+2)}h_{\rho\mu}\,.
\end{aligned}
\tag{A.7}
$$

The indices $\lambda_1\ldots\lambda_{r+1}$ are understood to be symmetrised. We can compare this expression to our original expression for $A_{(r)}$ and write down a series of recursion relations for the $\mathcal{Y}^{(r,s)}$. The easiest of these are for $s=r$. We find

$$
\mathcal{Y}^{(r,r)\lambda_1\ldots\lambda_r}_{\mu\qquad\nu_1\ldots\nu_r} = n_\mu\nabla_{\nu_1}v^{\lambda_1}h^{\lambda_2}_{\nu_2}\ldots h^{\lambda_r}_{\nu_r} + \left(n_\mu v^\sigma h^{\lambda_1}_{\nu_1} + \frac{r}{r+1}\left(h^\sigma_\mu h^{\lambda_1}_{\nu_1} - h_{\mu\nu_1}h^{\lambda_1\sigma}\right)\right)\mathcal{Y}^{(r-1,r-1)\lambda_2\ldots\lambda_r}_{\sigma\qquad\qquad\nu_2\ldots\nu_r}\,. \tag{A.8}
$$

The $\lambda_1,\ldots,\lambda_r$ and $\nu_1,\ldots,\nu_r$ are understood to be symmetrised. This recursion is particularly simple to solve and one simply gets

$$
\mathcal{Y}^{(r,r)\lambda_1\ldots\lambda_r}_{\mu\qquad\nu_1\ldots\nu_r} = r\,n_\mu\nabla_{\nu_1}v^{\lambda_1}h^{\lambda_2}_{\nu_2}\ldots h^{\lambda_r}_{\nu_r}\,. \tag{A.9}
$$

Plugging this result in, we can obtain a recursion relation for $s=r-1$, leading to

$$
\begin{aligned}
\mathcal{Y}^{(r,r-1)\lambda_1\ldots\lambda_r}_{\mu\qquad\nu_1\ldots\nu_{r-1}} &= (r-1)\left(n_\mu v^\sigma + \frac{r}{r+1}h^\sigma_\mu\right)\mathcal{R}^{\lambda_1}{}_{\nu_1\sigma\rho}h^{\lambda_2}_{\nu_2}\ldots h^{\lambda_{r-1}}_{\nu_{r-1}}h^{\lambda_r\rho} \\
&+ \left(n_\mu v^\sigma h^{\lambda_1}_{\nu_1} + \frac{r}{r+1}\left(h^\sigma_\mu h^{\lambda_1}_{\nu_1} - h_{\mu\nu_1}h^{\lambda_1\sigma}\right)\right)\mathcal{Y}^{(r-1,r-2)\lambda_2\ldots\lambda_r}_{\sigma\qquad\qquad\nu_2\ldots\nu_{r-1}}\,,
\end{aligned}
\tag{A.10}
$$

where $\mathcal{R}^\lambda{}_{\rho\mu\nu}$ was defined in Eq. (5.10). We are unable to solve this recursion explicitly. All further recursion relations are decisively harder. For $1<s<r-1$, we get

$$
\begin{aligned}
\mathcal{Y}^{(r,s)\lambda_1\ldots\lambda_r}_{\mu\qquad\nu_1\ldots\nu_s} &= \left(n_\mu v^\sigma h^{\lambda_1}_{\nu_1} + \frac{r}{r+1}\left(h^\sigma_\mu h^{\lambda_1}_{\nu_1} - h_{\mu\nu_1}h^{\lambda_1\sigma}\right)\right)\mathcal{Y}^{(r-1,s-1)\lambda_2\ldots\lambda_r}_{\sigma\qquad\qquad\nu_2\ldots\nu_s} \\
&+ \left(n_\mu v^\sigma + \frac{r}{r+1}h^\sigma_\mu\right)\Bigg[\left(v^\rho F^n_{\sigma\nu} - 2\delta^\rho_{[\sigma}\nabla_{\nu]}\right)\mathcal{Y}^{(r-1,s)\lambda_1\ldots\lambda_{r-1}}_{\rho\qquad\qquad\nu_1\ldots\nu_s}h^{\lambda_r\nu} \\
&+ \sum_{t=s}^{r-1}2\mathcal{Y}^{(r-1,t)\lambda_1\ldots\lambda_{r-1}}_{[\sigma\qquad\qquad\rho_1\ldots\rho_t}\mathcal{Y}^{(t,s)\rho_1\ldots\rho_t}_{\nu]\qquad\nu_1\ldots\nu_s}h^{\lambda_r\nu}\Bigg]\,.
\end{aligned}
\tag{A.11}
$$

