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Foliated-exotic duality in fractonic BF theories

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Abstract

There has been proposed two continuum descriptions of fracton systems: foliated quantum field theories (FQFTs) and exotic quantum field theories. Certain fracton systems are believed to admit descriptions by both, and hence a duality is expected between such a class of FQFTs and exotic QFTs. In this paper we study this duality in detail for concrete examples in $2+1$ and $3+1$ dimensions. In the examples, both sides of the continuum theories are of BF-type, and we find the explicit correspondences of gauge-invariant operators, gauge fields, parameters, and allowed singularities and discontinuities. This deepens the understanding of dualities in fractonic quantum field theories.

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1 Introduction

A Fracton phase is a new kind of phase of matter that exhibits excitations with restricted mobility, which can only move in certain dimensional submanifolds (see [1,2] for reviews). The characteristic excitations are called fractons, lineons, and planons, depending on the spatial dimension of the excitation. Such fracton models, studied as lattice models in condensed matter physics [3–6], have various novel properties: a new type of symmetry and the exponential growth of ground state degeneracy in terms of the linear sizes of the system. The fracton systems are not only theoretically interesting in its own right, but expected to be applied to quantum information [3,7,8] and gravity [9].

While fracton phases first appeared in lattice systems, one would also expect a continuum description in the low-energy limit of a lattice system. There have been proposed such descriptions by continuum quantum field theories (QFTs) in various situations [10–26]. The QFTs do not have the Lorentz invariance or even the full rotational invariance, and can have the discontinuous field configurations. In the low-energy descriptions, the gapped excitations are not dynamical and arise as the gauge-invariant defects. The identification and construction of these QFTs are based on the subsystem symmetry, which is one of the generalizations of symmetry. A subsystem symmetry is a symmetry that acts on a spatial submanifold, e.g. a plane along a particular directions, and can have different values on each submanifold [12].[1]

For lattice models, some fracton models can be written as foliated fracton phases [6,28–30]. A foliation is a decomposition of a manifold and regarding it as a stack of an infinite number of submanifolds. For example, the X-cube model [5], which is a gapped fracton lattice model in $3+1$ dimensions, can be written as a stack of the $(2+1)$-dimensional toric codes [31] by using foliations [6]. For QFTs, there are fractonic QFTs coupled to foliations, which are called foliated quantum filed theories (FQFTs) [20–22]. On the other hand, some fractonic QFTs can be written as tensor gauge theories [10,13–16,32] respecting the lattice rotational symmetries, which we call the exotic QFTs [19]. The continuum QFT description of the X-cube model can be written as BF-type theories in terms of both a foliated QFT in the flat foliations[2] and an exotic QFT [11,15]. The foliated and exotic descriptions are believed to represent the same physics, but the duality between them has not been made clear.

In this paper, we will consider the foliated and exotic BF-type theories in $2+1$ and $3+1$ dimensions. In $2+1$ dimensions, the BF-type theories are the continuum description of the $\mathbb{Z}_N$ plaquette Ising model (see [33] for a review) and the $\mathbb{Z}_N$ lattice tensor gauge theory [13]. In $3+1$ dimensions, the BF-type theories are the continuum description of the X-cube model

[1] While a subsystem symmetry is similar to a higher form symmetry [27] as its corresponding symmetry operator has codimension higher than one, the operator is not topological in the directions out of the submanifold.

[2] The foliation is characterized by a foliation filed e. The foliation is flat when e is flat, i.e., $de = 0$. See Section 2.1.1.

and the $\mathbb{Z}_N$ lattice tensor gauge theory [15].

The goal of this paper is to show the explicit correspondences of the gauge fields and parameters between the foliated BF theory and the exotic BF theory, completing the previous observation made in [22]. We will see that both foliated and exotic BF theories have the same type of gauge-invariant operators and subsystem symmetries, and by matching the operators, we will derive the correspondences of the fields and parameters. It is novel to exhibit the explicit correspondences between the foliated fields, including the bulk fields, and the exotic tensor gauge fields. This establishes the duality between the foliated and exotic BF theories, which we call the foliated-exotic duality.

The organization of the rest of the paper is as follows. In Section 2, we will discuss the BF-type theories in $2+1$ dimensions. In Section 2.1, we will consider the foliated BF theory with attention to singularities and discontinuities. The foliated BF Lagrangian in the flat foliations[3] is

$$\mathcal{L}_f = \sum_{k=1}^{2} \frac{iN}{2\pi}(dB^k + b) \wedge A^k \wedge dx^k + \frac{iN}{2\pi} b \wedge da\,. \tag{1}$$

In Section 2.2, we will review the exotic BF theory [13]. The exotic BF Lagrangian[4] is

$$\mathcal{L}_e = \frac{iN}{2\pi}\phi^{12}(\partial_0 A_{12} - \partial_1\partial_2 A_0)\,. \tag{2}$$

In Section 2.3, we show the explicit correspondences between them by matching the gauge-invariant operators. In order to match the gauge-invariant operators, we need to modify the strip operators in the foliated BF theory. The modification turns out to be only by an operator that is not remotely detectable [34, 35]. The correspondences of the gauge fields and parameters are shown in Table 1. In the correspondences, the singularities and discontinuities are also matched. In Section 3, we will discuss BF-type theories in $3+1$ dimensions as in the case of $2+1$ dimensions. In Section 3.1, we will review the foliated BF theory [20–22]. The foliated BF Lagrangian in the flat foliations is

$$\mathcal{L}_f = \sum_{k=1}^{3} \frac{iN}{2\pi}(dB^k + b) \wedge A^k \wedge dx^k + \frac{iN}{2\pi} b \wedge da\,. \tag{3}$$

In Section 3.2, we will review the exotic BF theory [11, 15]. The exotic BF Lagrangian is

$$\mathcal{L}_e = \frac{iN}{2\pi}\sum_{i,j}\left(\frac{1}{2}A_{ij}(\partial_0\hat{A}^{ij} - \partial_k\hat{A}_0^{k(ij)}) + \frac{1}{2}A_0\,\partial_i\partial_j\hat{A}^{ij}\right). \tag{4}$$

In Section 3.3, we show the explicit correspondences between them. The correspondences of the gauge fields and parameters are shown in Table 2, 3. In Appendix A, we will consider the electric-magnetic dual descriptions of the BF-type theories in $2+1$ dimensions.

Along the way, we find that there are gauge-invariant operators that cannot be remotely detected by other spatially placed operators, but represents a time-like symmetry [36]. This makes a contrast to the case of ordinary topological order or topological field theory, where every operator is remotely detectable.

The establishment of the foliated-exotic duality deepens the understanding of both of the continuum descriptions of the fractonic systems. In general it is not known when a fractonic system admits a description by a foliated or an exotic QFT, and this result will be a clue in this interesting question. It would also serve as a starting point of exploring more general dualities in quantum field theories without Lorentz invariance.

[3] The superscripts k index the directions of the foliations. The subscripts in Table 1, 2 and 3 are the spatial indices.

[4] In the exotic theories, the superscripts and subscripts are the spacetime indices. As the metric is flat, we do not need to distinguish them.

Table 1: The correspondences of the gauge fields and parameters between the foliated BF theory and the exotic BF theory in $2+1$ dimensions.

The foliated BF theory		The exotic BF theory	
Gauge fields and parameters	Gauge transformations	Gauge fields and parameters	Gauge transformations
a_0	$\partial_0\lambda$	A_0	$\partial_0\alpha$
$A_0^k+\partial_0 a_k$ $(k=1,2)$	$\partial_0\partial_k\lambda$ $(k=1,2)$	$\partial_k A_0$ $(k=1,2)$	$\partial_k\partial_0\alpha$ $(k=1,2)$
$A_i^k+\partial_i a_k$ $((k,i)=(1,2),(2,1))$	$\partial_i\partial_k\lambda$ $((k,i)=(1,2),(2,1))$	A_{12}	$\partial_1\partial_2\alpha$
λ	$2\pi\xi^1+2\pi\xi^2$	α	$2\pi\tilde{n}^1+2\pi\tilde{n}^2$
ξ^k $(k=1,2)$		$\tilde{n}^k$ $(k=1,2)$	
B^1-B^2	$2\pi m^1-2\pi m^2$	ϕ^{12}	$2\pi\tilde{m}^1-2\pi\tilde{m}^2$
m^k $(k=1,2)$		$\tilde{m}^k$ $(k=1,2)$	

2 BF-type Theory in 2+1 Dimensions

In this section, we consider two BF-type theories in $2+1$ dimensions: a foliated BF theory and an exotic BF theory. Both of the theories are the continuum descriptions of the $\mathbb{Z}_N$ plaquette Ising model (see [33] for a review) and the $\mathbb{Z}_N$ lattice tensor gauge theory [13], both of which have subsystem symmetries and excitations of fractons. These two BF-type theories represent the same physics and we will show the explicit duality between them.

We take a three-torus of lengths l^0, l^1, l^2 as the spacetime and the coordinates (x^0, x^1, x^2) on it, with x^0 regarded as the Euclidean time. We consider theories that are Lorentz non-invariant and not fully rotation invariant. Instead, the spacetime symmetry is the spatial 90 degree rotational symmetry and the time translation as lattice models have. In the foliated theory the discrete rotational symmetry is not manifest, while in the exotic theory it is ex-

Table 2: The correspondences of the gauge fields and parameters between the foliated BF theory and the exotic BF theory in $3+1$ dimensions (the A-type and bulk fields).

The foliated BF theory		The exotic BF theory	
Gauge fields and parameters	Gauge transformations	Gauge fields and parameters	Gauge transformations
a_0	$\partial_0\lambda$	A_0	$\partial_0\alpha$
$A_0^k+\partial_0 a_k$ $(k=1,2,3)$	$\partial_0\partial_k\lambda$ $(k=1,2,3)$	$\partial_k A_0$ $(k=1,2,3)$	$\partial_k\partial_0\alpha$ $(k=1,2,3)$
$A_i^k+\partial_i a_k$ $(k\neq i,$ $k,i\in\{1,2,3\})$	$\partial_i\partial_k\lambda$ $(k\neq i,$ $k,i\in\{1,2,3\})$	A_{ki} $(k\neq i,$ $k,i\in\{1,2,3\})$	$\partial_k\partial_i\alpha$ $(k\neq i,$ $k,i\in\{1,2,3\})$
λ	$2\pi\xi^1+2\pi\xi^2+2\pi\xi^3$	α	$2\pi\tilde{n}^1+2\pi\tilde{n}^2+2\pi\tilde{n}^3$
ξ^k $(k=1,2,3)$		$\tilde{n}^k$ $(k=1,2,3)$	

plicit. In spite of the continuity of the spacetime, these theories can have discontinuous field configurations.

2.1 2+1d Foliated BF Theory

We will discuss a foliated BF theory in $2+1$ dimensions. This is the $2+1$d version of the $3+1$d foliated QFT studied in [20–22].

2.1.1 Foliation and Foliated Gauge Fields

We consider a QFT on the d-dimensional manifold that is regarded as a stack of an infinite number of $(d-1)$-dimensional submanifolds. These submanifolds are called leaves and such a decomposition of a manifold is called a codimension-one foliation. A QFT on such a manifold is called a foliated QFT (FQFT) [21].

A codimension-one foliation is characterized by a nonzero one-form foliation field e. The foliation field e is orthogonal to the leaves of the foliation. For the foliation to be well-defined,

Table 3: The correspondences of the gauge fields and parameters between the foliated BF theory and the exotic BF theory in $3+1$ dimensions (the B-type fields).

The foliated BF theory		The exotic BF theory	
Gauge fields and parameters	Gauge transformations	Gauge fields and parameters	Gauge transformations
$B^i_0 - B^j_0$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$\partial_0(\lambda^i - \lambda^j)$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$\hat{A}^{k(ij)}_0$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$\partial_0\hat{\alpha}^{k(ij)}$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$
$B^i_k - B^j_k$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$\partial_k(\lambda^i - \lambda^j)$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$\hat{A}^{ij}$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$\partial_k\hat{\alpha}^{k(ij)}$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$
$\lambda^i - \lambda^j$ $((i,j)=(1,2),$ $(2,3),(3,1))$	$2\pi m^i - 2\pi m^j$ $((i,j)=(1,2),$ $(2,3),(3,1))$	$\hat{\alpha}^{k(ij)}$ $((i,j,k)=(1,2,3),$ $(2,3,1),(3,1,2))$	$2\pi\tilde{m}^i - 2\pi\tilde{m}^j$ $((i,j)=(1,2),$ $(2,3),(3,1))$
m^k $(k=1,2,3)$		$\tilde{m}^k$ $(k=1,2,3)$	

e must satisfy the constraint

$$e \wedge de = 0\,. \tag{5}$$

The foliation field has a gauge redundancy under the transformation $e \to \gamma e$, where γ is a scalar function. Using this redundancy, we can locally write the foliation field as $e = df$, where f is a scalar function. We can consider f as a coordinate that specifies the leaves of the foliation. For example, we consider the flat foliation in $2+1$ dimensions that decomposes a (x^1, x^2)-plane into an infinite number of lines along the x^1 direction. Then the foliation field can be written as $e = dx^2$ locally. We can also consider multiple simultaneous foliations indexed by k $(k = 1, 2, ..., n_f)$, where each foliation field is e^k. In the following, we consider the flat foliations $e^k = dx^k$.

