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Mirror symmetry for five-parameter Hulek-Verrill manifolds

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Abstract

We study the mirrors of five-parameter Calabi-Yau threefolds first studied by Hulek and Verrill in the context of observed modular behaviour of the zeta functions for Calabi-Yau manifolds. Toric geometry allows for a simple explicit construction of these mirrors, which turn out to be familiar manifolds. These are elliptically fibred in multiple ways. By studying the singular fibres, we are able to identify the rational curves of low degree on the mirror manifolds. This verifies the mirror symmetry prediction obtained by studying the mirror map near large complex structure points. We undertake also an extensive study of the periods of the Hulek-Verrill manifolds and their monodromies. On the mirror, we compute the genus-zero and -one instanton numbers, which are labelled by 5 indices, as $h^{1,1} = 5$. There is an obvious permutation symmetry on these indices, but in addition there is a surprising repetition of values. We trace this back to an S_6 symmetry made manifest by certain constructions of the complex structure moduli space of the Hulek-Verrill manifold. Among other consequences, we see in this way that the moduli space has six large complex structure limits. It is the freedom to expand the prepotential about any one of these points that leads to this symmetry in the instanton numbers. An intriguing fact is that the group that acts on the instanton numbers is larger than S_6 and is in fact an infinite hyperbolic Coxeter group, that we study. The group orbits have a 'web' structure, and with certain qualifications the instanton numbers are only nonzero if they belong to what we term 'positive webs'. This structure has consequences for instanton numbers at all genera.

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1 Introduction

1.1 Preamble

In this paper, we study mirror symmetry for a family of Calabi-Yau manifolds associated to the root lattice A_4, a family first investigated in relation to the modularity of its zeta-function by Hulek and Verrill [1]. Apart from the modular properties, these manifolds are of considerable interest due to their high degree of symmetry: the five parameter complex structure moduli space of these manifolds has an S_6 symmetry, which leads to an infinite group of symmetries among the instanton numbers, realised in terms of a Coxeter group. The symmetries also allow many simplifications which make computations that are usually too computationally expensive for multiparameter manifolds accessible.

Singular Hulek-Verrill varieties comprise a five-parameter family, parametrised by

$$\phi \;=\; \left(\varphi^0, \varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^5\right),$$

and denoted[1] $\widehat{\text{HV}}_{\phi}$. They are embedded in the projective torus $\mathbb{T}^4 = \mathbb{P}^4 \setminus \{X_1 \cdots X_5 = 0\}$ as the vanishing loci of

$$\left(X_1 + X_2 + X_3 + X_4 + X_5\right)\left(\frac{\varphi^1}{X_1} + \frac{\varphi^2}{X_2} + \frac{\varphi^3}{X_3} + \frac{\varphi^4}{X_4} + \frac{\varphi^5}{X_5}\right) \;=\; \varphi^0\,. \tag{1.1}$$

These varieties admit toric compactifications, which we will review briefly in §2. Of particular interest are small projective resolutions HV of $\widehat{\text{HV}}$, which have smooth projective Calabi-Yau models [1]. We concentrate mostly on analysing these, and call them simply *Hulek-Verrill manifolds*.

A reformulation of (1.1) turns out to be very useful, whereby these manifolds are realised

[1] We often do not display the parameter ϕ explicitly.

Table 1: Index conventions in each section.

Section	Index Convention
§2	Varies by subsection.
§3	Greek indices run from 0 to 5. Latin indices run from 1 to 5.
§4	Greek indices run from 0 to 5. Latin indices run from 1 to 5.
§7	Latin indices run from 0 to 4. Distinct indices are understood to take distinct values.

as a toric compactification of the locus $P_1 = P_2 = 0$, the intersection of two polynomials

$$P^1(X) = \sum_{\mu=0}^{5} X_\mu\,, \qquad P^2(X;\phi) = \sum_{\mu=0}^{5} \frac{\varphi^\mu}{X_\mu} \tag{1.2}$$

on a torus $\mathbb{T}^5$. This is seen by eliminating X_0, which immediately returns us to (1.1). We denote these manifolds by $\mathrm{HV}_{(\varphi^0,\ldots,\varphi^5)}$, or more compactly by HV.

These manifolds have received attention in the physics literature, since the periods of these manifolds (and their analogues in each dimension) are related to the banana Feynman graphs [2]. The particular manifolds $\mathrm{HV}_{(1/\varphi,1,1,1,1,1)}$ and their quotients exhibit, for certain values of φ, rank-two attractor points with interesting number theoretic properties. These attractor varieties were identified in [3].

The mirror-symmetric counterpart to the work of [3] involves a IIA setup. In [4], nonperturbative solutions were given to the attractor equations which involved instanton numbers, or Gromov-Witten invariants, giving a hint of microstate counting. This motivates us to study the geometry of the mirror Hulek-Verrill manifold focusing especially on aspects related to counting microstates of D4-D2-D0 brane systems on the manifold.

In studying the periods of HV we are naturally led to consider integrals of products of Bessel functions, similar to those considered in [5, 6]. We find additional motivation for the present work in the connection between the manifolds HV and this topic.

While this paper was in preparation we received [2], which has overlap with the present work.

1.2 Outline of the paper

The analysis of the Hulek-Verrill manifolds presented in this paper occasionally becomes somewhat involved. This being so, we give below a brief overview of the contents and main results of each section. Where possible, we strive to keep different sections relatively independent.

A comment on indices

We adopt specific index conventions in various sections of the paper. While these conventions are strictly followed in their respective sections, they are not consistently applied throughout the paper. These are set out in Table 1.

Toric geometry of mirror Hulek-Verrill manifolds

In §2, we briefly review the toric construction of the singular Hulek-Verrill manifolds $\widehat{\text{HV}}$ as first discussed in [1]. Then we proceed to find a toric description of its small resolution. We use the method of Batyrev and Borisov [7,8] to find the toric description of the mirror Hulek-Verrill manifolds HΛ. Somewhat surprisingly, these mirror manifolds turn out to be familiar spaces [9,10], given by the complete intersection

$$\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}. \tag{1.3}$$

Parenthetically, we note that this manifold is itself a remarkable split [11] of the tetraquadric,

$$\begin{matrix}\mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\end{matrix}\begin{bmatrix}2\\ 2\\ 2\\ 2\end{bmatrix}. \tag{1.4}$$

Subfamilies exist that admit a $\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry, or a subgroup thereof. The symmetry has a simple description: denoting the coordinates in each of these projective spaces by $Y_{i,0}$ and $Y_{i,1}$, the symmetries act for all i as

$$S\,:\,Y_{i,a}\mapsto Y_{i+1,a}\,,\qquad U\,:\,Y_{i,a}\mapsto(-1)^a Y_{i,a}\,,\qquad V\,:\,Y_{i,0}\longleftrightarrow Y_{i,1}\,. \tag{1.5}$$

We write the most general expressions for the polynomials defining manifolds invariant under these symmetries. In particular, the manifold invariant under $\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$ is given as the simultaneous vanishing locus of

$$\begin{aligned} Q^1 &= \frac{A_0}{5}\,m_{00000}+A_1\,m_{11000}+A_2\,m_{10100}+A_3\,m_{11110}\,,\\ Q^2 &= \frac{A_0}{5}\,m_{11111}+A_1\,m_{11100}+A_2\,m_{11010}+A_3\,m_{10010}\,, \end{aligned} \tag{1.6}$$

where m_{abcde} are $\mathbb{Z}_5$ invariant multidegree $(1,1,1,1,1)$ polynomials:

$$m_{abcde} \;=\; \sum_{i\in\mathbb{Z}_5} Y_{i,a}Y_{i+1,b}Y_{i+2,c}Y_{i+3,d}Y_{i+4,e}\,. \tag{1.7}$$

It will turn out to be occasionally useful to consider the singular mirror Hulek-Verrill manifolds $\widehat{\text{H}\Lambda}$, which can be obtained by using the contraction procedure of [11], or equivalently by blowing down 24 degree-1 lines which are parallel to one of the $\mathbb{P}^1$'s. In this way, we obtain a family of singular varieties, which are birational to mirrors of the singular Hulek-Verrill manifolds $\widehat{\text{HV}}$ found by using Batyrev's method [12].

Periods of the five-parameter family

Section 3 deals with the periods of HV, which describe the variation of the Hodge structure as a function of moduli space coordinates. We study the five-parameter family (2.2). The overall scaling of coordinates φ^μ does not affect the vanishing locus, and thus we can identify the moduli space[2] with $\mathbb{P}^5$. The manifolds are singular on the loci where one of the coordinates vanishes,

$$E_\mu = \left\{ \left(\varphi^0, \varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^5\right) \in \mathbb{P}^5 \,\middle|\, \varphi^\mu = 0 \right\}, \tag{1.8}$$

and also on the conifold locus

$$\Delta \overset{\text{def}}{=} \prod_{\epsilon_i \in \{\pm 1\}} \left(\sqrt{\varphi^0} + \epsilon_1 \sqrt{\varphi^1} + \epsilon_2 \sqrt{\varphi^2} + \epsilon_3 \sqrt{\varphi^3} + \epsilon_4 \sqrt{\varphi^4} + \epsilon_5 \sqrt{\varphi^5} \right) = 0\,. \tag{1.9}$$

Often it is necessary to work on an affine patch, for which we most often choose $\varphi^0 = 1$. Results obtained in this patch apply in any patch $\varphi^i = 1$, with Latin indices running from 1 to 5, after making a suitable permutation of indices.