Finally for $s=1$ we have

$$
\begin{aligned}
\mathcal{Y}^{(r,1)\lambda_1\ldots\lambda_r}_{\mu\qquad\nu_1} &= \left(n_\mu v^\sigma + \frac{r}{r+1}h^\sigma_\mu\right)\Bigg[\left(v^\rho F^n_{\sigma\nu} - 2\delta^\rho_{[\sigma}\nabla_{\nu]}\right)\mathcal{Y}^{(r-1,1)\lambda_1\ldots\lambda_{r-1}}_{\rho\qquad\qquad\nu_1\ldots\nu_1}h^{\lambda_r\nu} \\
&+ \sum_{t=1}^{r-1}2\mathcal{Y}^{(r-1,t)\lambda_1\ldots\lambda_{r-1}}_{[\sigma\qquad\qquad\rho_1\ldots\rho_t}\mathcal{Y}^{(t,1)\rho_1\ldots\rho_t}_{\nu]\qquad\nu_1}h^{\lambda_r\nu}\Bigg]\,.
\end{aligned}
\tag{A.12}
$$

### A.3 Multipole algebra in curved space

Let us explicitly compute the commutator of variations $[\delta_{\hat{\chi}'}, \delta_{\hat{\chi}}]$ in terms of the generator-decomposition of $\delta_{\hat{\chi}}$ given in Eq. (4.15). We find