A FQFT is a QFT coupled to foliation fields e^k as backgrounds. A FQFT contains foliated gauge fields that can have discontinuous configurations. We consider two types of $U(1)$ foliated gauge fields for each foliation k [22]. One is the foliated A-type $(1+1)$-form gauge field $\tilde{A}^k$ that obeys $\tilde{A}^k \wedge e^k = 0$.[5] $\tilde{A}^k$ can have one-form delta function singularities in the x^k direction

[5]The words A-type and B-type are the notation used only in this paper. In [21], A^k and B^k denote what we call $\tilde{A}^k$ and B^k in this paper, but in [22], the symbols A^k and B^k are swapped compared to those in [21].

Table 4: The foliated A-type $(1+1)$-form gauge field and its gauge parameters.

Gauge field and parameters	Constraints	Gauge transformations	Singularities and discontinuities
$(1+1)$-form $\tilde{A}^k$	$\tilde{A}^k \wedge e^k = 0$	$\tilde{A}^k \to \tilde{A}^k + d\tilde{\zeta}^k$	one-form delta functions $\delta(x^k - x_0^k)\,dx^k$
$(0+1)$-form $\tilde{\zeta}^k$	$\tilde{\zeta}^k \wedge e^k = 0$	$\tilde{\zeta}^k \to \tilde{\zeta}^k + 2\pi d\xi^k$	one-form delta functions $\delta(x^k - x_0^k)\,dx^k$
x^k-dependent function $\xi^k \in \mathbb{Z}$			zero-form step functions $\theta(x^k - x_0^k)$

as $\delta(x^k - x_0^k)\,dx^k$. The gauge transformation of $\tilde{A}^k$ is

$$\tilde{A}^k \to \tilde{A}^k + d\tilde{\zeta}^k\,, \tag{6}$$

where $\tilde{\zeta}^k$ is a $(0+1)$-form gauge parameter satisfying $\tilde{\zeta}^k \wedge e^k = 0$. The gauge parameter $\tilde{\zeta}^k$ has its own gauge transformation $\tilde{\zeta}^k \to \tilde{\zeta}^k + 2\pi d\xi^k$, where ξ^k is a x^k-dependent function valued in integers. The gauge parameter $\tilde{\zeta}^k$ can have one-form delta function singularities in the x^k direction, while the gauge parameters ξ^k can have zero-form step function discontinuities $\theta(x^k - x_0^k)$ in the x^k direction. For flat foliations $e^k = dx^k$, $\tilde{\zeta}^k$ can be locally written as $\zeta^k dx^k$, where ζ^k is a zero-form gauge parameter. The foliated A-type $(1+1)$-form gauge fields and its gauge parameters are summarized in Table 4. The other foliated gauge field is the foliated B-type gauge field B^k. In the foliated BF theory in $2+1$ dimensions, B^k is a zero-form gauge field that can have zero-form step function discontinuities in the x^k direction. The gauge transformation of B^k is

$$B^k \to B^k + 2\pi m^k - \mu\,, \tag{7}$$

where the gauge parameter m^k is a x^k-dependent function valued in integers and μ is a zero-form bulk gauge parameter. m^k can have zero-form step function discontinuities in the x^k direction. The foliated B-type zero-form gauge fields and its gauge parameters are summarized in Table 5.

2.1.2 2+1d Foliated BF Lagrangian

The foliated BF theory is a FQFT containing foliated gauge fields and bulk ordinary gauge fields with interactions among them. The foliated BF Lagrangian is

$$\mathcal{L}_f = \sum_{k=1}^{n_f} \frac{iM_k}{2\pi}(dB^k + n_k b) \wedge \tilde{A}^k + \frac{iN}{2\pi} b \wedge da\,, \tag{8}$$

Table 5: The foliated B-type zero-form gauge field and its gauge parameters.

Gauge field and parameter	Gauge transformations	Singularities and discontinuities
zero-form B^k	$B^k \to B^k + 2\pi m^k - \mu$	zero-form step functions $\theta(x^k - x^k_0)$
x^k-dependent function $m^k \in \mathbb{Z}$		zero-form step functions $\theta(x^k - x^k_0)$

where $\tilde{A}^k$ is an A-type $(1+1)$-form foliated gauge field satisfying $\tilde{A}^k \wedge e^k = 0$, B^k is a B-type zero-form foliated gauge field, a and b are one-form gauge fields, and N, M_k, and n_k are integers. These fields are $U(1)$ gauge fields and the gauge symmetry $U(1)$ is Higgsed down to $\mathbb{Z}_N$ or $\mathbb{Z}_{M_k}$. The first term $\sum_{k=1}^{n_f} \frac{iM_k}{2\pi} dB^k \wedge \tilde{A}^k$ is a stack of $1+1$d BF theories for each foliations, the third term $\frac{iN}{2\pi} b \wedge da$ is a bulk $2+1$d BF theory, and the second term is interactions between the foliated fields and the bulk fields.

Let us discuss the special case where the foliations are flat, $n_f = 2$ (i.e., $e^k = dx^k$ for $k = 1,2$), $M_k = N$ and $n_k = 1$. In this case, the foliated gauge field $\tilde{A}^k$ can be written as $\tilde{A}^k = A^k \wedge dx^k$, where A^k is a one-form gauge field. In this special case, the foliated Lagrangian can be written as

$$\mathcal{L}_f = \sum_{k=1}^{2} \frac{iN}{2\pi} (dB^k + b) \wedge A^k \wedge dx^k + \frac{iN}{2\pi} b \wedge da\,. \tag{9}$$

The equations of motion are

$$\frac{N}{2\pi}(dB^k + b) \wedge dx^k = 0\,, \tag{10a}$$
$$\frac{N}{2\pi} db = 0\,, \tag{10b}$$
$$\frac{N}{2\pi} dA^k \wedge dx^k = 0\,, \tag{10c}$$
$$\sum_{k=1}^{2} \frac{N}{2\pi} A^k \wedge dx^k + \frac{N}{2\pi} da = 0\,. \tag{10d}$$

The gauge transformations are

$$A^k \wedge dx^k \to A^k \wedge dx^k + d\zeta^k \wedge dx^k\,, \tag{11a}$$
$$B^k \to B^k + 2\pi m^k - \mu\,, \tag{11b}$$
$$a \to a + d\lambda - \sum_{k=1}^{2} \zeta^k dx^k\,, \tag{11c}$$
$$b \to b + d\mu\,, \tag{11d}$$

Table 6: Singularities and discontinuities of the bulk gauge field a and its gauge parameter.

Gauge field and parameter	Gauge transformation	Terms including singularities and discontinuities
a_0	$a_0 \to a_0 + \partial_0 \lambda$	$f_0^1(x^0, x^2)\theta(x^1 - x_0^1) + f_0^2(x^0, x^1)\theta(x^2 - x_0^2)$
a_1	$a_1 \to a_1 + \partial_1 \lambda - \zeta^1$	$f_1^1(x^0, x^2)\delta(x^1 - x_0^1) + f_1^2(x^0, x^1)\theta(x^2 - x_0^2)$
a_2	$a_2 \to a_2 + \partial_2 \lambda - \zeta^2$	$f_2^1(x^0, x^2)\theta(x^1 - x_0^1) + f_2^2(x^0, x^1)\delta(x^2 - x_0^2)$
λ	$\lambda \to \lambda + 2\pi\xi^1 + 2\pi\xi^2$	$g^1(x^0, x^2)\theta(x^1 - x_0^1) + g^2(x^0, x^1)\theta(x^2 - x_0^2)$

where ζ^k, m^k and μ are the gauge parameters explained in Section 2.1.1, and λ are zero-form bulk gauge parameters that also have their own gauge transformations. The gauge transformation of λ is $\lambda \to \lambda + 2\pi\xi^1 + 2\pi\xi^2$, where ξ^k are x^k-dependent functions valued in integers explained in Section 2.1.1. Note that while ξ^k are the parameters for the transformation of ζ^k, the constant modes of ξ^k do not affect ζ^k and rather make λ a $U(1)$-valued function. The equations of motion and the gauge transformations imply that the bulk fields a, b and their gauge parameters can have singularities and discontinuities. The singularities and discontinuities of a are shown in Table 6, where f_i^k and g^k are some continuous functions with appropriate periodicity conditions.

Integrating the fields out and considering specific field configurations, we can show that the following quantities are quantized:

$$\oint_{C_1^0} a \in \frac{2\pi}{N}\mathbb{Z}, \tag{12a}$$

$$\oint_{S_2^k} A^k \wedge dx^k \in \frac{2\pi}{N}\mathbb{Z}, \tag{12b}$$

$$\oint_{C_1} b \in \frac{2\pi}{N}\mathbb{Z}, \tag{12c}$$

$$B^1 - B^2 \in \frac{2\pi}{N}\mathbb{Z}, \tag{12d}$$

where C_1^0 is a closed one-dimensional loop along the time x^0 direction, C_1 is an arbitrary closed one-dimensional loop, and S_2^k is a two-dimensional strip with a fixed width along the x^k direction. For example for (12b), there is a configuration

$$B^1 = 2\pi j \frac{x^2}{l^2}(\theta(x^1 - x_1^1) - \theta(x^1 - x_2^1)), \tag{13}$$

where j is an integer. This configuration is periodic in x^2 up to the gauge transformation (11b). With this configuration, we have

$$\begin{aligned}\oint_{C_1^0\times C_1^1\times C_1^2} dB^1\wedge A^1\wedge dx^1 &= \oint_{C_1^0\times C_1^1\times C_1^2} 2\pi j\left(\theta(x^1-x_1^1)-\theta(x^1-x_2^1)\right)\frac{1}{l^2}dx^2\wedge A^1\wedge dx^1\\ &= \frac{2\pi j}{l^2}\oint_{C_1^2}dx^2\oint_{C_1^0\times[x_1^1,x_2^1]}A^1\wedge dx^1\,,\end{aligned} \tag{14}$$

where C_1^2 is a closed one-dimensional loop along the x^2 direction. If we use the equation of motion (10c) and perform the sum over j in this configuration as a part of the path-integral in terms of B^1, we get

$$\sum_{j\in\mathbb{Z}}\exp\left[\frac{iNj}{l^2}\oint_{C_1^2}dx^2\oint_{C_1^0\times[x_1^1,x_2^1]}A^1\wedge dx^1\right]=\sum_{j\in\mathbb{Z}}\exp\left[iNj\oint_{C_1^0\times[x_1^1,x_2^1]}A^1\wedge dx^1\right]. \tag{15}$$

Then $N\oint_{C_1^0\times[x_1^1,x_2^1]}A^1\wedge dx^1$ must be in $2\pi\mathbb{Z}$; the configuration of A^1 not satisfying this condition does not contribute to the path integral. Again from the equation of motion (10c), C_1^0 can be deformed into C_1^{02} that is a closed loop in the (x^0,x^2)-plane.

2.1.3 Gauge-Invariant Operators

Let us consider the gauge-invariant operators, which describe excitations moving in spacetime.

The first one is

$$F^q[C_1^0]=\exp\left[iq\oint_{C_1^0}a\right], \tag{16}$$

where q is an integer. From (12a), we can see that $F^N[C_1^0]=1$, and thus $F^q[C_1^0]$ is a $\mathbb{Z}_N$ operator: $F^{q+N}=F^q$. The deformation of C_1^0 would break the gauge invariance of the operator under the transformation ζ^k. If the contour were C_1^{02} in the (x^0,x^2)-plane, under the gauge transformation of a, the defect operator would become

$$F^q[C_1^{02}]\to\exp\left[iq\oint_{C_1^{02}}\left\{dx^0\partial_0\lambda+dx^2(\partial_2\lambda-\zeta^2)\right\}\right]F^q[C_1^{02}], \tag{17}$$

which would not be gauge invariant. Since C_1^0 is a line in the time direction, this one-dimensional operator is the defect operator that describes a fracton, which cannot move in space.

The second one is

$$V^q[x]=\exp\left[iq(B^1-B^2)\right], \tag{18}$$

where q is an integer again. From (12d), we can see that $V^N[x]=1$ and thus $V^q[x]$ is also a $\mathbb{Z}_N$ operator: $V^{q+N}=V^q$. The point operator $V^q[x]$ is the symmetry operator that generates a $\mathbb{Z}_N$ electric global symmetry, which is a subsystem symmetry.

The third ones are

$$W_k^q[S_2^k]=\exp\left[iq\oint_{S_2^k}A^k\wedge dx^k\right],\quad k=1,2, \tag{19}$$

where q is an integer again. Similarly from (12b), $W_k^q[S_2^k]^N = 1$ and thus $W_k^q[S_2^k]$ are $\mathbb{Z}_N$ operators: $W^{q+N} = W^q$. These two-dimensional strip operators describe a dipole of fractons separated in the x^k direction, which can move in the other direction in space, like a lineon. If S_2^k are in the (x^1, x^2)-plane, these operators become the symmetry operators that generate $\mathbb{Z}_N$ dipole global symmetries, which are also subsystem symmetries.