We begin the investigation by recalling a series expansion for the fundamental period [13, 14],

$$\varpi^{(0);0}(\varphi) = \sum_{n=0}^{\infty} \sum_{\deg(\boldsymbol{p})=n} \binom{n}{\boldsymbol{p}}^2 \boldsymbol{\varphi}^{\boldsymbol{p}}\,, \tag{1.10}$$

where $\boldsymbol{p} = (p_1, \ldots, p_5)$ is a five-component multi-index, $\deg(\boldsymbol{p})$ is the sum $p_1 + \cdots + p_5$, and

$$\binom{n}{\boldsymbol{p}} = \frac{n!}{p_1! p_2! p_3! p_4! p_5!} \tag{1.11}$$

is the multinomial coefficient. By $\boldsymbol{x}^{\boldsymbol{p}}$ we mean the product $x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4} x_5^{p_5}$. The superscript (0) in $\varpi^{(0);0}$ refers to the coordinate patch $\varphi^0 = 1$.

On seeking the differential equations obeyed by this period, we are led to the system

$$\mathcal{L}_i \varpi^{(0);0}(\boldsymbol{\varphi}) \overset{\text{def}}{=} \left(\frac{1}{\varphi^0} (\Theta + 1)^2 - \frac{1}{\varphi^i} \theta_i^2 \right) \varpi^{(0);0}(\boldsymbol{\varphi}) = 0\,, \quad \text{with} \quad \theta_i = \varphi^i \frac{\partial}{\partial \varphi^i}\,, \; \Theta = \sum_{i=1}^{5} \theta_i\,.$$

These constitute a partial Picard-Fuchs system, giving 32 solutions among which we find the 12 periods.[3] These are the components of the vector

$$\varpi^{(0)} = (\varpi^{(0);0}, \varpi^{(0);i}, \varpi^{(0)}_j, \varpi^{(0)}_0)^T\,, \qquad i = 1, \ldots, 5\,. \tag{1.12}$$

By a simple separation-of-variables argument, it can be shown that integrals of Bessel functions of the following form furnish a basis of solutions:

$$\frac{\varphi^0}{\mathrm{i}\pi} \int_0^\infty \mathrm{d}z\, z\, B_0\left(\sqrt{\varphi^0} z\right) \prod_{i=1}^{5} B_i\left(\sqrt{\varphi^i} z\right), \tag{1.13}$$

where $B_i(\zeta)$ is either $K_0(\zeta)$ or $\mathrm{i}\pi I_0(\zeta)$. Naïvely there are $2^6 = 64$ integrals of this type. However, at a generic point in the moduli space there are exactly 32 such integrals that converge. The analytic continuation of each integral outside of its domain of convergence can be written as a linear combination of integrals of the general form (1.13) that converge in the new region.

[2]Note that two points in $\mathbb{P}^5$ can correspond to biholomorphic manifolds. There exists a 'fundamental domain' in $\mathbb{P}^5$, where the points are in one-to-one correspondence with distinct biholomorphism classes. This issue does not affect our studies.

[3]12 is the dimension of the third cohomology of HV.

There is an additional differential operator which, together with those above, completely fixes the periods. After setting $\varphi^0 = 1$, this takes the form of a polynomial in Θ with coefficients that are polynomials in φ_μ. In principle this operator is determined by the recurrence methods of [14], but for fully general φ^i these recurrence relations cannot be solved in a practical amount of time. It is possible, however, to choose constants s^i and specialise the parameters to $\varphi^i = s^i\varphi$, thus restricting to lines in the moduli space, and write a differential operator in terms of φ that governs the variation of the periods along these lines. In many cases, it is possible to find this remaining operator on these lines, and in our examples this operator obtained via the methods of [14] turns out to factorise.[4] We give an example of such an operator in §3.3.

Despite lacking the explicit form of the general Picard-Fuchs system, we can fix the 12 periods among the 32 solutions of the partial system by imposing boundary conditions. These are found by matching the asymptotics of the solutions to the asymptotics near the large complex structure point predicted by mirror symmetry. We also give explicit series expansions for these periods near the large complex structure point.

Mirror map and large complex structure

The large complex structure points are located at the loci where all but one of the coordinates a_μ vanish. Near the large complex structure point with $\varphi^0 \neq 0$, the period vector in the integral basis can be written in terms of the prepotential $\mathcal{F}$ as

$$\Pi^{(0)} = \begin{pmatrix} \Pi^{(0)}_0 \\ \Pi^{(0)}_i \\ \Pi^{(0);0} \\ \Pi^{(0);i} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z^0}\mathcal{F} \\ \frac{\partial}{\partial z^i}\mathcal{F} \\ z^0 \\ z^i \end{pmatrix}, \qquad \mathcal{F}(z^0,\ldots,z^5) = -\frac{1}{3!}\sum_{a,b,c=0}^{5} Y_{abc}\frac{z^a z^b z^c}{z^0} + (z^0)^2\sum_{\boldsymbol{p}\neq\boldsymbol{0}} n_{\boldsymbol{p}}\,\mathrm{Li}_3(q^{\boldsymbol{p}})\,.$$

The Y_{abc} are topological quantities which we compute in §4 and the $n_{\boldsymbol{p}}$ are the genus-0 instanton numbers of multidegree $\boldsymbol{p}$. We find the following relation between the integral basis period vector $\Pi^{(0)}$ and the period vector $\varpi^{(0)}$ in the Frobenius basis of§3:

$$\Pi^{(0)} = \rho\,\nu^{-1}\,\varpi^{(0)}\,, \tag{1.14}$$

with matrices

$$\rho = \begin{pmatrix} -\frac{1}{3}Y_{000} & \mathbf{1}^T & \mathbf{0}^T & 1 \\ \mathbf{1} & \mathbb{0} & -\mathbb{I} & \mathbf{0} \\ 1 & \mathbf{0}^T & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I} & \mathbb{0} & \mathbf{0} \end{pmatrix}, \qquad \text{and} \qquad \nu = \mathrm{diag}(1,(2\pi\mathrm{i})\mathbf{1},(2\pi\mathrm{i})^2\mathbf{1},(2\pi\mathrm{i})^3)\,.$$

Here, and in what follows, $\mathbf{1}$ denotes the vector $(1,1,1,1,1)^T$ and $\mathbf{0}$ the vector $(0,0,0,0,0)^T$. The unit matrix is denoted by $\mathbb{I}$, while $\mathbb{0}$ is a 5×5 zero matrix.

With the period vectors in the integral basis in hand, we can compute the instanton numbers by studying the Yukawa couplings y_{ijk}. These are given by the formula

$$y_{ijk} = -(\Pi^{(0)})^T\Sigma\,\partial_{ijk}\Pi^{(0)}\,, \tag{1.15}$$

[4] This is a consequence of the fact that while the procedure in [14] gives a recurrence of minimal order, the degrees of the polynomial coefficients are not minimised.

but also have the following expansions in terms of the instanton numbers:

$$y_{ijk} \;=\; Y_{ijk} + \sum_{n=1}^{\infty} \sum_{\deg(\boldsymbol{p})=n} \frac{p_i\, p_j\, p_k\, n_{\boldsymbol{p}}\, \boldsymbol{q}^{\boldsymbol{p}}}{1-\boldsymbol{q}^{\boldsymbol{p}}}\,, \qquad \text{where} \qquad q^i \;=\; \mathrm{e}^{2\pi \mathrm{i} t^i}\,. \tag{1.16}$$

Due to the permutation symmetry of the parameters φ^i, we can express many quantities in terms of the elementary symmetric polynomials. This results in a significantly less complicated series expressions which are far more amenable to computation. While we are still unable to reach the degrees possible in one-parameter computations, we find for genus-0 the instanton numbers up to a total degree of 29, which we collect in appendix E.

In addition, we are able to compute the genus-1 instanton numbers by constructing the genus-1 prepotential using the expressions in [15]. Rather pleasantly, the prepotential turns out to be conceptually simpler than on the quotients studied in [3]. This is largely due to the fact that the distinct singular points on the moduli space of the quotient are replaced by the irreducible singular locus $\Delta = 0$ on the moduli space of HV. The limiting factor is the number of genus-0 instanton numbers we are able to compute, since those are needed to extract the genus-1 numbers from the prepotential. We are thus able to compute the genus-1 instanton numbers up to total degree 29, and we give these in §E.2. The instanton numbers to order 16 are tabulated in the text, while longer tables giving the numbers up to degree 29 are to be found in appendix E.

Having computed the instanton numbers to a high degree, a rich array of patterns becomes evident. We are able to explain some of the observed invariances of instanton numbers using the freedom to expand the prepotential about any one of the six large complex structure limits.

Duality webs

In this section we elaborate on the symmetries discovered amongst the instanton numbers. We find that these symmetries correspond to an infinite Lorentzian Coxeter group $\mathcal{W} \subset \mathrm{Sp}(12, \mathbb{Z})$. An S_5 subgroup is immediately manifest, as there is a permutation symmetry in the five Kähler structure moduli of HΛ. The complex structure moduli space of HV can be parametrised with six homogeneous coordinates, which leads to additional identities between the instanton numbers given not by permuting indices, but the duality operation

$$(i,j,k,l,m) \mapsto (-i+j+k+l+m, j, k, l, m)\,. \tag{1.17}$$

By acting on a single multi-index $\boldsymbol{I} = (i,j,k,l,m)$ with a sequence of permutations and this duality, we can form orbits that we term a 'web'. A multi-index is said to be positive if all of its entries are nonnegative and at least one is positive. A web is positive if every multi-index is positive. We conjecture that these webs have a surprisingly simple description: up to a permutation they are in one-to-one correspondence with *source vectors* $\boldsymbol{I}$, which are defined by the condition $\deg \boldsymbol{I} \geqslant 3 \max \boldsymbol{I}$.