$$\begin{aligned}
[\delta_{\hat{\chi}'}, \delta_{\hat{\chi}}] = \delta_{[\hat{\chi}', \hat{\chi}]} &+ \chi^{\mu}\chi'^{\nu}\Big(e^a_\mu e^b_\nu [\mathrm{P}_a, \mathrm{P}_b] - (n_\mu e^a_\nu - n_\nu e^a_\mu)[\mathrm{H}, \mathrm{P}_a] + i\mathcal{C}_{\mu\nu}\Big) \\
&+ \frac{1}{2}\Big((\omega^{ab} + \chi^{\mu}\omega^{ab}{}_{\mu})\chi'^{\nu} n_\nu - (\omega'^{ab} + \chi'^{\mu}\omega^{ab}{}_{\mu})\chi^{\nu} n_\nu\Big)[\mathrm{M}_{ab}, \mathrm{H}] \\
&- \frac{1}{2}\Big((\Omega^{ab} + \chi^{\mu}\omega^{ab}{}_{\mu})\chi'^{\nu} e^c_\nu - (\Omega'^{ab} + \chi'^{\mu}\omega^{ab}{}_{\mu})\chi^{\nu} e^c_\nu\Big)\Big([\mathrm{M}_{ab}, \mathrm{P}_c] - 2i\delta_{ac}\mathrm{P}_b\Big) \\
&+ \frac{1}{4}\Big(\Omega^{ab} + \chi^{\mu}\omega^{ab}{}_{\mu}\Big)\Big(\Omega'^{cd} + \chi'^{\mu}\omega^{cd}{}_{\mu}\Big)\Big([\mathrm{M}_{ab}, \mathrm{M}_{cd}] - 4i\delta_{ac}\mathrm{M}_{bd}\Big) \\
&- \frac{1}{2}\Big(\Omega^{ab} + \chi^{\mu}\omega^{ab}{}_{\mu}\Big)\Big(\psi'^{c} + \chi'^{\nu}A^c_\nu\Big) \\
&- \Big(\Omega'^{ab} + \chi'^{\mu}\omega^{ab}{}_{\mu}\Big)\Big(\psi^{c} + \chi^{\nu}A^c_\nu\Big)\Big([\mathrm{M}_{ab}, \mathrm{D}_c] - 2i\delta_{ac}\mathrm{D}_b\Big) \\
&- \Big(\chi^{\mu} n_\mu(\Lambda' + \chi'^{\nu}A_\nu) - \chi'^{\mu} n_\mu(\Lambda + \chi^{\nu}A_\nu)\Big)[\mathrm{H}, \mathrm{Q}] \\
&+ \Big(\chi^{\mu} e^a_\mu(\Lambda' + \chi'^{\nu}A_\nu) - \chi'^{\mu} e^a_\mu(\Lambda + \chi^{\nu}A_\nu)\Big)[\mathrm{P}_a, \mathrm{Q}] \\
&- \frac{1}{2}\Big((\Omega^{ab} + \chi^{\mu}\omega^{ab}{}_{\mu})(\Lambda' + \chi'^{\nu}A_\nu) - (\Omega'^{ab} + \chi'^{\mu}\omega^{ab}{}_{\mu})(\Lambda + \chi^{\nu}A_\nu)\Big)[\mathrm{M}_{ab}, \mathrm{Q}] \\
&+ \Big((\psi^{a} + \chi^{\mu}A^a_\mu)(\Lambda' + \chi'^{\nu}A_\nu) - (\psi'^{a} + \chi'^{\mu}A^a_\mu)(\Lambda + \chi^{\nu}A_\nu)\Big)[\mathrm{D}_a, \mathrm{Q}] \\
&- \Big(\chi^{\mu} n_\mu(\psi'^{a} + \chi'^{\nu}A^a_\nu) - \chi'^{\mu} n_\mu(\psi^{a} + \chi^{\nu}A^a_\nu)\Big)\Big([\mathrm{H}, \mathrm{D}_a] + \frac{i}{2}e^\rho_a e^{b\sigma}\$_\nu h_{\rho\sigma}\mathrm{D}_b\Big) \\
&+ \Big(\chi^{\mu} e^a_\mu(\psi'^{b} + \chi'^{\nu}A^b_\nu) - \chi'^{\mu} e^a_\mu(\psi^{b} + \chi^{\nu}A^b_\nu)\Big)\Big([\mathrm{P}_a, \mathrm{D}_b] - i\delta_{ab}\mathrm{Q}\Big) \\
&+ (\psi^{a} + \chi^{\mu}A^a_\mu)(\psi'^{b} + \chi'^{\nu}A^b_\nu)[\mathrm{D}_a, \mathrm{D}_b]\,.
\end{aligned} \tag{A.13}$$

The curvature $\mathcal{C}_{\mu\nu}$ is defined in Eq. (4.17). Requiring the RHS of Eq. (A.13) to be just $\delta_{[\hat{\chi}', \hat{\chi}]}$, we can read out the commutation relations of the curved space dipole algebra given in Eq. (4.16). This computation can be generalised to higher multipole moments as well.

## B Background coupling of scalar charge theory

The covariant Lagrangian is given as

$$\begin{aligned}
\mathcal{L} = \frac{i}{2}\Big(\Phi^* v^\mu \mathrm{D}_\mu \Phi - \Phi v^\mu \mathrm{D}_\mu \Phi^*\Big) &- \lambda_S\Big(h^{\mu\nu}\mathrm{D}_{\mu\nu}(\Phi^*, \Phi^*) + \gamma\Phi^{*2}\Big)\Big(h^{\rho\sigma}\mathrm{D}_{\rho\sigma}(\Phi, \Phi) + \gamma\Phi^2\Big) \\
&- \lambda_T h^{\mu\rho}h^{\nu\sigma}\mathrm{D}_{\mu\nu}(\Phi^*, \Phi^*)\mathrm{D}_{\rho\sigma}(\Phi, \Phi) - V(\Phi^*\Phi)\,.
\end{aligned} \tag{B.1}$$