These two types of symmetry operators are the charged objects under the other symmetry. That is, $V^p[x]$ and $W_k^q[S_2^k]$ satisfy the following relations at equal time:

$$V^p[x]\, W_k^q[S_2^k] = \mathrm{e}^{2\pi i pq/N}\, W_k^q[S_2^k]\, V^p[x], \quad \text{if} \quad x_1^k < x^k < x_2^k, \tag{20}$$

when S_2^k is $[x_1^1, x_2^1] \times C_1^2$ $(k=1)$ or $C_1^1 \times [x_1^2, x_2^2]$ $(k=2)$, where C_1^r $(r=1,2)$ is a closed loop in the x^r-plane. We can derive this relation using the canonical commutation relation

$$\left[B^1(x^0,x^1,x^2), A_2^1(x^0,y^1,y^2)\right] = -\frac{2\pi i}{N}\delta^2(x^1-y^1, x^2-y^2), \tag{21a}$$

$$\left[B^2(x^0,x^1,x^2), A_1^2(x^0,y^1,y^2)\right] = +\frac{2\pi i}{N}\delta^2(x^1-y^1, x^2-y^2). \tag{21b}$$

All the other commutators are zero.

In addition, the bulk $2+1$d BF theory has a gauge-invariant operator

$$T^q[C_1] = \exp\left[iq\oint_{C_1} b\right]. \tag{22}$$

From (12c), this b operator is also a $\mathbb{Z}_N$ operator: $T^{q+N} = T^q$. This operator has the winding action on the gauge-invariant operator (16) as

$$T^p[C_1]\ \cdot F^q[C_1^0] = \mathrm{e}^{-2\pi i pq/N} F^q[C_1^0], \tag{23}$$

when C_1 surrounds C_1^0 [37–40]. Without the defect operator F^q inside C_1, the b operator T^q becomes trivial, which corresponds to a time-like symmetry [36].[6] For the later purpose, it will be convenient to consider the case when C_1 is a rectangle $C_1^{12,\mathrm{rect}}(x_1^1, x_2^1, x_1^2, x_2^2)$ in the space. In this case, using the equation of motion (10a), the integral in the definition of T^q can be performed as

$$\begin{aligned} T^q\left[C_1^{12,\mathrm{rect}}(x_1^1,x_2^1,x_1^2,x_2^2)\right] &= \exp\left[iq\oint_{C_1^{12,\mathrm{rect}}}(-\partial_1 B^2 dx^1 - \partial_2 B^1 dx^2)\right] \\ &= \exp\left[-iq\Delta_{12}(B^1-B^2)(x_1^1,x_2^1,x_1^2,x_2^2)\right], \end{aligned} \tag{24}$$

where $\Delta_{12} f(x_1^1,x_2^1,x_1^2,x_2^2) = f(x_2^1,x_2^2) - f(x_2^1,x_1^2) - f(x_1^1,x_2^2) + f(x_1^1,x_1^2)$. This quadrupole operator is a product of the gauge-invariant operators V^q localized at the corners of the rectangle.

Note that the operator T^q cannot be remotely detected by an operator within a spatial slice, as the fracton operator F^q cannot be bent to braid with T^p. This implies that there is no physical excitation corresponding to the operator T^p in this situation. This contrasts with the case of usual topological field theory where every non-trivial line operator corresponds to a physical excitation.

[6] A time-like symmetry acts nontrivially on a Hilbert space in the presence of time-like defects. Without the defect operator, the time-like symmetry operator becomes trivial. We thank Pranay Gorantla for his comments on the relations between the b operators and the time-like symmetries.

2.2 2+1d Exotic BF Theory

In this section, we review the exotic BF theory in $2+1$ dimensions, which is the $\mathbb{Z}_N$ tensor gauge theory in [13]. In Section 2.3 we will see that this exotic theory is equivalent to the foliated BF theory discussed in the previous section.

2.2.1 Tensor Gauge Fields

We will discuss an exotic theory that is not Lorentz invariant and has only the 90 degree rotational invariance. Such theories can have tensor gauge fields, each of which is in a representation of the 90 degree rotation group $\mathbb{Z}_4$. Irreducible representations of $\mathbb{Z}_4$ are one-dimensional ones $\mathbf{1}_n$ $(n=0,\pm1,2)$, where n is the spin. The exotic BF theory contains a compact scalar ϕ^{12} in the representation $\mathbf{1}_2$ and a $U(1)$ tensor gauge field (A_0, A_{12}) in the representation $(\mathbf{1}_0, \mathbf{1}_2)$. Their gauge transformations are

$$A_0 \to A_0 + \partial_0 \alpha\,, \tag{25a}$$
$$A_{12} \to A_{12} + \partial_1 \partial_2 \alpha\,, \tag{25b}$$
$$\phi^{12} \to \phi^{12} + 2\pi \tilde{m}^1 - 2\pi \tilde{m}^2\,, \tag{25c}$$

where α is a $U(1)$-valued gauge parameter: $\alpha \sim \alpha + 2\pi$, in the representation $\mathbf{1}_0$, and $\tilde{m}^k$ are x^k-dependent functions valued in integers. The gauge parameter α has its own gauge transformation: $\alpha \to \alpha + 2\pi\tilde{n}^1 + 2\pi\tilde{n}^2$, where $\tilde{n}^k$ are x^k-dependent functions valued in integers. A_{12} can have delta function singularities and A_0, α and ϕ^{12} can have step function discontinuities as in Table 7, where $\tilde{f}_0^k$, $\tilde{f}_{12}^k$, $\tilde{g}^k$, and $\tilde{h}^k$ are some continuous functions with appropriate periodicity conditions. For example, the following configurations are allowed:

$$A_{12} = 2\pi \frac{x^0}{l^0}\left[\frac{1}{l^2}\delta(x^1 - x_0^1) + \frac{1}{l^1}\delta(x^2 - x_0^2) - \frac{1}{l^1 l^2}\right], \tag{26}$$

$$\phi^{12} = 2\pi \left[\frac{x^2}{l^2}\theta(x^1 - x_0^1) + \frac{x^1}{l^1}\theta(x^2 - x_0^2) - \frac{x^1 x^2}{l^1 l^2}\right]. \tag{27}$$

The gauge parameters $\tilde{m}^k$ and $\tilde{n}^k$ can have step function discontinuities in the x^k direction.

2.2.2 2+1d Exotic BF Lagrangian

The exotic BF Lagrangian is

$$\mathcal{L}_e = \frac{iN}{2\pi}\phi^{12}(\partial_0 A_{12} - \partial_1\partial_2 A_0)\,. \tag{28}$$

The equations of motion are

$$\frac{N}{2\pi}\partial_1\partial_2\phi^{12} = 0\,, \tag{29a}$$
$$\frac{N}{2\pi}\partial_0\phi^{12} = 0\,, \tag{29b}$$
$$\frac{N}{2\pi}(\partial_0 A_{12} - \partial_1\partial_2 A_0) = 0\,. \tag{29c}$$

Table 7: Singularities and discontinuities of the tensor gauge fields and their gauge parameters.

Gauge fields and parameter	Gauge transformation	Terms including singularities and discontinuities
A_0	$A_0 \to A_0 + \partial_0 \alpha$	$\tilde{f}_0^1(x^0,x^2)\theta(x^1-x_0^1)+\tilde{f}_0^2(x^0,x^1)\theta(x^2-x_0^2)$
A_{12}	$A_{12} \to A_{12} + \partial_1\partial_2 \alpha$	$\tilde{f}_{12}^1(x^0,x^2)\delta(x^1-x_0^1)+\tilde{f}_{12}^2(x^0,x^1)\delta(x^2-x_0^2)$
α	$\alpha \to \alpha + 2\pi\tilde{n}^1 + 2\pi\tilde{n}^2$	$\tilde{g}^1(x^0,x^2)\theta(x^1-x_0^1)+\tilde{g}^2(x^0,x^1)\theta(x^2-x_0^2)$
ϕ^{12}	$\phi^{12} \to \phi^{12} + 2\pi\tilde{m}^1 - 2\pi\tilde{m}^2$	$\tilde{h}^1(x^0,x^2)\theta(x^1-x_0^1)+\tilde{h}^2(x^0,x^1)\theta(x^2-x_0^2)$

Integrating specific configurations out, we can show that the following quantities are quantized:

$$\oint_{C_1^0} dx^0 A_0 \in \frac{2\pi}{N}\mathbb{Z}\,, \tag{30a}$$

$$\oint_{S_2^1} (dx^0 dx^1 \partial_1 A_0 + dx^2 dx^1 A_{12}) \in \frac{2\pi}{N}\mathbb{Z}\,, \tag{30b}$$

$$\oint_{S_2^2} (dx^0 dx^2 \partial_2 A_0 + dx^1 dx^2 A_{12}) \in \frac{2\pi}{N}\mathbb{Z}\,, \tag{30c}$$

$$\phi^{12} \in \frac{2\pi}{N}\mathbb{Z}\,, \tag{30d}$$

where C_1^0 is a closed one-dimensional loop along the time x^0 direction, and S_2^k is a two-dimensional strip with a fixed width extended along the x^k direction. For example for (30b), there is a configuration

$$\phi^{12} = 2\pi j \frac{x^2}{l^2}\left[\theta(x^1-x_1^1)-\theta(x^1-x_2^1)\right], \tag{31}$$

where j is an integer. Then we have

$$\oint_{C_1^1} dx^1 \oint_{C_1^2} dx^2 \partial_2 \phi^{12} \partial_1 A_0 = 2\pi j \oint_{C_1^1} dx^1 (\theta(x^1-x_1^1)-\theta(x^1-x_2^1)) \frac{1}{l^2} \oint_{C_1^2} dx^2 \partial_1 A_0\,. \tag{32}$$

Integrating this configuration out and using the equation of motion (29c), this part of partition function becomes

$$\sum_{j\in\mathbb{Z}} \exp\left[\frac{iNj}{l^2}\oint_{C_1^2} dx^2 \oint_{C_1^0\times[x_1^1,x_2^1]} dx^0 dx^1 \partial_1 A_0\right] = \sum_{j\in\mathbb{Z}} \exp\left[iNj \oint_{C_1^0\times[x_1^1,x_2^1]} dx^0 dx^1 \partial_1 A_0\right], \tag{33}$$

and the term $N\oint_{C_1^0\times[x_1^1,x_2^1]} dx^0 dx^1 \partial_1 A_0$ must be an integer for the configuration to contribute. Again using the equation of motion (29c), C_1^0 can be deformed into C_1^{02} and the term becomes $\oint_{C_1^{02}\times[x_1^1,x_2^1]}(dx^0 dx^1\partial_1 A_0 + dx^2 dx^1 A_{12})$.

2.2.3 Gauge-Invariant Operators

Let us discuss gauge-invariant operators. The defect operator that describes fractons is

$$\tilde{F}^q[C_1^0] = \exp\left[iq\oint_{C_1^0} dx^0 A_0\right]. \tag{34}$$

As in the case of the foliated BF theory, the deformation of C_1^0 would break the gauge invariance of the operator. The symmetry operator that generates a $\mathbb{Z}_N$ electric global symmetry is

$$\tilde{V}^q[x] = \exp\left[iq\phi^{12}\right]. \tag{35}$$

The strip operators that describe a dipole of fractons are

$$\tilde{W}_1^q\left[S_2^1\right] = \exp\left[iq\oint_{S_2^1}(dx^0 dx^1\partial_1 A_0 + dx^2 dx^1 A_{12})\right], \tag{36a}$$

$$\tilde{W}_2^q\left[S_2^2\right] = \exp\left[iq\oint_{S_2^2}(dx^0 dx^2\partial_2 A_0 + dx^1 dx^2 A_{12})\right]. \tag{36b}$$

If S_2^k are in the (x^1,x^2)-plane, $\tilde{W}_k^q$ are the symmetry operators that generate $\mathbb{Z}_N$ dipole global symmetries. These gauge-invariant operators are $\mathbb{Z}_N$ operators: q is an element of $\mathbb{Z}_N$.

The two types of symmetry operators satisfy the following relations

$$\tilde{V}^p[x]\,\tilde{W}_k^q[S_2^k] = \mathrm{e}^{2\pi i pq/N}\,\tilde{W}_k^q[S_2^k]\,\tilde{V}^p[x]\,,\quad \text{if}\quad x_1^k < x^k < x_2^k\,. \tag{37}$$

We can derive this relation using the canonical commutation relations at equal time:

$$\left[\phi^{12}(x^0,x^1,x^2), A_{12}(x^0,y^1,y^2)\right] = -\frac{2\pi i}{N}\delta^2(x^1-y^1,x^2-y^2)\,. \tag{38}$$

All the other commutators are zero. These symmetries in the exotic BF theory have the same structure as the foliated BF theory discussed in Section 2.1.

In addition, there is a gauge-invariant operator that can detect the fracton defect operator:

$$\tilde{T}^q\left[C_1^{12,\mathrm{rect}}(x_1^1,x_2^1,x_1^2,x_2^2)\right] = \exp\left[-iq\Delta_{12}\phi^{12}(x_1^1,x_2^1,x_1^2,x_2^2)\right]. \tag{39}$$

This quadrupole operator is a product of the gauge-invariant operators $\tilde{V}^q$ localized at the corners of the rectangle, which is a time-like symmetry [36]. The operator $\tilde{T}^p$ can detect the fracton operator $\tilde{F}^q$:

$$\tilde{T}^p\left[C_1^{12,\mathrm{rect}}(x_1^1,x_2^1,x_1^2,x_2^2)\right]\cdot\tilde{F}^q[C_1^0] = \mathrm{e}^{-2\pi i pq/N}\tilde{F}^q[C_1^0], \tag{40}$$

when $C_1^{12,\mathrm{rect}}(x_1^1,x_2^1,x_1^2,x_2^2)$ surrounds C_1^0.