We observe and prove that the genus-0 numbers $n_{\boldsymbol{I}}$ are non-vanishing only if $\boldsymbol{I}$ belongs to a positive web, or a certain exceptional half web $\mathbb{W}_+$, the positive elements of the web containing $(1,0,0,0,0)$. At genus one and beyond, the instanton number for a degree $\boldsymbol{I}$ is nonzero only if $\boldsymbol{I}$ belongs to a positive web.

Monodromies

In §6, we turn to computing the monodromies around the singular loci $\varphi^0\varphi^1\varphi^2\varphi^3\varphi^4\varphi^5 = 0$ and $\Delta = 0$. As sugested by the fact that (1.13) is a function of $\sqrt{\varphi^\mu}$, this is most conveniently done by first classifying the singularities in coordinates $\sqrt{\varphi^\mu}$. Then the singular locus $\Delta = 0$ becomes a reducible union of codimension-1 hyperplanes of the form

$$D_I \;=\; \left\{ (\varphi^0, \ldots, \varphi^5) \in \mathbb{P}^5 \;\middle|\; \sum_{\mu \in I} \sqrt{\varphi^\mu} = \sum_{\nu \in I^c} \sqrt{\varphi^\nu} \right\}, \qquad I \subset \{0, \ldots, 5\}. \tag{1.18}$$

The monodromies around these loci can be found by numerically integrating the Picard-Fuchs equations on a path circling these loci. Alternatively, one can find the linear relations between analytically continued Bessel function integrals in different regions, and use this to compute the monodromies. While the former approach is too difficult with arbitrary paths due to the complicated nature of the complete Picard-Fuchs system, we can integrate along various lines on which the Picard-Fuchs operator can be found as discussed above. By studying various different lines and using symmetry, we can use the resulting 'reduced' monodromy matrices to piece together the full monodromies.

What makes this computation simpler than it first appears is the fact that the monodromy matrix around a conifold locus should be expressible in terms of a single vector:

$$\mathrm{M}_I \;=\; \mathrm{I}_{12} - \boldsymbol{w}_I (\Sigma \boldsymbol{w}_I)^T \,. \tag{1.19}$$

Here $\boldsymbol{w}$ is a 12-component vector that gives the integral basis components of the three-cycle vanishing at the conifold locus. Consequently, the vector $\boldsymbol{w}$ should also obey the symmetries relevant to each locus.

At first, we study the periods in the patch $\varphi^0 = 1$, although later we find it useful to consider other patches as well. To find the partial monodromy matrices, we study lines of the form

$$\left(\varphi^0, \ldots, \varphi^5\right) \;=\; \left(1, s^1\varphi, \ldots, s^5\varphi\right), \tag{1.20}$$

where $s^1, \ldots, s^5$ are constants. To make the numerical computations tractable, we take at least two s^i equal. To be concrete, consider the simple case where $s^1 \neq s^2 = s^3 = s^4 = s^5$. Then, by symmetry

$$\begin{aligned} \Pi^{(0);2} &= \Pi^{(0);3} = \Pi^{(0);4} = \Pi^{(0);5}\,, \\ \Pi^{(0)}_2 &= \Pi^{(0)}_3 = \Pi^{(0)}_4 = \Pi^{(0)}_5\,, \end{aligned}$$

and there are 6 independent periods, which form a vector $\widehat{\Pi}^0$.

$$\widehat{\Pi}^0 \;=\; \begin{pmatrix} \Pi^{(0)}_0 \\ \Pi^{(0)}_1 \\ \Pi^{(0)}_2 \\ \Pi^{(0);0} \\ \Pi^{(0);1} \\ \Pi^{(0);2} \end{pmatrix}. \tag{1.21}$$

In the general case the monodromy matrices M can be written as

$$\mathrm{M} \;=\; (\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{10}, \boldsymbol{u}_{11})\,, \tag{1.22}$$

where $\boldsymbol{u}_i$ are 12-component column vectors

$$\boldsymbol{u}_i \;=\; (u_i^0, u_i^1, \ldots, u_i^{10}, u_i^{11})^T\,. \tag{1.23}$$

Since some of the periods are equal on the line $(\varphi^0, \ldots \varphi^5) = (1, s_1\varphi, s_2\varphi, \ldots, s_2\varphi)$, we cannot find the full monodromy matrces M directly by computing monodromies around the singular points on the line. Instead, we find reduced monodromy matrices $\widehat{\mathrm{M}}$ which give the monodromy of the vector $\widehat{\Pi}^0$. These matrices take the form

$$\widehat{\mathrm{M}} \;=\; (\hat{\boldsymbol{u}}_0,\, \hat{\boldsymbol{u}}_1,\, \hat{\boldsymbol{u}}_2 + \hat{\boldsymbol{u}}_3 + \hat{\boldsymbol{u}}_4 + \hat{\boldsymbol{u}}_5,\, \hat{\boldsymbol{u}}_6,\, \hat{\boldsymbol{u}}_7,\, \hat{\boldsymbol{u}}_8 + \hat{\boldsymbol{u}}_9 + \hat{\boldsymbol{u}}_{10} + \hat{\boldsymbol{u}}_{11})\,, \tag{1.24}$$

where the $\hat{\boldsymbol{u}}_i$ are 6 component column vectors

$$\hat{\boldsymbol{u}}_i \;=\; \left(u_i^0,\, u_i^1,\, u_i^2,\, u_i^6,\, u_i^7,\, u_i^8\right)^T\,. \tag{1.25}$$

By considering several lines and using symmetry arguments to simplify the computations, we are able to gain enough information to completely fix the full monodromy matrices.

Around a conifold locus, given the vector $\boldsymbol{w}$

$$\boldsymbol{w} \;=\; (w_0,\, w_1,\, w_2, \ldots,\, w_2,\, w_7,\, w_8,\, w_9, \ldots,\, w_9)\,, \tag{1.26}$$

the reduced 6×6 matrix $\widehat{\mathrm{M}}$ takes the form

$$\widehat{\mathrm{M}} \;=\; \mathrm{I}_6 - \widehat{\boldsymbol{w}}\big(\widehat{\Sigma}\widehat{\boldsymbol{w}}\big)^T\,, \qquad \widehat{\boldsymbol{w}} = (w_0,\, w_1,\, w_2,\, w_7,\, w_8,\, w_9)\,. \tag{1.27}$$

The reduced intersection matrix $\widehat{\Sigma}$ is given by

$$\widehat{\Sigma} \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{pmatrix}. \tag{1.28}$$

In this way we find 16 of the 32 vectors corresponding to the vanishing loci:

$$\begin{aligned} \boldsymbol{w}_{\{0\}} \;&=\; (\;\;\, 0,0,0,0,0,0, \;\;\; 1, \;\;\; 0,0,0,0,0)\,,\\ \boldsymbol{w}_{\{0,1\}} \;&=\; (-2,0,0,0,0,0, \;\;\; 1,-1,0,0,0,0)\,,\\ \boldsymbol{w}_{\{0,1,2\}} \;&=\; (\;\;\, 4,0,0,2,2,2,-1, \;\;\; 1,1,0,0,0)\,, \end{aligned} \tag{1.29}$$

with the vectors of the form $\boldsymbol{w}_{\{0,i\}}$ obtained by effecting the permutation $(2, i+1)(8, i+7)$ on the components of $\boldsymbol{w}_{\{0,1\}}$. Similarly, the vectors of the form $\boldsymbol{w}_{\{0,i,j\}}$ are obtained from $\boldsymbol{w}_{\{0,1,2\}}$ by using the permutation $(2, i+1)(3, i+2)(8, i+7)(9, i+8)$. The remaining 16 vectors are most conveniently obtained by studying the other patches where $\varphi^i = 1$. For example, consider the patch $\varphi^1 = 1$. Near the large complex structure point at $\varphi^0 = \varphi^2 = \cdots = \varphi^5 = 0$, we have, in the natural integral basis, the period vector Π^1, which is obtained by replacing the φ^1-dependence in Π^0 by φ^0 and vice versa. By symmetry, in this basis, the monodromy around this locus is

$$\boldsymbol{w}_{\{1\}} \;=\; (0,0,0,0,0,0,1,0,0,0,0,0)\,. \tag{1.30}$$

To find the corresponding monodromy matrix in the original basis of Π^0, we just need to find the relation between these two bases. We find the transition matrix $\mathrm{T}_{\Pi^1\Pi^0}$ (6.7) which takes us from one base to another. With this, we are able to find the monodromy matrix $\mathrm{M}_{\{1\}}$ in the original basis:

$$\mathrm{M}_{\{1\}} = \mathrm{T}^{-1}_{\Pi^1\Pi^0}\Big(\mathrm{I}_{12} - \boldsymbol{w}_{\{1\}}\big(\Sigma\boldsymbol{w}_{\{1\}}\big)^T\Big)\mathrm{T}_{\Pi^1\Pi^0} = \mathrm{T}^{-1}_{\Pi^1\Pi^0}\mathrm{M}_{\{0\}}\mathrm{T}_{\Pi^1\Pi^0}\,. \tag{1.31}$$

The other monodromy matrices of the form $\mathrm{M}_{\{i\}}$, $\mathrm{M}_{\{i,j\}}$ and $\mathrm{M}_{\{i,j,k\}}$ are found in a similar manner.

Counting curves on the mirror Hulek-Verrill manifold

In §7 we use use elementary geometric methods in tandem with the Kodaira classification of singular elliptic fibres [16, 17] to directly count curves of certain multidegrees on generic manifolds in the family $\mathrm{H}\Lambda$.

Counting of these curves is based on the observation that $\mathrm{H}\Lambda$ can be viewed as an elliptic fibration with base $\mathbb{P}^1 \times \mathbb{P}^1$. While the generic fibre is an elliptic curve, it is possible to find the discriminant locus corresponding to base points above which the fibres are singular. According to Kodaira's classification, the fibres over nodes of the discriminant locus are unions of two rational curves. By classifying these fibres, we find all rational curves of degrees ≤ 3, and some of the higher-degree curves.