The non-local covariant derivatives were defined before. The variation of the Lagrangian with respect to the background fields is given as

$$\begin{aligned}
\frac{1}{\sqrt{\gamma}}\delta(\sqrt{\gamma}\,\mathcal{L}) = &\Big(\mathcal{L}v^\mu - \frac{i}{2}(\Phi^* v^\mu v^\nu \mathrm{D}_\nu \Phi - \Phi v^\mu v^\nu \mathrm{D}_\nu \Phi^*) + 2\mathcal{B}_{\rho\nu}v^\rho h^{\mu\nu}\Big)\delta n_\mu \\
&+ \left(\frac{1}{2}\mathcal{L}h^{\mu\nu} - \frac{i}{2}\Big(\Phi^* v^\mu h^{\nu\rho}\mathrm{D}_\rho \Phi - \Phi v^\mu h^{\nu\rho}\mathrm{D}_\rho \Phi^*\Big) + \mathcal{B}_{\rho\sigma}h^{\rho\mu}h^{\sigma\nu}\right)\delta h_{\mu\nu} \\
&+ q\Phi^*\Phi\, v^\mu \delta A_\mu - \mathcal{A}^{*\mu\nu}\delta\mathcal{X}_{\mu\nu} - \mathcal{A}^{\mu\nu}\delta\mathcal{X}^*_{\mu\nu}\,,
\end{aligned} \tag{B.2}$$

where we have defined

$$\mathcal{X}_{\mu\nu} = \Phi\, \mathrm{D}_{(\mu}\mathrm{D}_{\nu)}\Phi - \mathrm{D}_\mu\Phi\, \mathrm{D}_\nu\Phi + \frac{1}{2} iq\Phi^2 a_{\mu\nu}\,,$$
$$\mathcal{A}^{\mu\nu} = \lambda_S\Big(h^{\rho\sigma}\mathcal{X}_{\rho\sigma} + \gamma\Phi^2\Big)h^{\mu\nu} + \lambda_T h^{\mu\rho}h^{\nu\sigma}\mathcal{X}_{\rho\sigma}\,,$$
$$\mathcal{B}_{\mu\nu} = \lambda_S\Big(h^{\rho\sigma}\mathcal{X}^*_{\rho\sigma} + \gamma\Phi^{*2}\Big)\mathcal{X}_{\mu\nu} + \lambda_S\Big(h^{\rho\sigma}\mathcal{X}_{\rho\sigma} + \gamma\Phi^2\Big)\mathcal{X}^*_{\mu\nu} + 2\lambda_T h^{\rho\sigma}\mathcal{X}^*_{\rho(\mu}\mathcal{X}_{\nu)\sigma}\,. \tag{B.3}$$

All the technicalities from the variation are captured in the variation of $\mathcal{X}_{\mu\nu}$; we get

$$\mathcal{A}^{*\mu\nu}\delta\mathcal{X}_{\mu\nu} = \frac{1}{2}iq\Phi^2\mathcal{A}^{*\mu\nu}\delta a_{\mu\nu} - iq\Phi^2\,\mathcal{A}^{*\mu\nu}\nabla_\mu\delta A_\nu$$
$$- \Phi\mathrm{D}_\lambda\Phi\Big(v^\lambda\mathcal{A}^{*\mu\nu}\nabla_\mu\delta n_\nu + \frac{1}{2}\left(2h^{\lambda(\mu}\mathcal{A}^{*\nu)\rho} - h^{\lambda\rho}\mathcal{A}^{*\mu\nu}\right)\nabla_\rho\delta h_{\mu\nu} \tag{B.4}$$
$$+ \frac{1}{2}h^{\lambda\rho}\mathcal{A}^{*\mu\nu}(\$_v h_{\mu\nu})\delta n_\rho - v^\mu h^{\lambda\sigma}F^n_{\sigma\rho}\mathcal{A}^{*\rho\nu}\delta h_{\mu\nu}\Big), \tag{B.5}$$

where we have used the variation of the connection given in Eq. (3.21). After ignoring some total derivative terms, we get