2.3 Correspondences in 2+1 Dimensions

The 2+1d foliated BF theory explained in Section 2.1 and the 2+1d exotic BF theory explained in Section 2.2 are equivalent in case that $e^k = dx^k$, $M_k = N$ and $n_k = 1$ ($k = 1, 2$), which we call the foliated-exotic duality. We identify the gauge-invariant operators in the foliated BF theory with those of the exotic BF theory. By matching the gauge-invariant operators, we can derive the correspondences of the gauge fields and parameters.

As noted in [22], some foliated theories with the flat condition

$$d\left(\sum_{k=1}^{2}\frac{N}{2\pi}A^k\wedge dx^k\right)=0 \tag{41}$$

correspond to the tensor gauge theories. In the foliated BF theory explained in Section 2.1, integrating b out leads to the equation of motion (10d) that becomes the flat condition (41). In the following, we will explicitly see how the gauge-invariant operators are identified and the gauge fields and parameters match under the condition (41). The precise correspondences include the bulk gauge field a in the FQFT side in a non-trivial way. We also see that the allowed singularities and discontinuities of the fields and parameters match between the two sides.

First, let us consider the fracton defect operators. We identify the operators $F^q[C_1^0]$ with $\tilde{F}^q[C_1^0]$ defined in (16) and (34):

$$\exp\left[iq\oint_{C_1^0}a\right]\simeq\exp\left[iq\oint_{C_1^0}dx^0A_0\right], \tag{42}$$

which leads to the field correspondence

$$a_0\simeq A_0\,. \tag{43}$$

The gauge transformations of a_0 and A_0 explained in (11c) and (25a) are

$$a_0\to a_0+\partial_0\lambda\,, \tag{44a}$$
$$A_0\to A_0+\partial_0\alpha\,, \tag{44b}$$

from which we obtain the gauge parameter correspondence

$$\lambda\simeq\alpha\,. \tag{45}$$

Moreover, the gauge transformations of λ and α are

$$\lambda\to\lambda+2\pi\xi^1+2\pi\xi^2\,, \tag{46a}$$
$$\alpha\to\alpha+2\pi\tilde{n}^1+2\pi\tilde{n}^2\,, \tag{46b}$$

which can be matched by

$$\xi^k\simeq\tilde{n}^k\,. \tag{47}$$

In these correspondences, one can check that their singularities and discontinuities are also matched.

The equations of motion (10d) in components are

$$\frac{N}{2\pi}(A_0^1+\partial_0a_1-\partial_1a_0)=0\,, \tag{48a}$$
$$\frac{N}{2\pi}(A_0^2+\partial_0a_2-\partial_2a_0)=0\,, \tag{48b}$$
$$\frac{N}{2\pi}(A_2^1-A_1^2+\partial_2a_1-\partial_1a_2)=0\,. \tag{48c}$$

These equations of motion, when combined with (43), imply

$$A_0^1 + \partial_0 a_1 \simeq \partial_1 A_0 \,, \tag{49a}$$
$$A_0^2 + \partial_0 a_2 \simeq \partial_2 A_0 \,. \tag{49b}$$

Note that the gauge transformations by ζ^k in the left hand sides cancel out, and thus these are consistent with the correspondence (45).

Next, let us consider the strip operators. We want to identify the operators $W_k^q[S_2^k]$ with $\tilde{W}_k^q[S_2^k]$ defined in (19), (36a) and (36b), but the gauge transformations of the exponents are not matched and the field correspondences would be inconsistent with (49a) and (49b). Therefore, we define the modified gauge-invariant strip operators $W_{k,\text{mod}}^q[S_2^k]$ as

$$W_{k,\text{mod}}^q[S_2^k] = \exp\left[iq \oint_{S_2^k} \left(A^k \wedge dx^k + d(a_k dx^k)\right)\right], \quad k = 1, 2\,, \tag{50}$$

where the exponents are quantized:

$$\oint_{S_2^k} \left(A^k \wedge dx^k + d(a_k dx^k)\right) \in \frac{2\pi}{N}\mathbb{Z}\,, \tag{51}$$

and therefore $W_{k,\text{mod}}^q[S_2^k]$ are $\mathbb{Z}_N$ operators. For example for $k=1$, to show this quantization, we consider a configuration

$$b = 2\pi j \frac{1}{l^2}(\theta(x^1 - x_1^1) - \theta(x^1 - x_2^1))dx^2\,, \tag{52}$$

where j is an integer. Using the equation of motion (10c), we have

$$\begin{aligned}
\oint_{C_1^0\times C_1^1\times C_1^2} b \wedge \left(\sum_{k=1}^{2} A^k \wedge dx^k + da\right) &= \frac{2\pi j}{l^2}\oint dx^2 \oint_{C_1^0\times[x_1^1,x_2^1]} (A_0^1 + \partial_0 a_1 - \partial_1 a_0)\, dx^0 \wedge dx^1 \\
&= 2\pi j \oint_{C_1^0\times[x_1^1,x_2^1]} (A_0^1 + \partial_0 a_1 - \partial_1 a_0)\, dx^0 \wedge dx^1\,.
\end{aligned} \tag{53}$$

From the quantization (12a), the term $\oint_{C_1^0} a_0 dx^0$ is in $2\pi\mathbb{Z}/N$. Integrating this configuration out, this part of partition function becomes

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \exp\left[iNj \oint_{C_1^0\times[x_1^1,x_2^1]} (A_0^1 + \partial_0 a_1)\, dx^0 \wedge dx^1\right] \\
&= \sum_{j\in\mathbb{Z}} \exp\left[iNj \oint_{C_1^0\times[x_1^1,x_2^1]} (A^1 \wedge dx^1 + d(a_1 dx^1))\right].
\end{aligned} \tag{54}$$

Then $N \oint_{C_1^0\times[x_1^1,x_2^1]}(A^1 \wedge dx^1 + d(a_1 dx^1))$ must be in $2\pi\mathbb{Z}$. Again using the equation of motion (10c), we can deform the C_1^0 into C^{02} and conclude the quantization (51).

Having prepared the operator (50), we identify the operators $W_k'^q[S_2^k]$ with $\tilde{W}_k^q[S_2^k]$:

$$\exp\left[iq \oint_{S_2^1} \left(A^1 \wedge dx^1 + d(a_1 dx^1)\right)\right] \simeq \exp\left[iq \oint_{S_2^1} (dx^0 dx^1 \partial_1 A_0 + dx^2 dx^1 A_{12})\right], \tag{55a}$$

$$\exp\left[iq \oint_{S_2^2} \left(A^2 \wedge dx^2 + d(a_2 dx^2)\right)\right] \simeq \exp\left[iq \oint_{S_2^2} (dx^0 dx^2 \partial_2 A_0 + dx^1 dx^2 A_{12})\right], \tag{55b}$$

which leads to the field correspondences

$$A_2^1 + \partial_2 a_1 \simeq A_{12}\,, \tag{56a}$$
$$A_1^2 + \partial_1 a_2 \simeq A_{12}\,, \tag{56b}$$

and also (49a) and (49b) again. The terms $\partial_2 a_1$ and $\partial_1 a_2$ make the gauge transformations match with those of A_{12} under the gauge parameter correspondence (45).

Note that the ratio of $W^q_{k,\text{mod}}$ to W^q_k,

$$W^q_{k,\text{mod}}[S_2^k](W^q_k[S_2^k])^{-1} = \exp\left[iq\oint_{S_2^k} d(a_k dx^k)\right] \tag{57}$$

is trivial when a_k is single-valued on S_2^k by applying the Stokes' theorem. This means that there is no operator that braids with the ratio operator and thus it does not describe a physical excitation. This subtlety is tied to the fractonic nature of the system since in a non-fractonic topological field theory every operator corresponds to a physical excitation [34, 35].

Lastly, let us consider the $\mathbb{Z}_N$ electric global symmetry operators. We identify $V^q[x]$ with $\tilde{V}^q[x]$ defined in (18) and (35):

$$\exp\left[iq(B^1 - B^2)\right] \simeq \exp\left[iq\phi^{12}\right]. \tag{58}$$

Then we can derive the field correspondence

$$B^1 - B^2 \simeq \phi^{12}\,. \tag{59}$$

The gauge transformations by μ cancel out in the left-hand side. From the gauge transformations (11b) and (25c), we obtain

$$m^1 - m^2 \simeq \tilde{m}^1 - \tilde{m}^2\,. \tag{60}$$

Considering the discontinuities, we can see that

$$m^1 \simeq \tilde{m}^1\,, \quad m^2 \simeq \tilde{m}^2\,. \tag{61}$$

Under the correspondence (59), the time-like symmetry operator $T^q[C_1^{12,\text{rect}}]$ defined (24) corresponds to $\tilde{T}^q[C_1^{12,\text{rect}}]$ defined in (39). Note that on a Hilbert space with fracton defect operators, the b operator $T^q[C_1]$ is a product of $T^q[C_1^{12,\text{rect}}]$ surrounding the defects that are surrounded by C_1.

Under these correspondences, the equations of motion are also matched. Moreover, after integrating b out, and then using the correspondences, the Lagrangians (9) and (28) are exactly matched.

3 BF-type Theory in 3+1 Dimensions

In this section, we consider a foliated BF theory and an exotic BF theory in $3+1$ dimensions. Both of the theories are the continuum descriptions of the $\mathbb{Z}_N$ X-cube model [5] and the $\mathbb{Z}_N$ lattice tensor gauge theory [15]. As in the case of 2+1 dimensions, these two BF-type theories represent the same physics and we will show the explicit duality between them. Basically the discussion proceeds in parallel with that in $2+1$ dimensions in Section 2.

We take a four-torus of lengths l^0, l^1, l^2, l^3 as a spacetime and the coordinates (x^0, x^1, x^2, x^3) on it, with x^0 regarded as the Euclidean time. The spatial symmetry is the S_4 group generated by the 90 degree rotations along one of the axes, as the cubic lattice has.

Table 8: The foliated B-type one-form gauge field and its gauge parameters.

Gauge field and parameters	Constraint	Gauge transformation	Singularities and Discontinuities
one-form B^k		$B^k \to B^k + d\lambda^k + \tilde{\beta}^k - \mu$	zero-form step functions $\theta(x^k - x_0^k)$
zero-form λ^k		$\lambda^k \to \lambda^k + 2\pi m^k + \nu$	zero-form step functions $\theta(x^k - x_0^k)$
$(0+1)$-form $\tilde{\beta}^k$	$\tilde{\beta}^k \wedge e^k = 0$	$\tilde{\beta}^k \to \tilde{\beta}^k - 2\pi d m^k$	one-form delta functions $\delta(x^k - x_0^k)\, dx^k$
one-form μ		$\mu \to \mu + d\nu$	zero-form step functions
x^k-dependent function $m^k \in \mathbb{Z}$			zero-form step functions $\theta(x^k - x_0^k)$

3.1 3+1d Foliated BF Theory

We review the foliated BF theory in $3+1$ dimensions [20–22].

3.1.1 Foliated Gauge Fields

In the foliated BF theory in $3+1$ dimensions, the foliated A-type $(1+1)$-form gauge field $\tilde{A}^k$ is almost the same as $(2+1)$-dimensional one, while the foliated B-type gauge field B^k is a one-form gauge field that can have zero-form step function discontinuities in the x^k direction. The gauge transformation of B^k is

$$B^k \to B^k + d\lambda^k + \tilde{\beta}^k - \mu, \tag{62}$$

where λ^k is a zero-form gauge parameter, $\tilde{\beta}^k$ is a $(0+1)$-form gauge parameter satisfying $\tilde{\beta}^k \wedge e^k = 0$, and μ is a one-form bulk gauge parameter. The gauge parameter λ^k has its own gauge transformation $\lambda^k \to \lambda^k + 2\pi m^k + \nu$, where m^k is a x^k-dependent function valued in integers, β^k also has its own gauge transformation $\tilde{\beta}^k \to \tilde{\beta}^k + 2\pi d m^k$, and μ also has its own gauge transformation $\mu \to \mu + d\nu$, where ν is a zero-form gauge parameter. The gauge parameters λ^k and m^k can have zero-form step function discontinuities in the x^k direction. The gauge parameters $\tilde{\beta}^k$ can have one-form delta function singularities in the x^k direction that cancel out the delta function terms in $d\lambda^k$. The foliated B-type one-form gauge fields and its gauge parameters are summarized in Table 8.