As the discriminant of the elliptic fibration is relatively simple for tetraquadrics, it is often beneficial to consider the singular manifolds $\widehat{\mathrm{H}\Lambda}_i$ obtained by blowing down 24 lines along i'th copy of $\mathbb{P}^1$ in the ambient space. On a generic manifold $\widehat{\mathrm{H}\Lambda}_i$, the discriminant locus has 200 nodes, of which $3 \times 24 = 72$ correspond to lines, 72 to quadrics, and 56 to cubics. We obtain all curves up to degree 3 in this way. In addition, the fibres containing lines and quadrics also contain degree 5 and 4 curves, respectively, as the second component. These account for all rational curves with multidegrees $(0, 0, 1, 2, 2)$, $(0, 0, 1, 1, 2)$, and permutations thereof.

In this way we confirm the predictions from mirror symmetry, and provide details of the elliptic fibrations $\widehat{\mathrm{H}\Lambda}_i$ that may see future use in M/F-theory compactifications.

We collect some symbols that appear in multiple sections, together with their definitions, in Table 2.

Table 2: Some symbols that are used throughout the paper with references to where they are defined.

Symbol	Definition/Description	Ref.
φ	The coordinates $(\varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^5)$ on the complex structure space of HV.	(2.1)
HV	The family of Hulek-Verrill manifolds.	(2.4)
$\mathrm{H\Lambda}$	The family of mirror Hulek-Verrill manifolds, which are complete intersections in $\left(\mathbb{P}^1\right)^5$.	(2.4)
$\widehat{\mathrm{HV}}$	Family of singular manifolds birational to HV.	(2.4)
$\widehat{\mathrm{H\Lambda}}$	Family of singular manifolds birational to $\mathrm{H\Lambda}$.	(2.4)
$\widehat{\mathrm{H\Lambda}}_i$	Families of singular manifolds birational to $\mathrm{H\Lambda}$, obtained by projecting out the i'th $\mathbb{P}^1$ coordinate axis.	(7.1)
P	Laurent polynomial defining $\widehat{\mathrm{HV}}$ in $\mathbb{T}^4$.	(2.1)
P^1, P^2	Laurent polynomials defining the small resolution of $\widehat{\mathrm{HV}}$ in $\mathbb{P}^5$.	(2.2)
Q^1, Q^2	Multidegree $(1,1,1,1,1)$ polynomials that together define $\mathrm{H\Lambda}$ in $\left(\mathbb{P}^1\right)^5$.	(2.17)
$\widehat{Q}$	A multidegree $(2,2,2,2)$ polynomial defining $\widehat{\mathrm{H\Lambda}}$ in $\left(\mathbb{P}^1\right)^4$.	(2.13)
$\widehat{Q}^i$	A multidegree $(2,2,2,2)$ polynomial defining $\widehat{\mathrm{H\Lambda}}_i$ in $\left(\mathbb{P}^1\right)^4$.	(7.3)
E	The locus in $\mathbb{P}^5$ where any of the homogeneous coordinates vanish.	(3.1)
E_μ	The irreducible component of E on which the μ'th homogenous coordinate vanishes.	(3.2)
D_I	Irreducible components of the discriminant locus $\Delta = 0$ in variables $\sqrt{\varphi^\mu}$.	(3.4)
Π	The period vector of the Hulek-Verrill manifold expressed in the integral symplectic basis. A superscript as in $\Pi^{(\mu)}$ denotes the expansion about the μ'th large complex structure point.	(4.1)
ϖ	The HV period vector in the Frobenius basis. A superscript as in $\varpi^{(\mu)}$ denotes the expansion about the μ'th large complex structure point.	(3.29)
$\pi^{(\mu)}$	The HV period vector in the μ'th "Bessel integral basis".	(3.23)
$\mathrm{T}_{u\,v}$	The matrix effecting the basis change between period vectors u, v.	Various
M_s	Matrix giving the monodromy transformation of Π about the locus s.	§6
Δ	In §2 and appendix A, a polytope. In §7, the discriminant of an elliptic fibration.	Various
Δ	The discriminant. $\Delta = 0$ is the conifold locus in the moduli space of HV.	(2.15)

2 Toric geometry and mirror symmetry

We review the construction of Hulek and Verrill's manifold [1] following in part [3]. The starting point of their analysis is the five-parameter family $\widehat{\mathrm{HV}}_{(\varphi^0,\ldots,\varphi^5)}$ of singular varieties embedded in the projective torus $\mathbb{T}^4 = \mathbb{P}^4 \setminus \{X_1 \cdots X_5 = 0\}$ as the vanishing locus of

$$P(\boldsymbol{X};\varphi) = \left(X_1 + X_2 + X_3 + X_4 + X_5\right)\left(\frac{\varphi^1}{X_1} + \frac{\varphi^2}{X_2} + \frac{\varphi^3}{X_3} + \frac{\varphi^4}{X_4} + \frac{\varphi^5}{X_5}\right) - \varphi^0 \,. \tag{2.1}$$

These varieties can be compactified by using the standard methods of toric geometry (see for example [18]), giving in general a variety with 30 singularities. Outside of the discriminant locus[5] these have small resolutions, which constitute a smooth family that we call Hulek-Verrill manifolds $\mathrm{HV}_{(\varphi^0,\ldots,\varphi^5)}$.

Particularly interesting examples of such manifolds are provided by a highly symmetric one-parameter subfamily, where $\varphi^0 = 1$ and $\varphi^1 = \cdots = \varphi^5 = \varphi$. These are characterised by a $\mathbb{Z}_5 \times \mathbb{Z}_2$ symmetry, with the group action on the coordinates generated by

$$\mathfrak{A} : X_i \mapsto X_{i+1}\,, \qquad \mathfrak{B} : X_i \mapsto \frac{1}{X_i}\,,$$

where the indices are understood to take values in $\mathbb{Z}_5$. The action on the manifold is free outside of the points $\varphi \in \{\frac{1}{25}, \frac{1}{9}, 1\}$ in moduli space where fixed points are present. This allows one to take a quotient with respect to these symmetries to get a one-parameter family of Calabi-Yau manifolds, which are smooth for moduli outside these isolated points.

As noted in [1], the varieties on $\mathbb{T}^4$ defined by (2.1) are birational to complete intersection varieties in $\mathbb{P}^5$ defined as the vanishing locus of two polynomials:

$$P^1(\mathbf{X}) = \sum_{\mu=0}^{5} X_\mu\,, \qquad P^2(\mathbf{X};\varphi) = \sum_{\mu=0}^{5} \frac{\varphi^\mu}{X_\mu}\,. \tag{2.2}$$

This innocuous transformation turns out to be useful for finding the (non-singular) mirror manifolds $\mathrm{H}\Lambda$ of the (non-singular) Hulek-Verrill Manifolds HV. Combined with the methods of Batyrev and Borisov [7,8,12], which we briefly review in §2.2, this allows finding the mirror Calabi-Yau manifold as a subvariety of a suitable toric variety.

By standard methods of toric geometry, we can find the mirror manifolds $\widehat{\mathrm{H}\Lambda}$ and $\mathrm{H}\Lambda$ of $\widehat{\mathrm{HV}}$ and HV. As expected, we find that $\widehat{\mathrm{H}\Lambda}$ is singular and birational to $\mathrm{H}\Lambda$. Figure 1 outlines the pairings.

$$\begin{array}{ccc} \mathrm{HV} & \longleftrightarrow & \mathrm{H}\Lambda \\ \updownarrow & & \updownarrow \\ \widehat{\mathrm{HV}} & \longleftrightarrow & \widehat{\mathrm{H}\Lambda} \end{array}$$

Figure 1: Relations between the various families of manifolds we study: the singular Hulek-Verrill manifolds are denoted by $\widehat{\mathrm{HV}}$, Hulek-Verrill manifolds by HV, the singular mirror Hulek-Verrill manifolds by $\widehat{\mathrm{H}\Lambda}$, and mirror Hulek-Verrill manifolds by $\mathrm{H}\Lambda$. The horizontal arrows denote mirror maps, and the vertical arrows birational equivalences (blow-ups/-downs).

[5] The situation is a little more involved on the discriminant locus, for details see [1].

2.1 The polytopes corresponding to singular varieties

Table 3: Quantities associated to the lattices $\widehat{N}$, N, $\widehat{N}^* = \widehat{M}$, and $N^* = M$.

Quantity	$\widehat{N}, N$	$\widehat{M}, M$
Basis	$\boldsymbol{e}_i$	$\boldsymbol{e}^i$
Coordinates on $\mathbb{T}$	X_i	Y_i
Coordinates on $N_{\mathbb{R}}/M_{\mathbb{R}}$	x_i	y_i
Cox coordinates	ξ_n	η_n
Polytopes	$\widehat{\Delta}^*$, Δ_1, Δ_2, Δ, ∇^*	$\widehat{\Delta}$, ∇_1, ∇_2, ∇, Δ^*
Polytope vertex labels	v_n	u_n
Polytope face labels	ρ_n	τ_n

We group the symbols denoting various polytopes, Cox coordinates, and other related information by their associated lattices in Table 3. The lattices $\widehat{N}$ and $\widehat{M}$ associated to the singular varieties $\widehat{\text{HV}}$ and $\widehat{\text{H}\Lambda}$ are four-dimensional, and consequently for them the index i runs from 1 to 4. The lattices N and M are five-dimensional and for them the indices take values $i = 1, \ldots, 5$.