$$\mathcal{A}^{*\mu\nu}\delta\mathcal{X}_{\mu\nu} = -\left(\frac{1}{2}h^{\lambda\mu}\$_v h_{\rho\sigma}\mathcal{A}^{*\rho\sigma}\Phi\mathrm{D}_\lambda\Phi - \nabla'_\nu\left(\Phi\, v^\lambda\mathrm{D}_\lambda\Phi\mathcal{A}^{*\mu\nu}\right)\right)\delta n_\mu$$
$$+ \left(v^\mu h^{\lambda\sigma}F^n_{\sigma\rho}\mathcal{A}^{*\rho\nu}\Phi\mathrm{D}_\lambda\Phi + \frac{1}{2}\nabla'_\rho\left(\left(2h^{\lambda(\mu}\mathcal{A}^{*\nu)\rho} - h^{\lambda\rho}\mathcal{A}^{*\mu\nu}\right)\Phi\mathrm{D}_\lambda\Phi\right)\right)\delta h_{\mu\nu}$$
$$+ \nabla'_\nu\left(iq\mathcal{A}^{*\mu\nu}\Phi^2\right)\delta A_\mu + \frac{1}{2}iq\mathcal{A}^{*\mu\nu}\Phi^2\delta a_{\mu\nu}\,. \tag{B.6}$$

Using these, we can read out the conserved currents. The $U(1)$ monopole and dipole currents are given simply as

$$J^\mu = q\Phi^*\Phi\, v^\mu - \nabla'_\nu\left(iq\mathcal{A}^{*\mu\nu}\Phi^2 - iq\Phi^{*2}\mathcal{A}^{\mu\nu}\right),$$
$$J^{\mu\nu} = -iq\mathcal{A}^{*\mu\nu}\Phi^2 + iq\Phi^{*2}\mathcal{A}^{\mu\nu}\,. \tag{B.7}$$

On the other hand, the energy current, momentum density, and stress tensor are given as

$$\epsilon^\mu = v^\mu\left(\frac{i}{2}\left(\Phi^* v^\nu\mathrm{D}_\nu\Phi - \Phi v^\nu\mathrm{D}_\nu\Phi^*\right) - \mathcal{L}\right) - 2\mathcal{B}_{\rho\nu}v^\rho h^{\mu\nu}$$
$$+ \nabla'_\nu\left(\mathcal{A}^{*\mu\nu}\Phi\, v^\lambda\mathrm{D}_\lambda\Phi + \Phi^* v^\lambda\mathrm{D}_\lambda\Phi^*\mathcal{A}^{\mu\nu}\right)$$
$$- h^{\lambda\mu}\left(\mathcal{A}^{*\rho\sigma}\Phi\mathrm{D}_\lambda\Phi + \Phi^*\mathrm{D}_\lambda\Phi^*\mathcal{A}^{\rho\sigma}\right)\frac{1}{2}\$_v h_{\rho\sigma} - iq\left(\mathcal{A}^{*\mu\nu}\Phi^2 - \Phi^{*2}\mathcal{A}^{\mu\nu}\right)v^\rho F_{\rho\nu}\,,$$
$$\pi^\mu = -\frac{i}{2}\left(\Phi^* h^{\mu\nu}\mathrm{D}_\nu\Phi - \Phi h^{\mu\nu}\mathrm{D}_\nu\Phi^*\right) - h^{\lambda\sigma}F^n_{\sigma\rho}\left(\mathcal{A}^{*\rho\mu}\Phi\mathrm{D}_\lambda\Phi + \Phi^*\mathrm{D}_\lambda\Phi^*\mathcal{A}^{\rho\mu}\right),$$
$$\tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}} = \mathcal{L}h^{\mu\nu} + 2\mathcal{B}_{\rho\sigma}h^{\rho\mu}h^{\sigma\nu}$$
$$- \nabla'_\rho\left(\left(2h^{\lambda(\mu}\mathcal{A}^{*\nu)\rho} - h^{\lambda\rho}\mathcal{A}^{*\mu\nu}\right)\Phi\mathrm{D}_\lambda\Phi + \Phi^*\mathrm{D}_\lambda\Phi^*\left(2h^{\lambda(\mu}\mathcal{A}^{\nu)\rho} - h^{\lambda\rho}\mathcal{A}^{\mu\nu}\right)\right)$$
$$- iq\left(\mathcal{A}^{*\mu\rho}\Phi^2 - \Phi^{*2}\mathcal{A}^{\mu\rho}\right)A^\nu{}_\rho\,, \tag{B.8}$$