3.1.2 3+1d Foliated BF Lagrangian

The foliated BF Lagrangian is similar to (8), where B^k is a B-type one-form foliated gauge field and b is a two-form gauge field. In the special case where the foliations are flat, $n_f = 3$ (i.e., $e^k = dx^k$ $(k = 1, 2, 3)$), $M_k = N$ and $n_k = 1$, the foliated Lagrangian can be written as

$$\mathcal{L}_f = \sum_{k=1}^{3} \frac{iN}{2\pi}(dB^k + b) \wedge A^k \wedge dx^k + \frac{iN}{2\pi} b \wedge da \,. \tag{63}$$

The equations of motion are

$$\frac{N}{2\pi}(dB^k + b) \wedge dx^k = 0 \,, \tag{64a}$$
$$\frac{N}{2\pi} db = 0 \,, \tag{64b}$$
$$\frac{N}{2\pi} dA^k \wedge dx^k = 0 \,, \tag{64c}$$
$$\sum_{k=1}^{3} \frac{N}{2\pi} A^k \wedge dx^k + \frac{N}{2\pi} da = 0 \,. \tag{64d}$$

The gauge transformations are

$$A^k \wedge dx^k \to A^k \wedge dx^k + d\zeta^k \wedge dx^k \,, \tag{65a}$$
$$B^k \to B^k + d\lambda^k + \beta^k dx^k - \mu \,, \tag{65b}$$
$$a \to a + d\lambda - \sum_{k=1}^{3} \zeta^k dx^k \,, \tag{65c}$$
$$b \to b + d\mu \,, \tag{65d}$$

where ζ^k, λ^k, $\beta^k dx^k = \tilde{\beta}^k$ and μ are the gauge parameters explained in Section 2.1.1 and Section 3.1.1, and λ are zero-form bulk gauge parameters that also have their own gauge transformations. The gauge transformation of λ is $\lambda \to \lambda + 2\pi\xi^1 + 2\pi\xi^2 + 2\pi\xi^3$, where ξ^k are x^k-dependent functions valued in integers explained in Section 2.1.1. As in the case of $2+1$ dimensions, the bulk fields a, b and their gauge parameters can have singularities and discontinuities.

Integrating the fields out and considering specific field configurations, we can show that the following quantities are quantized:

$$\oint_{C_1^0} a \in \frac{2\pi}{N}\mathbb{Z} \,, \tag{66a}$$
$$\oint_{S_2^k} A^k \wedge dx^k \in \frac{2\pi}{N}\mathbb{Z} \,, \tag{66b}$$
$$\oint_{S_2} b \in \frac{2\pi}{N}\mathbb{Z} \,, \tag{66c}$$
$$\oint_{C_1^q} \sum_{k=1}^{3} q_k B^k \in \frac{2\pi}{N}\mathbb{Z} \,, \tag{66d}$$

where C_1^0 is a closed one-dimensional loop along the time x^0 direction, S_2 is an arbitrary closed two-dimensional surface, and S_2^k is a two-dimensional strip with a fixed width along the x^k direction. The charges q_k are integers that satisfy $\sum_{k=1}^{3} q_k = 0$ and C_1^q is a one-dimensional loop supported on the intersection of leaves with nonzero q_k.

3.1.3 Gauge-Invariant Operators

Let us discuss the gauge-invariant operators, which describe excitations moving in spacetime.

The first one is

$$F^q[C_1^0] = \exp\left[iq \oint_{C_1^0} a\right], \tag{67}$$

$F^q[C_1^0]$ is a $\mathbb{Z}_N$ operator: $F^{q+N} = F^q$. Since C_1^0 is a line in the time direction, this one-dimensional operator is the defect operator that describes a fracton, which cannot move in space.

The second one is

$$L[C_1^q] = \exp\left[i \oint_{C_1^q} \sum_{k=1}^{3} q_k B^k\right], \tag{68}$$

where $q = (q_k)_k$ is a charge vector where the components are integers satisfying $\sum_{k=1}^3 q_k = 0$, and C_1^q is a closed one-dimensional line in an intersection of leaves, each of which is a single leaf for foliation k with $q_k \neq 0$. From (66d), we can see that $L[C_1^{(Nq)}] = 1$. Any charge vector q can be spanned by $q^1 = (0, 1, -1)$ and $q^2 = (-1, 0, 1)$, and we define $q^3 = -q^1 - q^2 = (1, -1, 0)$, and then the corresponding operators are

$$L_1^q[C_1^{01}] = \exp\left[iq \oint_{C_1^{01}} (B^2 - B^3)\right], \tag{69a}$$

$$L_2^q[C_1^{02}] = \exp\left[iq \oint_{C_1^{02}} (B^3 - B^1)\right], \tag{69b}$$

$$L_3^q[C_1^{03}] = \exp\left[iq \oint_{C_1^{03}} (B^1 - B^2)\right], \tag{69c}$$

where C_1^{0k} is a closed one-dimensional loop in the (x^0, x^k)-plane. The one-dimensional operator $L_k^q[C_1^{0k}]$ describes a x^k-lineon that can only move along the x^k direction. Therefore $L[C_1^q]$ is written as a product of the lineon operators. From (66d), we can see that $L_k^q[C_1^{0k}]$ are $\mathbb{Z}_N$ operators: $L_k^{q+N} = L_k^q$. The line operators $L_k^q[C_1^k]$ are the symmetry operators that generate $\mathbb{Z}_N$ tensor global symmetries, which are subsystem symmetries.

The third ones are

$$W_k^q[S_2^k] = \exp\left[iq \oint_{S_2^k} A^k \wedge dx^k\right], \quad k = 1, 2, 3. \tag{70}$$

Similarly, $W_k^q[S_2^k]$ are $\mathbb{Z}_N$ operators: $W^{q+N} = W^q$. These two-dimensional strip operators describe a dipole of fractons separated along the x^k direction, which can move in the other directions in space, like a planon. If S_2^k are in the space, these operators become the symmetry operators that generate $\mathbb{Z}_N$ dipole global symmetries, which are also subsystem symmetries.

These two types of symmetry operators are the charged objects under the other symmetry. That is, L^p and $W_k^q[S_2^k]$ satisfy the following relations at equal time:

$$L_2^p[C_1^2]\, W_1^q[S_2^1] = \mathrm{e}^{2\pi i pq I(C_1^2, S_2^1)/N}\, W_1^q[S_2^1]\, L_2^p[C_1^2], \quad \text{if} \quad x_1^1 < x^1 < x_2^1, \tag{71a}$$

$$L_3^p[C_1^3]\, W_1^q[S_2^1] = \mathrm{e}^{2\pi i pq I(C_1^3, S_2^1)/N}\, W_1^q[S_2^1]\, L_3^p[C_1^3], \quad \text{if} \quad x_1^1 < x^1 < x_2^1, \tag{71b}$$

where $S_2^1 = [x_2^1, x_1^1] \times C_1^{23}$, and I is the intersection number. Similar relations holds in the other directions.

The forth one is

$$K_{12}^q[C_1^{03} \times \mathcal{C}_1^{12}] = \exp\left[iq \oint_{C_1^{03}\times\mathcal{C}_1^{12}} (A^1 \wedge dx^1 + A^2 \wedge dx^2)\right], \tag{72}$$

where $\mathcal{C}_1^{12}$ is a one-dimensional line connecting (x_1^1, x_1^2) to (x_2^1, x_2^2) in the (x^1, x^2)-plane. K_{12}^q is the strip operator that describes a dipole of fractons at (x_1^1, x_1^2, x^3) and (x_2^1, x_2^2, x^3), which can move in the x^3 direction, like a x^3-lineon. Using the Stokes' theorem, the equations of motion (64c), $\mathcal{C}_1^{12}$ can be deformed to $[x_1^1, x_2^1] \times \{x_1^2\} + \{x_2^1\} \times [x_1^2, x_2^2]$, and this special case we write $\tilde{K}_3^q$ as

$$\begin{aligned} K_{12}^q[C_1^{03} \times \mathcal{C}_1^{12}] =& \exp\left[iq \oint_{C_1^{03}\times[x_1^1,x_2^1]\times\{x_1^2\}} A^1 \wedge dx^1\right] \exp\left[iq \oint_{C_1^{03}\times\{x_2^1\}\times[x_1^2,x_2^2]} A^2 \wedge dx^2\right] \\ =& W_1^q\left[C_1^{03} \times [x_1^1, x_2^1] \times \{x_1^2\}\right] W_2^q\left[C_1^{03} \times \{x_2^1\} \times [x_1^2, x_2^2]\right]. \end{aligned} \tag{73}$$

Similarly, we have the strip operators K_{23}^q and K_{31}^q.

In addition, the bulk $3+1$d BF theory has a gauge-invariant operator

$$T^q[S_2] = \exp\left[iq \oint_{S_2} b\right]. \tag{74}$$

From (66c), this surface operator is also a $\mathbb{Z}_N$ operator: $T^{q+N} = T^q$. This operator has the winding action on the gauge-invariant operator (67) as

$$T^p[S_2] \cdot F^q[C_1^0] = \mathrm{e}^{-2\pi i pq/N} F^q[C_1^0], \tag{75}$$

when S_2 surrounds C_1^0. Without the defect operator F^q, the b operator T^q becomes trivial, which corresponds to a time-like symmetry [36]. When S_2 is $S_2^{123,\text{cube}}$ that is the surface of $[x_1^1, x_2^1] \times [x_1^2, x_2^2] \times [x_1^3, x_2^3]$, using the equation of motion (64a), a part of the integral in the definition of T^q can be performed as

$$\begin{aligned} T^q[S_2^{123,\text{cube}}] =& \exp\left[iq \oint_{S_2^{123,\text{cube}}} \{-(\partial_1 B_2^3 + \partial_2 B_1^3) dx^1 dx^2 - (\partial_2 B_3^1 + \partial_3 B_2^1) dx^2 dx^3\right. \\ &\left. \qquad\qquad -(\partial_3 B_1^2 + \partial_1 B_3^2) dx^3 dx^1\}\right] \\ =& \exp\left[-iq \int_{x_1^1}^{x_2^1} \{\Delta_{23}(B_1^2 - B_1^3)(x_1^2, x_2^2, x_1^3, x_2^3)\}\, dx^1\right] \\ &\times \exp\left[-iq \int_{x_1^2}^{x_2^2} \{\Delta_{31}(B_2^3 - B_2^1)(x_1^3, x_2^3, x_1^1, x_2^1)\}\, dx^2\right] \\ &\times \exp\left[-iq \int_{x_1^3}^{x_2^3} \{\Delta_{12}(B_3^1 - B_3^2)(x_1^1, x_2^1, x_1^2, x_2^2)\}\, dx^3\right]. \end{aligned} \tag{76}$$

This cage operator is localized on the edges of the rectangular cuboid whose surface is $S_2^{123,\text{cube}}$. When S_2 is $S_2^{012,\text{cube}}(x_1^1, x_2^1, x_1^2, x_2^2)$ that is $C_1^0 \times C_1^{12,\text{rect}}(x_1^1, x_2^1, x_1^2, x_2^2)$ extended along the x^0

direction, the b operator can be written as

$$\begin{aligned} T^q[S_2^{012,\text{cube}}(x_1^1,x_2^1,x_1^2,x_2^2)] &= \exp\left[iq \oint_{S_2^{012,\text{cube}}} \{-(\partial_0 B_1^2 + \partial_1 B_0^2) dx^0 dx^1 \right. \\ &\qquad \left. - (\partial_2 B_0^1 + \partial_0 B_2^1) dx^2 dx^0 \} \right] \\ &= \exp\left[-iq \oint_{C_1^0} \Delta_{12}(B_0^1 - B_0^2)(x_1^1,x_2^1,x_1^2,x_2^2) dx^0 \right]. \end{aligned} \tag{77}$$

This operator describes a quadrupole of x^3-lineons $L_3^q[C_1^0]$ at the corners of the rectangle $C_1^{12,\text{rect}}(x_1^1,x_2^1,x_1^2,x_2^2)$, which is trivial. Similarly, we also have the operators that describe a quadrupole of x^1-lineons and a quadrupole of x^2-lineons.

As opposed to the $(2+1)$-dimensional case, the time-like T^q operator does correspond to physical excitations that are the lineons sitting at the corners of the time-slice of the operator. However the operator is trivial as it cannot be detected by a time-like symmetry.[7] This is also different from the case of an ordinary BF theory, in which a $(1+1)$-dimensional operator like T^q corresponds to a one-dimensional, or string, excitation.

3.2 3+1d Exotic BF Theory

We will review the exotic BF theory in $3+1$ dimensions [15].