Five-dimensional description

The polynomial $P(\mathbf{X}; \mathbf{a})$ contains 21 monomials in coordinates $X_1, \ldots, X_5$. Writing these monomials using multi-index notation defines 21 vectors $v_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5)$, $n = 0, \ldots, 20$, in $\mathbb{Z}^5$. We write X^{v_n} for the monomial

$$X^{v_n} = X_1^{v_n^1} X_2^{v_n^2} X_3^{v_n^3} X_4^{v_n^4} X_5^{v_n^5} .$$

The vectors v_n make up the set

$$\big\{(0,0,0,0,0)\big\} \cup \big\{\mathbf{e}_i - \mathbf{e}_j \mid i,j = 1,\ldots,5\,,\;\; i \neq j\big\} .$$

These vectors in fact lie in a four-dimensional sublattice

$$A_4 = \left\{ (n_1, n_2, n_3, n_4, n_5) \in \mathbb{Z}^5 \;\middle|\; \sum_{i=1}^{5} n_i = 0 \right\} \subset N \simeq \mathbb{Z}^5 ,$$

with $\mathbf{e}_i$ denoting the standard orthonormal basis for $\mathbb{Z}^5$. We take as basis for the sublattice A_4 the vectors

$$\mathbf{e}_{2,1}\,, \mathbf{e}_{3,2}\,, \mathbf{e}_{4,3}\,, \mathbf{e}_{5,4}\,, \qquad \text{where} \qquad \mathbf{e}_{i,j} = \mathbf{e}_i - \mathbf{e}_j\,.$$

The dual lattice can be realised as a sublattice of $M = N^* \simeq \mathbb{Z}^5$, with the basis given by

$$\mathbf{e}^{i+1,i} = \frac{i}{5} \sum_{t=i+1}^{5} \mathbf{e}^t - \frac{5-i}{5} \sum_{t=1}^{i} \mathbf{e}^t ,$$

where $\mathbf{e}_i$ and $\mathbf{e}^i$ are the canonical bases of $N \simeq \mathbb{Z}^5$ and $M \simeq \mathbb{Z}^5$. With these definitions we have that the canonical inner product gives a non-degenerate pairing:

$$\langle \mathbf{e}_{i+1,1}, \mathbf{e}^{j+1,j} \rangle = \delta_{ij}\,.$$

To find a convenient four-dimensional description for these lattices, we project $N \mapsto \widehat{N} \simeq \mathbb{Z}^4$ and $M \mapsto \widehat{M} \simeq (\mathbb{Z}^4)^*$ with

$$\begin{aligned} &\mathbf{e}_i \mapsto \mathbf{e}_i\,, \quad i = 1,\ldots,4\,, \qquad \mathbf{e}_5 \mapsto 0\,, \\ &\mathbf{e}^i \mapsto \mathbf{e}^i\,, \quad i = 1,\ldots,4\,, \qquad \mathbf{e}^5 \mapsto -\mathbf{e}^1 - \mathbf{e}^2 - \mathbf{e}^3 - \mathbf{e}^4\,. \end{aligned} \tag{2.3}$$

Four-dimensional description of $\widehat{\boldsymbol{\Delta}}$

An equivalent way of arriving at the form of the four-dimensional polytope starts with going to an affine patch, say $X_5 = 1$, where the polynomial $P(\mathbf{X}; \varphi)$ contains 21 monomials that are now of the form

$$1\,, \qquad X_i\,, \qquad \frac{1}{X_i}\,, \qquad \frac{X_i}{X_j}\,, \qquad i \neq j\,, \qquad i,j \neq 5\,. \tag{2.4}$$

These monomials correspond to lattice points in $\widehat{N}^4$ that are given by the 21 vectors in the set

$$\big\{(0,0,0,0)\big\} \cup \big\{ \pm \mathbf{e}_i \,\big|\, i = 1,\ldots,4\big\} \cup \big\{\mathbf{e}_i - \mathbf{e}_j \,\big|\, i,j = 1,\ldots,4\,,\ \ i \neq j\big\}\,.$$

For the numbering of these lattice points, see appendix A. The convex hull of these points in the real span $\widehat{N}_{\mathbb{R}}$ of $\widehat{N}$,

$$\widehat{\Delta} = \text{Conv}(u_0, \ldots, u_{20}) \tag{2.5}$$

is a four-dimensional reflexive polytope. The only internal lattice point is the origin u_0, and the vertices are exactly $u_1, \ldots, u_{20}$, which are the only lattice points in $\widehat{\Delta}$. The faces of $\widehat{\Delta}$ consist of 10 tetrahedra and 20 triangular prisms lying on the boundary planes defined by the equations

$$\delta_1 y_1 + \delta_2 y_2 + \delta_3 y_3 + \delta_4 y_4 + \epsilon_0 = 0\,, \qquad \delta_i \in \{0,1\}\,, \qquad \epsilon_0 \in \{-1,1\}\,.$$

For the labelling of the faces, see appendix A. The 20 triangular prisms break up into two $\mathbb{Z}_5 \times \mathbb{Z}_2$ transitive orbits under the actions $\mathfrak{A}$ and $\mathfrak{B}$ given in (2.6), and the tetrahedra form one such orbit. The facets meet as displayed in Figure 2. The polytope $\widehat{\Delta}$ defines a fan whose cones are exactly those supported by the faces of $\widehat{\Delta}$. This fan, however, is not simplicial, and consequently we wish to find a triangulation of $\widehat{\Delta}$, which corresponds to a smooth fan. We find that there are two triangulations that respect the $\mathbb{Z}_5 \times \mathbb{Z}_2$ symmetry. For the purposes of this work, the choice of triangulations does not make a difference. In particular, the family of mirror manifolds one finds does not depend on the triangulation, nor are any of the properties that we consider here, such as the location of singularities, affected by this choice. Therefore we will give just the first of these triangulations.

In the four-dimensional description, the action is a composition of the $\mathbb{Z}_5 \times \mathbb{Z}_2$ action in five dimensions and the projection to four dimensions. This gives

$$\begin{aligned} &\mathfrak{A} : (m_1, m_2, m_3, m_4) \mapsto \begin{cases} (\ \ 0, m_1, m_2, m_3)\,, & \text{if } \sum_{i=1}^4 m_i = 0\,, \\ (\pm 1, m_1, m_2, m_3)\,, & \text{if } \sum_{i=1}^4 m_i = \mp 1\,, \end{cases} \\ &\mathfrak{B} : \mathbf{e}^i \mapsto -\mathbf{e}^i\,. \end{aligned} \tag{2.6}$$

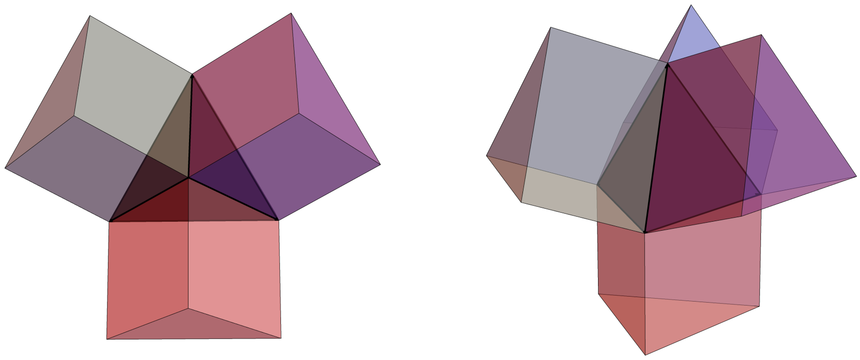

Figure 2: At each of the ten tetrahedra, four of the twenty prisms meet in the above configuration (depicted twice). For each pair of prisms above there is a third (not pictured) sharing a face with both. The altitudes of prisms that share a face are perpendicular. This figure corrects Fig. 13 of [3].

The cones in the first triangulation are given by

$$\begin{aligned}
\sigma_1 &= \Big\langle (1,0,0,0),\ (1,0,\ \ 0,-1),\ (1,0,-1,\ \ 0),\ (0,1,-1,0) \Big\rangle\,,\\
\sigma_2 &= \Big\langle (1,0,0,0),\ (0,1,\ \ 0,\ \ 0),\ (0,1,\ \ 0,-1),\ (0,1,-1,0) \Big\rangle\,,\\
\sigma_3 &= \Big\langle (1,0,0,0),\ (1,0,\ \ 0,-1),\ (0,1,\ \ 0,-1),\ (0,1,-1,0) \Big\rangle\,,\\
\sigma_4 &= \Big\langle (1,0,0,0),\ (1,0,-1,\ \ 0),\ (0,1,-1,\ \ 0),\ (0,0,-1,1) \Big\rangle\,,\\
\sigma_5 &= \Big\langle (1,0,0,0),\ (0,1,\ \ 0,\ \ 0),\ (0,1,-1,\ \ 0),\ (0,0,\ \ 0,1) \Big\rangle\,,\\
\sigma_6 &= \Big\langle (1,0,0,0),\ (0,0,\ \ 0,\ \ 1),\ (0,1,-1,\ \ 0),\ (0,0,-1,1) \Big\rangle\,,
\end{aligned} \tag{2.7}$$

and their images under $\mathbb{Z}_2 \times \mathbb{Z}_5$, together with the 10 simplicial cones supported by the tetrahedra. The cones σ_1, σ_2 and σ_3 correspond to the triangulation of the triangular prism $y_1 + y_2 = 1$ and σ_4, σ_5 and σ_6 give a triangulation of the prism $y_3 + y_5 = -1$.