The expressions are considerably simpler in flat space when all the background sources have been switched off. For the $U(1)$ monopole and dipole currents, we revert back to the expressions we found before

$$J^t = q\Phi^*\Phi\,,$$
$$J^i = -\partial_j\left(iq\mathcal{A}^{*ij}\Phi^2 - iq\Phi^{*2}\mathcal{A}^{ij}\right),$$
$$J^{ij} = -iq\mathcal{A}^{*ij}\Phi^2 + iq\Phi^{*2}\mathcal{A}^{ij}\,, \tag{B.9}$$

whereas for the energy density, energy current, momentum density, and stress tensor, we get

$$
\begin{aligned}
\epsilon^t &= \frac{i}{2}(\Phi^*\partial_t\Phi - \Phi\partial_t\Phi^*) - \mathcal{L}\,,\\
\epsilon^i &= -2\mathcal{B}_t^i + \partial_j\left(\mathcal{A}^{*ij}\Phi\partial_t\Phi + \Phi^*\partial_t\Phi^*\mathcal{A}^{ij}\right),\\
\pi^i &= -\frac{i}{2}\left(\Phi^*\partial^i\Phi - \Phi\partial^i\Phi^*\right),\\
\tau^{ij} &= \mathcal{L}\delta^{ij} + 2\mathcal{B}^{ij} - \partial_k\left(2\mathcal{A}^{*k(i}\Phi\partial^{j)}\Phi - \mathcal{A}^{*ij}\Phi\partial^k\Phi + 2\Phi^*\partial^{(i}\Phi^*\mathcal{A}^{j)k} - \Phi^*\partial^k\Phi^*\mathcal{A}^{ij}\right),
\end{aligned}
\tag{B.10}
$$

where $\mathcal{A}^{ij}$, $\mathcal{B}_{ij}$, and $\mathcal{B}_{ti}$ are defined as

$$
\begin{aligned}
\mathcal{X}_{ti} &= \Phi\,\partial_i\partial_t\Phi - \partial_i\Phi\,\partial_t\Phi\,,\\
\mathcal{X}_{ij} &= \Phi\,\partial_i\partial_j\Phi - \partial_i\Phi\,\partial_j\Phi\,,\\
\mathcal{A}^{ij} &= \lambda_S\left(\mathcal{X}^k{}_k + \gamma\Phi^2\right)\delta^{ij} + \lambda_T\mathcal{X}^{ij}\,,\\
\mathcal{B}_{ti} &= \lambda_S(\mathcal{X}^k{}_k + \gamma\Phi^2)^*\mathcal{X}_{ti} + \lambda_S(\mathcal{X}^k{}_k + \gamma\Phi^2)\mathcal{X}^*_{ti} + \lambda_T h^{\rho\sigma}\left(\mathcal{X}^*_{tk}\mathcal{X}_i{}^k + \mathcal{X}^*_{ki}\mathcal{X}_t{}^k\right),\\
\mathcal{B}_{ij} &= \lambda_S(\mathcal{X}^k{}_k + \gamma\Phi^2)^*\mathcal{X}_{ij} + \lambda_S(\mathcal{X}^k{}_k + \gamma\Phi^2)\mathcal{X}^*_{ij} + 2\lambda_T\mathcal{X}^*_{k(i}\mathcal{X}_{j)}{}^k\,.
\end{aligned}
\tag{B.11}
$$

Note that $\tau_{\mathrm{d}}^{ij}$ is identically zero in the absence of background fields.

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