3.2.1 Tensor Gauge Fields

In $3+1$ dimensions, the 90 degree rotations generate the orientation-preserving cubic group, which is isomorphic to the permutation group S_4. Then each tensor gauge field is in a representation of S_4. The irreducible representations of S_4 are classified as the following tensors (see Appendix in [14, 15]):

$$\begin{aligned} &\mathbf{1}: && S\,, \\ &\mathbf{1}': && T_{(ijk)}\,, \quad i\neq j\neq k\,, \\ &\mathbf{2}: && B_{[ij]k}\,, \quad i\neq j\neq k\,, \quad B_{[ij]k}+B_{[jk]i}+B_{[ki]j}=0\,, \\ & && B_{i(jk)}\,, \quad i\neq j\neq k\,, \quad B_{i(jk)}+B_{j(ki)}+B_{k(ij)}=0\,, \\ &\mathbf{3}: && V_i\,, \\ &\mathbf{3}': && E_{ij}\,, \quad i\neq j\,, \quad E_{ij}=E_{ji}\,, \end{aligned} \tag{78}$$

where i, j, k are 1 or 2 or 3, the indices $[ij]$ are antisymmetrized, the indices (ij) are symmetrized. The two bases of irreducible representation **2** are related as

$$B_{i(jk)} = B_{[ij]k} + B_{[ik]j}\,, \tag{79a}$$

$$B_{[ij]k} = \frac{1}{3}(B_{i(jk)} + B_{j(ik)})\,. \tag{79b}$$

In the following, sums are taken over 1, 2 and 3; for example, $\sum_{i,j,k} \hat{A}^{k(ij)} B_{k(ij)} = 2\hat{A}^{1(23)} B_{1(23)} + 2\hat{A}^{2(31)} B_{2(31)} + 2\hat{A}^{3(12)} B_{3(12)}$.
The exotic BF theory contains two $U(1)$ tensor gauge fields (A_0, A_{ij}) in $(\mathbf{1},\mathbf{3}')$ and $(\hat{A}_0^{i(jk)}, \hat{A}^{ij})$ in $(\mathbf{2},\mathbf{3}')$ as its fields. The gauge transformation of (A_0, A_{ij}) is

[7] A lineon can be detected by a time-like symmetry generated by belt operator, which will be introduced later. However there is no way to place the belt operator so that it detects the quadrupole $T^q[S_2^{012,\text{cube}}]$ without intersecting the belt operator with $S_2^{012,\text{cube}}$. Thus the operator $T^q[S_2^{012,\text{cube}}]$ is trivial.

$$A_0 \to A_0 + \partial_0 \alpha \,, \tag{80a}$$
$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha \,, \tag{80b}$$

where α is a $U(1)$-valued gauge parameter: $\alpha \sim \alpha + 2\pi$, in $\mathbf{1}$. The gauge parameter α has its own gauge transformation: $\alpha \to \alpha + 2\pi\tilde{n}^1 + 2\pi\tilde{n}^2 + 2\pi\tilde{n}^3$, where $\tilde{n}^k$ are x^k-dependent functions valued in integers. A_{ij} can have delta function singularities and have step function discontinuities, while A_0, α can have step function discontinuities as in Table 9, where $\tilde{f}_0^k$, $\tilde{f}_{ij}^k$, $\tilde{g}^k$, $\tilde{h}_0^{i(jk),l}$, $\tilde{h}^{ij,l}$ and $\tilde{s}^{i(jk),l}$ are some continuous functions with appropriate periodicity conditions.[8] The gauge parameter $\tilde{n}^k$ can have step function discontinuities in the x^k direction. The gauge-invariant electric and magnetic fields of (A_0, A_{ij}) are

$$E_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0 \,, \tag{81}$$
$$B_{[ij]k} = \partial_i A_{jk} - \partial_j A_{ik} \quad \text{or} \quad B_{k(ij)} = 2\partial_k A_{ij} - \partial_i A_{kj} - \partial_j A_{ki} \,. \tag{82}$$

The gauge transformation of the other field $(\hat{A}_0^{i(jk)}, \hat{A}^{ij})$ is

$$\hat{A}_0^{i(jk)} \to \hat{A}_0^{i(jk)} + \partial_0 \hat{\alpha}^{i(jk)} \,, \tag{83a}$$
$$\hat{A}^{ij} \to \hat{A}^{ij} + \partial_k \hat{\alpha}^{k(ij)} \,, \tag{83b}$$

where $\hat{\alpha}^{i(jk)}$ is a $U(1)$-valued gauge parameter in $\mathbf{2}$. The gauge parameter $\hat{\alpha}^{i(jk)}$ has its own gauge transformation: $\hat{\alpha}^{i(jk)} \to \hat{\alpha}^{i(jk)} + 2\pi\tilde{m}^j - 2\pi\tilde{m}^k$, where $\tilde{m}^k$ are x^k-dependent functions valued in integers. $\hat{A}_0^{i(jk)}$, $\hat{A}^{jk}$ and $\hat{\alpha}^{i(jk)}$ can have step function discontinuities in the x^j and x^k directions as in Table 9. The gauge parameter $\tilde{m}^k$ can have step function discontinuities in the x^k direction. The gauge-invariant electric and magnetic fields of $(\hat{A}_0^{i(jk)}, \hat{A}^{ij})$ are

$$\hat{E}^{ij} = \partial_0 \hat{A}^{ij} - \partial_k \hat{A}_0^{k(ij)} \,, \tag{84}$$
$$\hat{B} = \frac{1}{2} \sum_{i,j} \partial_i \partial_j \hat{A}^{ij} \,. \tag{85}$$

3.2.2 3+1d Exotic BF Lagrangian

The exotic BF Lagrangian is

$$\mathcal{L}_e = \frac{iN}{2\pi} \left(\frac{1}{2} \sum_{i,j} A_{ij} \hat{E}^{ij} + A_0 \hat{B} \right) . \tag{86}$$

Integrating by parts, and using $\sum_{i,j,k} (\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}) \hat{A}_0^{k(ij)} =$
$= \sum_{i,j,k} \partial_i A_{jk} (\hat{A}_0^{k(ij)} + \hat{A}_0^{i(jk)} + \hat{A}_0^{j(ki)}) = 0$ and (79a), the Lagrangian can be written as

$$\mathcal{L}_e = \frac{iN}{2\pi} \left(\frac{1}{6} \sum_{i,j,k} \hat{A}_0^{k(ij)} B_{k(ij)} + \frac{1}{2} \sum_{i,j} \hat{A}^{ij} E_{ij} \right) . \tag{87}$$

[8] Functions $f(x; \hat{x^k})$ mean x^k-independent functions.

Table 9: Singularities and discontinuities of the tensor gauge fields and their gauge parameters.

Gauge fields and parameters	Gauge transformation	Terms including singularities and discontinuities
A_0	$A_0 \to A_0 + \partial_0\alpha$	$\sum_{k=1}^{3} \tilde{f}_0^k(x;\hat{x^k})\theta(x^k - x_0^k)$
A_{ij}	$A_{ij} \to A_{ij} + \partial_i\partial_j\alpha$	$\tilde{f}_{ij}^i(x;\hat{x^i})\delta(x^i - x_0^i) + \tilde{f}_{ij}^j(x;\hat{x^j})\delta(x^j - x_0^j)$ $+\tilde{f}_{ij}^k(x;\hat{x^k})\theta(x^j - x_0^j) \quad (k \neq 0, i, j)$
α	$\alpha \to \alpha + 2\pi\tilde{n}^1 + 2\pi\tilde{n}^2 + 2\pi\tilde{n}^3$	$\sum_{k=1}^{3} \tilde{g}^k(x;\hat{x^k})\theta(x^k - x_0^k)$
$\hat{A}_0^{i(jk)}$	$\hat{A}_0^{i(jk)} \to \hat{A}_0^{i(jk)} + \partial_0\hat{\alpha}^{i(jk)}$	$\tilde{h}_0^{i(jk),j}(x;\hat{x^j})\theta(x^j - x_0^j)$ $+\tilde{h}_0^{i(jk),k}(x;\hat{x^k})\theta(x^k - x_0^k)$
$\hat{A}^{ij}$	$\hat{A}^{ij} \to \hat{A}^{ij} + \partial_k\hat{\alpha}^{k(ij)}$	$\tilde{h}^{ij,i}(x;\hat{x^i})\theta(x^i - x_0^i)$ $+\tilde{h}^{ij,j}(x;\hat{x^j})\theta(x^j - x_0^j)$
$\hat{\alpha}^{i(jk)}$	$\hat{\alpha}^{i(jk)} \to \hat{\alpha}^{i(jk)} + 2\pi\tilde{m}^j - 2\pi\tilde{m}^k$	$\tilde{s}^{i(jk),j}(x;\hat{x^j})\theta(x^j - x_0^j)$ $+\tilde{s}^{i(jk),k}(x;\hat{x^k})\theta(x^k - x_0^k)$

The equations of motion are

$$\frac{N}{2\pi}E_{ij} = 0\,, \tag{88a}$$
$$\frac{N}{2\pi}B_{k(ij)} = 0\,, \tag{88b}$$
$$\frac{N}{2\pi}\hat{E}^{ij} = 0\,, \tag{88c}$$
$$\frac{N}{2\pi}\hat{B} = 0\,. \tag{88d}$$

Integrating specific fields out, we can show that the following quantities are quantized.

$$\oint_{C_1^0} dx^0 A_0 \in \frac{2\pi}{N}\mathbb{Z}, \tag{89a}$$

$$\oint_{S_2^1} (dx^0 dx^1 \partial_1 A_0 + dx^2 dx^1 A_{12} + dx^3 dx^1 A_{31}) \in \frac{2\pi}{N}\mathbb{Z}, \tag{89b}$$

$$\oint_{S_2^2} (dx^0 dx^2 \partial_2 A_0 + dx^1 dx^2 A_{12} + dx^3 dx^2 A_{23}) \in \frac{2\pi}{N}\mathbb{Z}, \tag{89c}$$

$$\oint_{S_2^3} (dx^0 dx^3 \partial_3 A_0 + dx^1 dx^3 A_{31} + dx^2 dx^3 A_{23}) \in \frac{2\pi}{N}\mathbb{Z}, \tag{89d}$$

$$\oint_{C_1^{0k}} (dx^0 \hat{A}_0^{k(ij)} + dx^k \hat{A}^{ij}) \in \frac{2\pi}{N}\mathbb{Z}, \tag{89e}$$

where C_1^0 is a closed one-dimensional loop along the time x^0 direction, S_2^k is a two-dimensional strip with a fixed width along the x^k direction, and C_1^{01} is a closed one-dimensional loop in the (x^0, x^k)-plane.

3.2.3 Gauge-Invariant Operators

Let us discuss gauge-invariant operators. The defect operator that describes fractons is

$$\tilde{F}^q[C_1^0] = \exp\left[iq \oint_{C_1^0} dx^0 A_0 \right]. \tag{90}$$

As in the case of the foliated BF theory, the deformation of C_1^0 would break the gauge invariance of the operator.

The defect operators that describe lineons are

$$\tilde{L}_k^q[C_1^{0k}] = \exp\left[iq \oint_{C_1^{0k}} (dx^0 \hat{A}_0^{k(ij)} + dx^k \hat{A}^{ij}) \right], \quad k = 1,2,3. \tag{91}$$

If C_1^{0k} are along the x^k direction, $\tilde{L}_k^q$ are the symmetry operators that generate $\mathbb{Z}_N$ tensor global symmetries.

The strip operators that describe a dipole of fractons separated along the x^k directions are

$$\tilde{W}_1^q[S_2^1] = \exp\left[iq \oint_{S_2^1} (dx^0 dx^1 \partial_1 A_0 + dx^2 dx^1 A_{12} + dx^3 dx^1 A_{31}) \right], \tag{92a}$$

$$\tilde{W}_2^q[S_2^2] = \exp\left[iq \oint_{S_2^2} (dx^0 dx^2 \partial_2 A_0 + dx^1 dx^2 A_{12} + dx^3 dx^2 A_{23}) \right], \tag{92b}$$

$$\tilde{W}_3^q[S_2^3] = \exp\left[iq \oint_{S_2^3} (dx^0 dx^3 \partial_3 A_0 + dx^1 dx^3 A_{31} + dx^2 dx^3 A_{23}) \right]. \tag{92c}$$

If S_2^k are in the (x^1, x^2, x^3)-plane, $\tilde{W}_k^q$ are the symmetry operators that generate $\mathbb{Z}_N$ dipole global symmetries.

The two types of symmetry operators satisfy the following relations at equal time:

$$\tilde{L}_2^p[C_1^2]\, \tilde{W}_1^q[S_2^1] = \mathrm{e}^{2\pi i pq I(C_1^2, S_2^1)/N}\, \tilde{W}_1^q[S_2^1]\, \tilde{L}_2^p[C_1^2], \quad \text{if} \quad x_1^1 < x^1 < x_2^1, \tag{93a}$$

$$\tilde{L}_3^p[C_1^3]\, \tilde{W}_1^q[S_2^1] = \mathrm{e}^{2\pi i pq I(C_1^3, S_2^1)/N}\, \tilde{W}_1^q[S_2^1]\, \tilde{L}_3^p[C_1^3], \quad \text{if} \quad x_1^1 < x^1 < x_2^1, \tag{93b}$$

where $S_2^1 = [x_2^1, x_1^1] \times C_1^{23}$, and I is the intersection number. Similar relations holds in the other directions. These symmetries in the exotic BF theory have the same structure as the foliated BF theory discussed in Section 3.1.

In addition, we can consider the strip operator that describes a dipole of fractons at (x_1^1, x_1^2, x^3) and (x_2^1, x_2^2, x^3), which can move in the x^3 direction, like a x^3-lineon:

$$\tilde{K}_{12}^q[C_1^{03}\times \mathcal{C}_1^{12}] = \exp\left[iq\oint_{C_1^{03}\times \mathcal{C}_1^{12}}(dx^0dx^1\partial_1 A_0 + dx^0dx^2\partial_2 A_0 + dx^3dx^1 A_{31} + dx^3dx^2 A_{23})\right], \quad (94)$$

where $\mathcal{C}_1^{12}$ is a one-dimensional line connecting (x_1^1, x_1^2) to (x_2^1, x_2^2) in the (x^1, x^2)-plane. Using the Stokes' theorem, the equations of motion (88a), and $\frac{N}{2\pi}B_{[ij]k} = 0$ from (88b), $\mathcal{C}_1^{12}$ can be deformed to $[x_1^1, x_2^1]\times\{x_1^2\} + \{x_2^1\}\times[x_1^2, x_2^2]$, and in this special case we write $\tilde{K}_{12}^q$ as

$$\begin{aligned}\tilde{K}_{12}^q[C_1^{03}\times \mathcal{C}_1^{12}] =& \exp\left[iq\oint_{C_1^{03}\times[x_1^1,x_2^1]\times\{x_1^2\}}(dx^0dx^1\partial_1 A_0 + dx^3dx^1 A_{31})\right]\\ &\times\exp\left[iq\oint_{C_1^{03}\times\{x_2^1\}\times[x_1^2,x_2^2]}(dx^0dx^2\partial_2 A_0 + dx^3dx^2 A_{23})\right]\\ =&\tilde{W}_1^q\left[C_1^{03}\times[x_1^1,x_2^1]\times\{x_1^2\}\right]\tilde{W}_2^q\left[C_1^{03}\times\{x_2^1\}\times[x_1^2,x_2^2]\right].\end{aligned} \quad (95)$$

Similarly, we have the strip operators $\tilde{K}_{23}^q$ and $\tilde{K}_{31}^q$.