The dual polytope $\widehat{\Delta}^*$

The polytope $\widehat{\Delta}$ has a dual reflexive polytope $\widehat{\Delta}^*$ which is bounded by 20 planes

$$\epsilon_0 + x_i \;=\; 0\,, \qquad 1 + x_i - x_j \;=\; 0\,, \qquad i,j \in \{1,2,3,4\}\,, \quad i \neq j\,, \qquad \epsilon_0 \in \{-1,1\}\,.$$

These planes intersect $\widehat{\Delta}^*$ in 20 cubical faces. For the explicit numbering of the faces, which manifests the explicit duality between these faces and the vertices of $\widehat{\Delta}$, see appendix A. It follows that $\widehat{\Delta}^*$ is a convex hull of 31 lattice points that we label $v_0, \ldots, v_{30}$.

$$\widehat{\Delta}^* \;=\; \mathrm{Conv}(v_0, \ldots, v_{30}) \;=\; \mathrm{Conv}\Big(\Big\{ \pm(\delta_1, \delta_2, \delta_3, \delta_4) \;\Big|\; \delta_i \in \{0,1\} \Big\}\Big)\,. \tag{2.8}$$

The corresponding fan is again non-simplicial, and thus requires triangulation to give a non-singular ambient toric variety. Hulek and Verrill [1] consider a triangulation that is invariant under the S_5 permutations of the lattice coordinates. All of the top-dimensional cones in this triangulation are obtained by acting on the vertices of a single cone with S_5:

$$\left\{\varsigma\langle(1,0,0,0),(1,1,0,0),(1,1,1,0),(1,1,1,1)\rangle \,\middle|\, \varsigma\in S_5\right\}.$$

Note that the action of $\varsigma \in S_5$ on $\widehat{N}$ is subtle: we have to consider the action of S_5 on the five-dimensional lattice and then project this to back to the four-dimensional lattice. Doing this, one is left with the following action on the basis

$$\varsigma(\mathbf{e}_i) = \begin{cases} \mathbf{e}_{\varsigma(i)}\,, & \text{if } \varsigma(i) \neq 5\,, \\ -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4\,, & \text{if } \varsigma(i) = 5\,. \end{cases}$$

The triangulation data serves as input for Batyrev's formula [12] for the Hodge numbers of smooth members of the families of Calabi-Yau manifolds corresponding to the polytopes $\widehat{\Delta}$ and $\widehat{\Delta}^*$:

$$\begin{aligned} h^{12} &= \text{pts}\big(\widehat{\Delta}^*\big) - \sum_{\text{codim}\,\widehat{\Theta}^*=1} \text{int}\big(\widehat{\Theta}^*\big) + \sum_{\text{codim}\,\widehat{\Theta}=2} \text{int}\big(\widehat{\Theta}^*\big)\,\text{int}\big(\widehat{\Theta}\big) - 5\,, \\ h^{11} &= \text{pts}\big(\widehat{\Delta}\big) - \sum_{\text{codim}\,\widehat{\Theta}=1} \text{int}\big(\widehat{\Theta}\big) + \sum_{\text{codim}\,\widehat{\Theta}=2} \text{int}\big(\widehat{\Theta}^*\big)\,\text{int}\big(\widehat{\Theta}\big) - 5\,, \end{aligned}$$

where pts($\widehat{\Theta}$) and int($\widehat{\Theta}$) denote the number of lattice points and interior lattice points of $\widehat{\Theta}$. $\widehat{\Theta}$ and $\widehat{\Theta}^*$ are faces of $\widehat{\Delta}$ and $\widehat{\Delta}^*$, respectively. These formulae are manifestly compatible with mirror symmetry. From the toric descriptions for the manifolds $\widehat{\text{HV}}$ and $\widehat{\text{H}\Lambda}$, we find the Hodge numbers

$$h^{p,q}\big(\widehat{\text{HV}}\big) = \begin{matrix} &&& 1 &&& \\ && 0 && 0 && \\ & 0 && 26 && 0 & \\ 1 && 16 && 16 && 1 \\ & 0 && 26 && 0 & \\ && 0 && 0 && \\ &&& 1 &&& \end{matrix}\,, \qquad h^{p,q}\big(\widehat{\text{H}\Lambda}\big) = \begin{matrix} &&& 1 &&& \\ && 0 && 0 && \\ & 0 && 16 && 0 & \\ 1 && 26 && 26 && 1 \\ & 0 && 16 && 0 & \\ && 0 && 0 && \\ &&& 1 &&& \end{matrix}\,.$$

2.2 The method of Batyrev and Borisov

To find the small resolutions HV and HΛ of the singular manifolds related to the polytopes discussed above, we use the toric geometry methods pioneered by Batyrev and Borisov [7, 8, 12]. We briefly review this approach.[6]

Given a variety defined as a vanishing locus of the set of n Laurent polynomials $\{P^i\}_{i=1}^n$, one can study the intersection of affine hypersurfaces $V'(P^i) \overset{\text{def}}{=} \{P^i = 0\} \subset \mathbb{T}$. If the polytopes $\{\Delta_i\}_{i=1}^n$ corresponding to the polynomials P^i form a nef-partition of a reflexive polytope Δ, we can define an ambient space $\widehat{\mathbb{P}}_{\Delta^*} \supset \mathbb{T}$ corresponding to the fan associated to Δ^*. The toric variety $\widehat{\mathbb{P}}_{\Delta^*}$ has a partial desingularisation $\mathbb{P}_{\Delta^*}$, corresponding to a maximal projective triangulation of Δ^*. The surfaces $V'(P^i)$ have closures $\widehat{V}(P^i) \subset \widehat{\mathbb{P}}_{\Delta^*}$ and $V(P^i) \subset \mathbb{P}_{\Delta^*}$, and we can define the closures of the intersections $\widehat{\mathcal{M}} = \widehat{V}(P^1) \cap \ldots \cap \widehat{V}(P^n)$ and $\mathcal{M} = V(P^1) \cap \ldots \cap V(P^n)$.

[6] To keep the notation consistent throughout the paper, we adopt here notation that is slightly different from that of [7]. For example, their $\mathbb{P}_\Delta$ corresponds to our $\widehat{\mathbb{P}}_{\Delta^*}$.

It can be shown [7] that if $\widehat{\mathcal{M}}$ is non-empty and irreducible, and also $\dim \mathcal{M} \geq 3$, then $\mathcal{M}$ defined in this way is a smooth manifold.[7]

To find the mirror variety of the smooth manifold constructed in this way, we note that by the definition of a nef-partition

$$\Delta = \text{Mink}\big(\{\Delta_i\}_{i=1}^n\big),$$

where Mink denotes the Minkowski sum. In addition, we can define the convex hull of the union of the polytopes Δ_i:

$$\nabla^* \stackrel{\text{def}}{=} \text{Conv}\big(\{\Delta_i\}_{i=1}^n\big).$$

One can show [7] that the polytope ∇^* so defined is also a reflexive polytope. In particular, it has a well-defined dual polytope ∇. This, and the dual polytope Δ^* of Δ, can be shown to be expressible in terms of n smaller polytopes $\{\nabla_i\}_{i=1}^n$:

$$\nabla = \text{Mink}\big(\{\nabla_i\}_{i=1}^n\big), \qquad \Delta^* = \text{Conv}\big(\{\nabla_i\}_{i=1}^n\big),$$

where the sum is again a Minkowski sum, and $\{\nabla_i\}_{i=1}^n$ gives a nef-partition of ∇. Now we can define the mirror manifold of $\widehat{\mathcal{M}}$ as follows: first we use the polytopes ∇_i to define a set of polynomials $\{Q^i\}_{i=1}^n$ and a desingularisation $\mathbb{P}_{\nabla^*}$ corresponding to a maximal projective triangulation of ∇^*. Then the mirror manifold $\mathcal{W}$ of $\mathcal{M}$ can be expressed as the closure $V(Q^1) \cap \cdots \cap V(Q^n)$ of the variety $\{Q^1 = \cdots = Q^n = 0\} \subset \mathbb{T}$. Due to the way $\mathcal{W}$ is constructed, it follows that it is smooth and irreducible if and only if $\mathcal{M}$ is [7].

There is an algorithm for computing the Hodge numbers of varieties defined in this way [19, 20]. In the case of complete intersection varieties, it is more complicated than Batyrev's original formulae for the Hodge numbers [12]. We will not review the details here, and simply note that some computer algebra packages, such as PALP [21], provide an implementation of the algorithm.

2.3 The polytopes corresponding to small resolutions

Small polytopes Δ_1, Δ_2

To find the toric descriptions of the non-singular manifolds HV and HΛ, we study the polytopes $\Delta_1, \Delta_2 \subset \mathbb{Z}^5$. Their vertices correspond to monomials in the polynomials P^1 and P^2, defined in (2.2), that define on $\mathbb{P}^5$ a variety birational to $\widehat{\text{HV}}$. We work directly in an affine patch where $X_0 = 1$. Then the two polytopes can be expressed as

$$\Delta_1 = \text{Conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5), \qquad \Delta_2 = \text{Conv}(\mathbf{0}, -\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3, -\mathbf{e}_4, -\mathbf{e}_5) = -\Delta_1 .$$

These, and the other polytopes defined this subsection, are schematically represented in two dimensions in Figure 3. Using these two, we can construct two larger polytopes as their Minkowski sum and the convex hull of their union

$$\Delta \stackrel{\text{def}}{=} \text{Mink}(\Delta_1, \Delta_2), \qquad \nabla^* \stackrel{\text{def}}{=} \text{Conv}(\Delta_1, \Delta_2).$$

[7] The reader conversant in toric geometry will recognise this as the MPCP-desingularisation. For the present purposes it is enough to note that this desingularisation is obtained from a triangulation of the polytope, and preserves the canonical class of the manifold.

From the definition of convex hull, it follows immediately that the vertices of ∇^* are exactly $\pm\mathbf{e}_i$ with $i = 1, \ldots, 5$. Its 32 faces are the four-dimensional simplices of the form

$$\tau_n \;=\; \mathrm{Conv}\big(\epsilon_1\mathbf{e}_1,\, \epsilon_2\mathbf{e}_2,\, \epsilon_3\mathbf{e}_3,\, \epsilon_4\mathbf{e}_4,\, \epsilon_5\mathbf{e}_5\big)\,, \qquad \epsilon_i \in \{-1, 1\}$$

given by intersections of ∇^* with bounding planes

$$\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 + \epsilon_5 x_5 \;=\; 1\,, \qquad \epsilon_i \in \{-1, 1\}\,.$$

The polytope Δ contains in total 31 lattice points,

$$\big\{(0,0,0,0)\big\} \cup \big\{\pm\mathbf{e}_i \,\big|\, i = 1, \ldots, 5\big\} \cup \big\{\mathbf{e}_i - \mathbf{e}_j \,\big|\, i, j = 1, \ldots, 5,\;\; i \neq j\big\}\,.$$

Thus it can be written as a convex hull of 30 lattice points

$$\Delta \;=\; \mathrm{Conv}(v_1, \ldots, v_{30})\,.$$

Its only internal point is the origin, and it has 62 faces that are hypercubes, given by intersections with planes

$$\epsilon_0 + \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_4 + \delta_5 x_5 \;=\; 0\,, \qquad \text{with} \qquad \epsilon_0 \in \{-1, 1\}\,,\;\; \delta_i \in \{0, 1\}\,.$$

It can be shown that $\{\Delta_1, \Delta_2\}$ is a nef-partition of Δ.

Small polytopes ∇_1, ∇_2

Finally, to find the equations defining the mirror Hulek-Verrill manifold, we need the two polytopes ∇_1 and ∇_2. These can be obtained by first finding the duals of ∇^* and Δ. The polytope ∇ is a hypercube centred at the origin. Its vertices are given by the 32 points of the form

$$\nabla = \mathrm{Conv}\big(\{(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)\} \mid \epsilon_i \in \{-1, 1\}\big)\,.$$

The faces are the 10 four-dimensional hypercubes given by intersections with the planes

$$y_i = \pm 1\,.$$

The remaining polytope Δ^* has a slightly more complicated structure. It can be written as the convex hull of 62 vertices of the form

$$\Delta^* \;=\; \mathrm{Conv}\big(\big\{\pm(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \mid \delta_i \in \{0, 1\}\big\}\big)\,.$$

The labelling of all vertices in given in appendix A. It has 30 faces, given by intersections with the planes

$$1 \pm y_i \;=\; 0\,, \qquad 1 + y_i - y_j \;=\; 0\,.$$

Like their duals, ∇ and Δ^* can be given in terms of two smaller polytopes ∇_1 and ∇_2:

$$\nabla = \mathrm{Mink}(\nabla_1,\, \nabla_2)\,, \qquad \Delta^* = \mathrm{Conv}(\nabla_1,\, \nabla_2)\,.$$

Here ∇_1 and ∇_2 are hypercubes with one vertex at origin, given by

$$\nabla_1 \;=\; \mathrm{Conv}(\{(\delta_2, \ldots, \delta_5) \,|\, \delta_i \in \{0, 1\}\})\,, \qquad \nabla_2 \;=\; \mathrm{Conv}(\{-(\delta_1, \ldots, \delta_5) \,|\, \delta_i \in \{0, 1\}\}) \;= -\nabla_1\,.$$

By the prescription of Batyrev and Borisov, the ambient variety $\mathbb{P}_{\Delta^*}$ for the Hulek-Verrill manifold is given by triangulating Δ^*. We leave most of the details to the reader, but the upshot is that, as in [1], we can take the triangulation to be invariant under permutations $\varsigma \in S_5$ of the coordinates X_i as well as under the $\mathbb{Z}_2$ inversion symmetry $X_i \mapsto \frac{1}{X_i}$.

The fan associated to Δ^* consists of 720 top-dimensional cones. There are three simplicial cones σ_1, σ_2 and σ_3, whose images under S_5 and $\mathbb{Z}_2$ generate the whole fan. These are given by

$$\begin{aligned} \sigma_1 &= \Big\langle (\ \ 1,0,0,0,0), (\ \ 1, \ \ 1,0,0,0), (1,1,1,0,0), (1,1,1,1,0), (1,1,1,1,1)\Big\rangle, \\ \sigma_2 &= \Big\langle (-1,0,0,0,0), (\ \ 0, \ \ 0,0,0,1), (0,0,0,1,1), (0,0,1,1,1), (0,1,1,1,1)\Big\rangle, \\ \sigma_3 &= \Big\langle (-1,0,0,0,0), (-1,-1,0,0,0), (0,0,0,0,1), (0,0,0,1,1), (0,0,1,1,1)\Big\rangle. \end{aligned} \tag{2.9}$$

The first cone, together with the 119 distinct cones generated by permuting the coordinates, $\{\varsigma(\sigma_1) \mid \varsigma \in S_5\}$, give a triangulation of the hypercube ∇_1. The $\mathbb{Z}_2$ inversion symmetry acts on these cones by $\varsigma(\sigma_1) \mapsto \varsigma(-\sigma_1)$. The hypercube ∇_2 is triangulated by the $\mathbb{Z}_2$ image of $\{\varsigma(\sigma_1) \mid \varsigma \in S_5\}$. The rest of the polytope ∇ is triangulated by σ_2, σ_3, and their images under $S_5 \times \mathbb{Z}_2$. There are additional triangulations, but as in the four-dimensional case, the choice of triangulation does not affect the discussion in this paper.

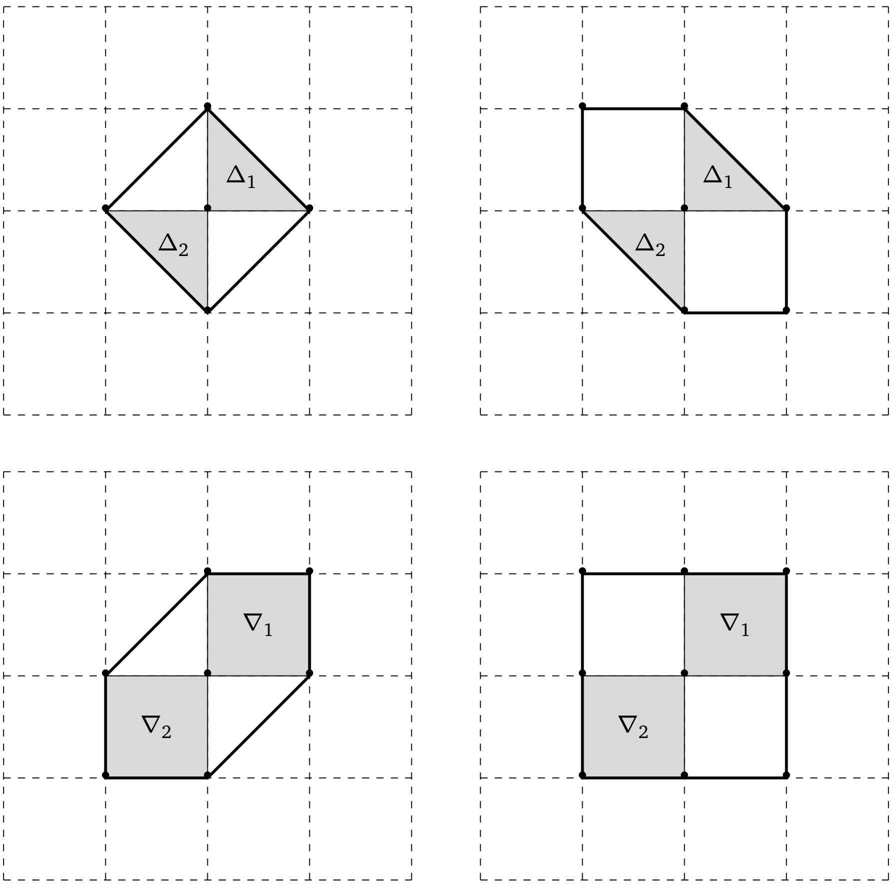

Figure 3: Two-dimensional analogues of the polytopes Δ, ∇, their duals, and their nef-partitions. Clockwise from top-left, we have Conv(Δ_1, Δ_2), Mink(Δ_1, Δ_2), Mink(∇_1, ∇_2), and Conv(∇_1, ∇_2).

Table 4: Quantities associated to the manifolds $\widehat{\text{HV}}$, $\widehat{\text{H}\Lambda}$, HV, and H$\Lambda$.

Quantity	$\widehat{\text{HV}}$	$\widehat{\text{H}\Lambda}$	HV	HΛ
Defining polynomials	P	Q	P^1, P^2	Q^1, Q^2
Polytopes defining monomials	$\widehat{\Delta}$	$\widehat{\Delta}^*$	Δ_1, Δ_2	∇_1, ∇_2
Ambient toric variety	$\mathbb{P}_{\widehat{\Delta}^*}$	$\mathbb{P}_{\widehat{\Delta}}$	$\mathbb{P}_{\Delta^*}$	$\mathbb{P}_{\nabla^*}$
Coordinates	$X_1, \ldots X_5$	$Y_1, \ldots, Y_4$	$X_0, \ldots, X_5$	$Y_0, \ldots, Y_4$

2.4 The Hulek-Verrill manifolds and their mirrors

Having studied the relevant lattice geometry, we are ready to turn to the toric geometry associated to the triangulations of the fans corresponding to the triangulated polytopes that were found in the previous sections. We will give both the singular manifolds $\widehat{\text{HV}}$ and $\widehat{\text{H}\Lambda}$ and their resolutions HV and HΛ as vanishing loci of a set of polynomials inside the relevant ambient toric variety. We also find some basic properties of these manifolds, which will be relevant in the following sections. The quantities associated to each manifold are summarised in Table 4.