Also, we can consider the strip operators that describe a dipole of x^1-lineons and a dipole of x^2-lineons, separated along the x^3 direction, which can move in the other directions in space, like a planon:

$$\tilde{P}_{3,1}^q[S_2^3] = \exp\left[iq\oint_{S_2^3}\left(dx^0dx^3\partial_3\hat{A}_0^{1(23)} + dx^1dx^3\partial_3\hat{A}^{23} - dx^2dx^3(\partial_3\hat{A}^{31} + \partial_2\hat{A}^{12})\right)\right], \quad (96)$$

$$\tilde{P}_{3,2}^q[S_2^3] = \exp\left[iq\oint_{S_2^3}\left(dx^0dx^3\partial_3\hat{A}_0^{2(31)} + dx^2dx^3\partial_3\hat{A}^{31} - dx^1dx^3(\partial_3\hat{A}^{23} + \partial_1\hat{A}^{12})\right)\right]. \quad (97)$$

Note that

$$\begin{aligned}\tilde{P}_{3,1}^q &= \{\tilde{P}_{3,2}^q\}^{-1}\exp\left[-iq\oint_{S_2^3}(dx^0dx^3\partial_3\hat{A}_0^{3(12)} + dx^2dx^3\partial_2\hat{A}^{12} + dx^1dx^3\partial_1\hat{A}^{12})\right]\\ &= \{\tilde{P}_{3,2}^q\}^{-1}\{\tilde{P}_{3,3}^q\}^{-1},\end{aligned} \quad (98)$$

where $\tilde{P}_{3,3}^q$ represents a dipole of x^3-lineons separated along the x^3 direction, which is trivial. Using the Stokes' theorem and the equations of motion (88c) and (88d), S_2^3 can be deformed to $C_1^{01}\times\{x^2\}\times[x_1^3, x_2^3]$, and in this special case we write $\tilde{P}_{3,1}^q[S_2^3]$ as

$$\begin{aligned}\tilde{P}_{3,1}^q[S_2^3] =& \exp\left[iq\oint_{C_1^{01}\times\{x^2\}\times[x_1^3,x_2^3]}(dx^0dx^3\partial_3\hat{A}_0^{1(23)} + dx^1dx^3\partial_3\hat{A}^{23})\right]\\ =& \exp\left[iq\oint_{C_1^{01}(x^2,x_2^3)}(dx^0\hat{A}_0^{1(23)} + dx^1\hat{A}^{23})\right]\\ &\times\exp\left[-iq\oint_{C_1^{01}(x^2,x_1^3)}(dx^0\hat{A}_0^{1(23)} + dx^1\hat{A}^{23})\right]\\ =&\tilde{L}_1^q\left[C_1^{01}(x^2,x_2^3)\right]\left\{\tilde{L}_1^q\left[C_1^{01}(x^2,x_1^3)\right]\right\}^{-1},\end{aligned} \quad (99)$$

where $C_1^{01}(x^2,x^3)$ is a closed one-dimensional loop in the (x^0,x^1)-plane at (x^2,x^3). Similarly, we have the strip operators $\tilde{P}^q_{k,i}$ for $(k,i)=(1,2),(1,3),(2,1),(2,3)$.

As in the case of $2+1$ dimensions, there is a gauge-invariant operator that can detect the fracton operator:

$$\begin{aligned}\tilde{T}^q\left[S_2^{123,\mathrm{cube}}\right] = \exp&\left[-iq\int_{x_1^1}^{x_2^1} dx^1\left\{\Delta_{23}\hat{A}^{23}(x_1^2,x_2^2,x_1^3,x_2^3)\right\}\right]\\ &\times\exp\left[-iq\int_{x_1^2}^{x_2^2} dx^2\left\{\Delta_{31}\hat{A}^{31}(x_1^3,x_2^3,x_1^1,x_2^1)\right\}\right]\\ &\times\exp\left[-iq\int_{x_1^3}^{x_2^3} dx^3\left\{\Delta_{12}\hat{A}^{12}(x_1^1,x_2^1,x_1^2,x_2^2)\right\}\right].\end{aligned}\tag{100}$$

This cage operator is localized on the edges of the rectangular cuboid whose surface is $S_2^{123,\mathrm{cube}}$. Without the defect operator $\tilde{F}^q$, $\tilde{T}^q$ becomes trivial, which corresponds to a time-like symmetry [36]. The operator $\tilde{T}^p$ can detect the fracton operator $\tilde{F}^q$:

$$\tilde{T}^p\left[S_2^{123,\mathrm{cube}}\right]\cdot\tilde{F}^q[C_1^0] = \mathrm{e}^{-2\pi ipq/N}\tilde{F}^q[C_1^0],\tag{101}$$

when $S_2^{123,\mathrm{cube}}$ surrounds C_1^0.

This theory has another time-like symmetry whose operator can detect the lineons: the belt operator

$$\begin{aligned}\tilde{U}^q_{[12]3}\left[S_2^{3,\mathrm{belt}}\right] = \exp&\left[iq\int_{x_1^2}^{x_2^2}\int_{x_1^3}^{x_2^3} dx^2dx^3\left(A_{23}(x_2^1)-A_{23}(x_1^1)\right)\right]\\ &\times\exp\left[-iq\int_{x_1^1}^{x_2^1}\int_{x_1^3}^{x_2^3} dx^1dx^3\left(A_{31}(x_2^2)-A_{31}(x_1^2)\right)\right],\end{aligned}\tag{102}$$

where $S_2^{3,\mathrm{belt}}$ is $C_1^{12,\mathrm{rect}}\times[x_1^3,x_2^3]$. Similarly, we also have $\tilde{U}^q_{[31]2}\left[S_2^{2,\mathrm{belt}}\right]$ and $\tilde{U}^q_{[23]1}\left[S_2^{1,\mathrm{belt}}\right]$. They act on the lineon operator as

$$\tilde{U}^p_{[12]3}\left[S_2^{3,\mathrm{belt}}\right]\cdot\tilde{L}_3^q[C_1^0] = \tilde{L}_3^q[C_1^0],\tag{103a}$$

$$\tilde{U}^p_{[31]2}\left[S_2^{2,\mathrm{belt}}\right]\cdot\tilde{L}_3^q[C_1^0] = \mathrm{e}^{2\pi ipq/N}\tilde{L}_3^q[C_1^0],\tag{103b}$$

$$\tilde{U}^p_{[23]1}\left[S_2^{1,\mathrm{belt}}\right]\cdot\tilde{L}_3^q[C_1^0] = \mathrm{e}^{-2\pi ipq/N}\tilde{L}_3^q[C_1^0],\tag{103c}$$

where $S_2^{123,\mathrm{cube}}$ that is the union of $S_2^{k,\mathrm{belt}}$ surrounds C_1^0. Similar relations also hold for $\tilde{L}_1^q$ and $\tilde{L}_2^q$.

These gauge-invariant operators are $\mathbb{Z}_N$ operators: q is an element of $\mathbb{Z}_N$.

3.3 Correspondences in 3+1 Dimensions

As in the case of the $2+1$d version, the $3+1$d foliated BF theory explained in Section 3.1 and the $3+1$d exotic BF theory explained in Section 3.2 are equivalent in case that $e^k=dx^k$, $M_k=N$ and $n_k=1$ $(k=1,2,3)$. We identify the gauge-invariant operators in the foliated BF theory with those of the exotic BF theory. By matching the gauge-invariant operators, we can derive the correspondences of the gauge fields and parameters.

First, let us consider the fracton defect operators. We identify the operators $F^q[C_1^0]$ with $\tilde{F}^q[C_1^0]$ defined in (67) and (90):

$$\exp\left[iq\oint_{C_1^0} a\right] \simeq \exp\left[iq\oint_{C_1^0} dx^0 A_0\right], \tag{104}$$

which leads to the field correspondence

$$a_0 \simeq A_0\,. \tag{105}$$

The gauge transformations of a_0 and A_0 explained in (65c) and (80a) are

$$a_0 \to a_0 + \partial_0\lambda\,, \tag{106a}$$
$$A_0 \to A_0 + \partial_0\alpha\,, \tag{106b}$$

from which we obtain the gauge parameter correspondence

$$\lambda \simeq \alpha\,. \tag{107}$$

Moreover, the gauge transformations of λ and α are

$$\lambda \to \lambda + 2\pi\xi^1 + 2\pi\xi^2 + 2\pi\xi^3\,, \tag{108a}$$
$$\alpha \to \alpha + 2\pi\tilde{n}^1 + 2\pi\tilde{n}^2 + 2\pi\tilde{n}^3\,, \tag{108b}$$

which can be matched by

$$\xi^k \simeq \tilde{n}^k\,. \tag{109}$$

In these correspondences, one can check that their singularities and discontinuities are also matched.

The equations of motion (64d) in components are

$$\frac{N}{2\pi}(A_0^k + \partial_0 a_k - \partial_k a_0) = 0\,, \quad k = 1,2,3\,, \tag{110a}$$
$$\frac{N}{2\pi}(A_i^k - A_k^i + \partial_i a_k - \partial_k a_i) = 0\,, \quad (k,i) = (1,2),(2,3),(3,1)\,. \tag{110b}$$

These equations of motion imply

$$A_0^k + \partial_0 a_k \simeq \partial_k A_0\,, \quad k = 1,2,3\,. \tag{111}$$

Note that the gauge transformations by ζ^k cancel out.

Next, let us consider the strip operators. As in the case of $2+1$ dimensions, we define the modified gauge-invariant strip operators $W^q_{k,\mathrm{mod}}[S_2^k]$ as

$$W^q_{k,\mathrm{mod}}[S_2^k] = \exp\left[iq\oint_{S_2^k}\left(A^k \wedge dx^k + d(a_k dx^k)\right)\right], \quad k = 1,2,3\,. \tag{112}$$

We identify the operators $W^q_{k,\mathrm{mod}}[S_2^k]$ with $\tilde{W}^q_k[S_2^k]$ defined in (112), (92a), (92b) and (92c):

$$\begin{gathered}\exp\left[iq\oint_{S_2^k}\left(A^k \wedge dx^k + d(a_k dx^k)\right)\right] \simeq \exp\left[iq\oint_{S_2^k}(dx^0 dx^k \partial_k A_0 + dx^i dx^k A_{ik} + dx^j dx^k A_{jk})\right], \\ (k,i,j) = (1,2,3),(2,3,1),(3,1,2)\,,\end{gathered} \tag{113}$$

which lead to the field correspondences

$$A_i^k + \partial_i a_k \simeq A_{ki}\,, \quad k \neq i\,, \quad k,i \in \{1,2,3\}\,, \tag{114}$$

and also (111) again. The terms $\partial_i a_k$ make the gauge transformations match with those of A_{ki} under the gauge parameter correspondence (107). Using the correspondences (111) and (114), we find the correspondence of gauge-invariant operators between K_{12}^q and $\tilde{K}_{12}^q$ defined in (72) and (94). To be precise, as we did for W_k^q, the operator K_{12}^q in the foliated side has to be modified as

$$K_{12,\mathrm{mod}}^q[C_1^{03} \times C_1^{12}] = \exp\left[iq \oint_{C_1^{03}\times C_1^{12}} \left(A^1 \wedge dx^1 + A^2 \wedge dx^2 + d(a_1 dx^1 + a_2 dx^2) \right) \right]. \tag{115}$$

Lastly, let us consider the lineon operators. We identify $L_k^q[C_{0k}^k]$ with $\tilde{L}_k^q[C_1^k 0k]$ defined in (69a)-(69c) and (91):

$$\exp\left[iq \oint_{C_1^{0k}} (B^i - B^j) \right] \simeq \exp\left[iq \oint_{C_1^{0k}} (dx^0 \hat{A}_0^{k(ij)} + dx^k \hat{A}^{ij}) \right]. \tag{116}$$

Then we can derive the field correspondences

$$B_0^i - B_0^j \simeq \hat{A}_0^{k(ij)}\,, \quad (i,j,k) = (1,2,3),(2,3,1),(3,1,2)\,, \tag{117a}$$

$$B_k^i - B_k^j \simeq \hat{A}^{ij}\,, \quad (i,j,k) = (1,2,3),(2,3,1),(3,1,2)\,. \tag{117b}$$

The gauge transformations by μ cancel out in the left-hand sides and the gauge transformations by β^k do not appear. Note that $B_0^2 - B_0^1 \simeq -\hat{A}_0^{3(21)}$, $B_3^2 - B_3^1 \simeq -\hat{A}^{21}$ and so on. These correspondences are consistent with the conditions $\hat{A}_0^{1(23)} + \hat{A}_0^{2(31)} + \hat{A}_0^{3(12)} = 0$. From (65b), (83a) and (83b), the gauge transformations are

$$B_0^i - B_0^j \to B_0^i - B_0^j + \partial_0(\lambda^i - \lambda^j)\,, \tag{118a}$$

$$B_k^i - B_k^j \to B_k^i - B_k^j + \partial_k(\lambda^i - \lambda^j)\,, \tag{118b}$$

$$\hat{A}_0^{k(ij)} \to \hat{A}_0^{k(ij)} + \partial_0 \hat{\alpha}^{k(ij)}\,, \tag{118c}$$

$$\hat{A}^{ij} \to \hat{A}^{ij} + \partial_k \hat{\alpha}^{k(ij)}\,, \tag{118d}$$

where $(i,j,k) = (1,2,3),(2,3,1),(3,1,2)$. Then we obtain the gauge parameter correspondences

$$\lambda^i - \lambda^j \simeq \hat{\alpha}^{k(ij)}\,, \quad (i,j,k) = (1,2,3),(2,3,1),(3,1,2)\,. \tag{119}$$

Moreover, the gauge transformations of λ^k and $\hat{\alpha}^{k(ij)}$ are

$$\lambda^k \to \lambda^k + 2\pi m^k + \nu\,, \tag{120a}$$

$$\hat{\alpha}^{k(ij)} \to \hat{\alpha}^{k(ij)} + 2\pi \tilde{m}^i - 2\pi \tilde{m}^j\,, \tag{120b}$$

which can be matched by

$$m^k \simeq \tilde{m}^k\,, \quad k = 1,2,3\,. \tag{121a}$$

Again in these correspondences, their singularities and discontinuities are matched. Using the correspondences (117a) and (117b), we find the gauge-invariant operator corresponding to $\tilde{P}_{3,1}^q$ defined in (96):

$$\begin{aligned} P_{3,1}^q[S_2^3] = \exp\Bigg[iq \oint_{S_2^3} \Big\{ & dx^0 dx^3 \partial_3 (B_0^2 - B_0^3) + dx^1 dx^3 \partial_3 (B_1^2 - B_1^3) \\ & - dx^2 dx^3 \left(\partial_3 (B_2^3 - B_2^1) + \partial_2 (B_3^1 - B_3^2) \right) \Big\} \Bigg]. \end{aligned} \tag{122}$$

Similarly, we also have $P^q_{k,i}$ for $(k,i) = (3,2), (1,2), (1,3), (2,1), (2,3)$.

Under the correspondence (117b), the cage time-like symmetry operator $T^q[S_2^{123,\text{cube}}]$ defined (76) corresponds to $\tilde{T}^q[S_2^{123,\text{cube}}]$ defined in (100). Note that on a Hilbert space with fracton defect operators, the b operator $T^q[S_2]$ is a product of $T^q[S_2^{123,\text{cube}}]$ surrounding the defects that are surrounded by S_2. In addition, using the correspondences (114), we can find the belt time-like symmetry operator in the foliated side corresponding to $\tilde{U}^q_{[12]3}\left[S_2^{3,\text{belt}}\right]$ defined in (102):

$$\begin{aligned} U^q_{[12]3,\text{mod}}\left[S_2^{3,\text{belt}}\right] = \exp&\left[iq\int_{x_1^2}^{x_2^2}\int_{x_1^3}^{x_2^3}\left(A_2^3(x_2^1)+\partial_2 a_3(x_2^1)-A_2^3(x_1^1)-\partial_2 a_3(x_1^1)\right)dx^2\wedge dx^3\right] \\ \times\exp&\left[-iq\int_{x_1^1}^{x_2^1}\int_{x_1^3}^{x_2^3}\left(A_1^3(x_2^2)+\partial_1 a_3(x_2^2)-A_1^3(x_1^2)-\partial_1 a_3(x_1^2)\right)dx^1\wedge dx^3\right], \end{aligned} \tag{123}$$

or non-modified one

$$\begin{aligned} U^q_{[12]3}\left[S_2^{3,\text{belt}}\right] = \exp&\left[iq\int_{x_1^2}^{x_2^2}\int_{x_1^3}^{x_2^3}\left(A_2^3(x_2^1)-A_2^3(x_1^1)\right)dx^2\wedge dx^3\right] \\ \times\exp&\left[-iq\int_{x_1^1}^{x_2^1}\int_{x_1^3}^{x_2^3}\left(A_1^3(x_2^2)-A_1^3(x_1^2)\right)dx^1\wedge dx^3\right]. \end{aligned} \tag{124}$$

Similarly, we can find the other belt operators $U^q_{[31]2}\left[S_2^{2,\text{belt}}\right]$ and $U^q_{[23]1}\left[S_2^{1,\text{belt}}\right]$.

There are other gauge-invariant operators, e.g. the one describes the creation of a quadrupole of fractons [14, 15] in the exotic side, and one can easily map them to the foliated side using the correspondences of the gauge fields.

4 Conclusion

In this paper, we have discussed the duality between the foliated BF theory and the exotic BF theory in 2+1 dimensions and 3+1 dimensions, and derived the explicit correspondences of the gauge fields and parameters by matching the gauge-invariant operators. The correspondences include the bulk field in the FQFT, and the singularities and discontinuities of the fields and the parameters are also consistent. The duality includes the correspondences of the time-like symmetries [36] in both sides.

One of the future directions is to consider the mixed 't Hooft anomalies [41] of the subsystem symmetries. The 't Hooft anomalies and the anomaly inflow [42] of the subsystem symmetries are studied [18,25], and the corresponding symmetry protected topological (SPT) phases are called subsystem SPT (SSPT) phases [18,43,44]. The foliated and exotic BF theory in this paper also have the mixed anomalies, and it would be interesting to consider how the duality incorporate the SSPT phases.

Another direction is to consider relations between gapless foliated and exotic QFTs such as the foliated scalar [19,22] and the ϕ theory [13,16]. Although in gapless theories the foliated and exotic theories will not represent the same physics, as pointed out in [22], they might be connected by a renormalization group flow.

In this work, we have considered the foliated BF theory in the flat foliations. It is not understood so far what exotic tensor gauge theory corresponds to the general foliated BF theory (8). If there are foliated-exotic dualities for more general classes of foliated theories, e.g. the ones studied in [20–22], it would provide a more general construction of exotic QFTs.

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A Electric-Magnetic Dual Description in 2+1 Dimensions

The ordinary BF theory in 1+1 dimensions is dual to the $\mathbb{Z}_N$ gauge theory realized as Higgsing of a charge-N scalar field coupled to a $U(1)$ gauge field [37–40], which is the electric-magnetic duality. Similarly the foliated BF theory described in Section 2.1 and the exotic BF theory described Section 2.2 can be written as such a dual description. These electric-magnetic dual theories are directly foliated-exotic dual to each other.

A.1 Foliated Gauge Theory

In the foliated BF theory, the term of a stack of $1+1$d BF theories can be written as the form of a $\mathbb{Z}_N$ gauge theory realized as Higgsing. The Lagrangian is

$$\mathcal{L}'_f = \sum_{k=1}^{n_f}\left[-\frac{i}{2\pi}U^k \wedge (d\Phi^k - NA^k)\wedge dx^k + \frac{iN}{2\pi}b\wedge A^k \wedge dx^k\right] + \frac{iN}{2\pi}b\wedge da\,, \tag{125}$$

where Φ^k is a compact scalar field and U^k is a one-form field, which is a Lagrangian multiplier. Φ^k can have delta function singularities in the x^k direction, and U^k can have zero-form step function discontinuities and one-form delta function singularities in the x^k direction. The gauge transformation is

$$\Phi^k \to \Phi^k + N\zeta^k + 2\pi\partial_k t^k\,, \tag{126a}$$

where t^k is a x^k-dependent function valued in integers that can have step function discontinuities in the x^k direction, and ζ^k is defined in Section 2.1.1 and Section 2.1.2. The equation of motion derived by integrating the Lagrangian multiplier out is

$$(d\Phi^k - NA^k)\wedge dx^k = 0\,. \tag{127}$$

Then the strip operators (19) can be written as

$$W_k^q[S_2^k] = \exp\left[i\frac{q}{N}\oint_{S_2^k} d\Phi^k \wedge dx^k\right], \quad k = 1,2\,. \tag{128}$$

We can dualize this theory to the foliated BF theory. By integrating Φ^k out, we can derive the equation of motion

$$\frac{N}{2\pi} dU^k \wedge dx^k = 0 \,. \tag{129}$$

Solving this equation locally, we can derive $U^k \wedge dx^k = dB^k \wedge dx^k$, where B^k is the foliated B-type zero-form field defined in Section 2.1.1. Then the Lagrangian becomes

$$\mathcal{L}'_f \to \sum_{k=1}^{n_f} \left[\frac{iN}{2\pi} dB^k \wedge A^k \wedge dx^k + \frac{iN}{2\pi} b \wedge A^k \wedge dx^k \right] + \frac{iN}{2\pi} b \wedge da \,, \tag{130}$$

which is equal to the foliated BF Lagrangian (9).

A.2 Exotic Gauge Theory

The exotic BF theory also has the form of the $\mathbb{Z}_N$ tensor gauge theory realized as Higgsing of a charge-N scalar field coupled to a $U(1)$ tensor gauge field [13]. This $\mathbb{Z}_N$ tensor theory is directly dual to the foliated $\mathbb{Z}_N$ gauge theory (125). The Lagrangian is

$$\mathcal{L}'_e = \frac{i}{2\pi} \hat{E}^{12} (\partial_1 \partial_2 \phi - N A_{12}) + \frac{i}{2\pi} \hat{B} (\partial_0 \phi - N A_0) \,, \tag{131}$$

where $(\hat{E}^{12}, \hat{B})$ in the representation $(\mathbf{1}_2, \mathbf{1}_0)$ is a Lagrangian multiplier and ϕ in the representation $\mathbf{1}_0$ is a compact scalar. The field ϕ can have configurations that are the sum of the terms proportional to step functions in the x^1 and x^2 directions. The gauge transformation is

$$\phi \to \phi + N\alpha + 2\pi \tilde{t}^1 + 2\pi \tilde{t}^2 \,, \tag{132}$$

where $\tilde{t}^k$ are x^k-dependent functions valued in integers that can have step function discontinuities in the x^k direction, and α is defined in Section 2.2.1.

The equations of motion derived by integrating the Lagrangian multiplier out are

$$\partial_1 \partial_2 \phi - N A_{12} = 0 \,, \tag{133a}$$
$$\partial_0 \phi - N A_0 = 0 \,. \tag{133b}$$

Then the strip operators (36a) and (36b) can be written as

$$\tilde{W}_1^q [S_2^1] = \exp\left[i \frac{q}{N} \oint_{S_2^1} (dx^0 dx^1 \partial_1 \partial_0 \phi + dx^2 dx^1 \partial_1 \partial_2 \phi) \right] , \tag{134a}$$
$$\tilde{W}_2^q [S_2^2] = \exp\left[i \frac{q}{N} \oint_{S_2^2} (dx^0 dx^2 \partial_2 \partial_0 \phi + dx^1 dx^2 \partial_1 \partial_2 \phi) \right] . \tag{134b}$$

We can dualize this theory to the exotic BF theory. Integrating ϕ out, we can derive the equation of motion

$$\partial_1 \partial_2 \hat{E}^{12} - \partial_0 \hat{B} = 0 \,. \tag{135}$$

Solving this equation locally, we can derive $\hat{E}^{12} = \partial_0 \phi^{12}$ and $\hat{B} = \partial_1 \partial_2 \phi^{12}$. Then the Lagrangian becomes

$$\mathcal{L}'_e \to \frac{iN}{2\pi} \phi^{12} \partial_0 A_{12} - \frac{iN}{2\pi} \phi^{12} \partial_1 \partial_2 A_0 \,, \tag{136}$$

which is equal to the exotic BF Lagrangian (28).

A.3 Correspondences

In the dual theories, from the strip operators (128), (134a), (134b), and the modification of $W_k^q[S_2^k]$ in (50), we identify

$$\exp\left[i\frac{q}{N}\oint_{S_2^1}\left(d\Phi^1\wedge dx^1+Nd(a_1dx^1)\right)\right]\simeq\exp\left[i\frac{q}{N}\oint_{S_2^1}(dx^0dx^1\partial_1\partial_0\phi+dx^2dx^1\partial_1\partial_2\phi)\right],\quad(137a)$$

$$\exp\left[i\frac{q}{N}\oint_{S_2^2}\left(d\Phi^2\wedge dx^2+Nd(a_2dx^2)\right)\right]\simeq\exp\left[i\frac{q}{N}\oint_{S_2^2}(dx^0dx^2\partial_2\partial_0\phi+dx^1dx^2\partial_1\partial_2\phi)\right],\quad(137b)$$

which lead to the correspondences of the scalar fields

$$\Phi^1+Na_1\simeq\partial_1\phi\,,\quad(138a)$$

$$\Phi^2+Na_2\simeq\partial_2\phi\,.\quad(138b)$$

The gauge transformations of Φ^k, a and ϕ are

$$\Phi^k\to\Phi^k+N\zeta^k+2\pi\partial_k t^k\,,\quad(139a)$$

$$a_k\to a_k+\partial_k\lambda-\zeta^k\,,\quad(139b)$$

$$\phi\to\phi+N\alpha+2\pi\tilde{t}^1+2\pi\tilde{t}^2\,.\quad(139c)$$

The gauge transformation by ζ^k cancel out in the left-hand sides. Then we derive the gauge parameter correspondences

$$\lambda\simeq\alpha\,,\quad(140a)$$

$$t^k\simeq\tilde{t}^k\,.\quad(140b)$$

Again the discontinuities are matched.

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