The singular Hulek-Verrill manifold $\widehat{\text{HV}}$

The ambient toric variety $\mathbb{P}_{\widehat{\Delta}^*}$ in which $\widehat{\text{HV}}$ can be embedded corresponds to the polytope $\widehat{\Delta}^*$. To the vertices we associate Cox coordinates $\xi_1, \ldots, \xi_{30}$. The ambient variety can then be given by the usual construction as

$$\mathbb{P}_{\widehat{\Delta}^*} = \frac{\mathbb{C}^{30} \setminus F}{(\mathbb{C}^*)^{26}}\,. \tag{2.10}$$

The scalings $(\mathbb{C}^*)^{26}$ correspond to linear relations between the vectors corresponding to the vertices of $\widehat{\Delta}^*$. F is the union of sets given by the simultaneous vanishing of Cox coordinates associated to rays not lying in the same cone. Excising this from $\mathbb{C}^{30}$ prior to quotienting in (2.10) ensures a well-defined toric variety.[8]

To study the Calabi-Yau manifold $\widehat{\text{HV}} \subset \mathbb{P}_{\widehat{\Delta}^*}$, we identify the coordinates $X_1, \ldots, X_4$ with the coordinates $\Xi_1, \ldots, \Xi_4$ on the torus, which we define in terms of Cox coordinates in appendix A. Then the Calabi-Yau manifold can be written as as a subset

$$\left\{ \sum_{i \neq j} a_{i,j} \frac{X_i}{X_j} + a_0 = 0 \right\} \subset \mathbb{P}_{\widehat{\Delta}^*}\,. \tag{2.11}$$

We are chiefly concerned with the five-parameter subfamily

$$a_{i,j} = a_j\,, \quad \text{for all} \quad i \neq j\,,$$

where the polynomial in (2.11) takes the form P given in (1.1). The generic manifold in this family contains 30 nodal singularities on $\widehat{\text{HV}} \setminus \mathbb{T}^4$, which can be seen by considering the local

[8] For technical details that we omit see the textbooks [18, 22], or the more physicist-oriented notes [23].

patches corresponding to the triangulation of the polytope $\widehat{\Delta}^*$ [1]. These singular varieties have resolutions $\mathrm{HV}_{(a_0,\dots,a_5)}$, which are smooth Calabi-Yau manifolds. We will discuss the toric description of these manifolds later in this section, where we show that the resolutions we find using toric methods have the same Hodge numbers and various other properties as the small resolutions studied by Hulek and Verrill. It may be possible to identify these, although we lack a formal proof. However, for the study of mirror symmetry in this paper it is enough to consider manifolds up to birational equivalence. In particular, the periods of the Hulek-Verrill manifold do not depend on the resolution of the subvariety of $\mathbb{T}^5$. Thus, in this paper, we *define* the Hulek-Verrill manifolds as the toric resolutions $\mathrm{HV}_{(a_0,\dots,a_5)}$ described in the following subsections. We then see that these are birational to the small resolutions studied by Hulek and Verrill and that their mirrors are given, via the Batyrev-Borisov construction, as the complete intersection manifolds we denote $\mathrm{H}\Lambda$.

The singular mirror Hulek-Verrill manifold $\widehat{\mathrm{H}\Lambda}$

We can use Batyrev's construction [12] to find the mirror manifolds of the singular Hulek-Verrill manifolds. These are of interest to us since some of the manifolds that concern us turn out to be singular. However, they are birational to the mirror manifolds of the small resolutions mentioned above. The construction of the resolved manifold in this way is somewhat complicated, but in §2.4 we give another method of finding this resolution.

We have already found the vertices of the dual polytope $\widehat{\Delta}^*$ in (2.8). These, together with the interior point, correspond to the monomials

$$1\,,\quad Y_i\,,\quad Y_iY_j\,,\quad Y_iY_jY_k\,,\quad Y_iY_jY_kY_l\,,\quad \frac{1}{Y_i}\,,\quad \frac{1}{Y_iY_j}\,,\quad \frac{1}{Y_iY_jY_k}\,,\quad \frac{1}{Y_iY_jY_kY_l}\,. \tag{2.12}$$

Each of the indices i,j,k,l are distinct and take values in $\{1,2,3,4\}$. The intersection of a generic mirror singular Hulek-Verrill manifold with the torus $\mathbb{T}^4$ is given by the closure of the vanishing locus

$$\widehat{Q} \stackrel{\text{def}}{=} \sum_{i,j,k,l=0}^{2} A_{i,j,k,l}Y_1^iY_2^jY_3^kY_4^l = 0\,. \tag{2.13}$$

One obtains this by taking the most general polynomial with monomials (2.12) and multiplying through by $Y_1Y_2Y_3Y_4$, which gives the same variety on $\mathbb{T}^4$.

Given the triangulation (2.7) of $\widehat{\Delta}$ discussed in §2.1, we can consider the local affine patches $\mathbb{A}_{\sigma_i}$ corresponding to the simplicial cones σ_i. Equivalently, we can choose suitable 4-tuples of the Cox coordinates η_i to act as the local coordinates on patches isomorphic to $\mathbb{A}^4$. It is only necessary to study the six local patches related to the fans given in (2.7) and a single patch generated by any tetrahedron. The other local patches are obtained from these by $\mathbb{Z}_5 \times \mathbb{Z}_2$ symmetry.

As an example, let us consider the cone σ_1. The coordinates associated to the generators of this cone are

$$x \stackrel{\text{def}}{=} \eta_{20}\,,\quad y \stackrel{\text{def}}{=} \eta_{19}\,,\quad z \stackrel{\text{def}}{=} \eta_{18}\,,\quad w \stackrel{\text{def}}{=} \eta_{14}\,.$$

Since the generators corresponding to these coordinates belong to the same simplicial cone, we can set the other coordinates to unity, and thus identify the local coordinates with those on the torus as

$$H^1 = xyz\,, \qquad H^2 = w\,, \qquad H^3 = \frac{1}{wz}\,, \qquad H^4 = \frac{1}{y}\,.$$

We can immediately find the local coordinates on

$$\mathfrak{A}\,\sigma_1 = \big\langle(-1,1,0,0),\ (0,1,0,0),\ (0,1,0,-1),\ (0,0,1,-1)\big\rangle\,,$$

by noting that the $\mathbb{Z}_5$ action on the Cox coordinates inherited from the action on the vertices maps

$$\eta_{20} \mapsto \eta_4 = x\,, \qquad \eta_{19} \mapsto \eta_{16} = y\,, \qquad \eta_{18} \mapsto \eta_{15} = z\,, \qquad \eta_{14} \mapsto \eta_{12} = w\,.$$

The equalities denote identifications with the coordinates on the affine patch $\mathbb{A}_{\mathfrak{A}\sigma_1}$. Thus on this patch, we can make the identifications with the torus coordinates as

$$H_1 = \frac{1}{x}\,, \qquad H_2 = xyz\,, \qquad H_3 = w\,, \qquad H_4 = \frac{1}{wz}\,.$$

Note that this corresponds to $\mathbb{Z}_5$ acting on the global coordinates as

$$H_i \mapsto H_{i+1}\,, \quad i \neq 4\,, \qquad\qquad H_4 \mapsto \frac{1}{H_1 H_2 H_3 H_4}\,,$$

which of course corresponds to the $\mathbb{Z}_5$ action $\mathbf{e}^i \mapsto \mathbf{e}^{i+1}$ of the five-dimensional lattice M, projected down to four dimensions by (2.3).

Writing the polynomial Q in global coordinates gives, for generic values of the moduli, an irreducible multidegree $(2,4,4,4)$ polynomial. A member of this family is generically smooth, but smooth members are not birational to mirrors of Hulek-Verrill manifolds $\mathrm{H}\Lambda_{(a_0,\ldots,a_5)}$.

Instead, it turns out that we must only consider those whose defining polynomials can be written in the form

$$\widehat{Q} = \alpha\,\delta - \beta\,\gamma\,,$$

where α, β, γ, and δ are multidegree $(1,1,1,1)$ polynomials in the coordinates $Y_1, \ldots, Y_4$. A manifold with this property has exactly 24 singularities, which can be resolved in order to obtain a smooth variety.

The Hulek-Verrill manifold HV

As we have already remarked, Hulek and Verrill noted that the singular variety $\widehat{\mathrm{HV}}_{(\varphi^0,\ldots,\varphi^5)}$ defined by the equation

$$\left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5} \frac{\varphi^i}{X_i}\right) = a_0 \tag{2.14}$$

on the toric variety $\mathbb{P}_{\widehat{\Delta}^*}$ is birational to the subvariety of $\mathbb{P}^5$ defined by the two polynomials

$$P^1 \stackrel{\text{def}}{=} \sum_{\mu=0}^{5} X_\mu\,, \qquad P^2 \stackrel{\text{def}}{=} \sum_{\mu=0}^{5} \frac{\varphi^\mu}{X_\mu}\,.$$

It is possible to develop this further by studying the two equations $P^1 = P^2 = 0$ on the torus $\mathbb{T}^5$ and finding the toric closure of this variety. This can be achieved using the techniques reviewed briefly in §2.2. In §2.1, we have studied the polytopes Δ_1 and Δ_2 whose vertices correspond to the monomials in P^1 and P^2, and found the polytope Δ^* which gives the ambient space $\mathbb{P}_{\Delta^*}$. The Cox coordinates and coordinate scalings defining the ambient variety are given in appendix A.

We can analyse this variety further by specialising to various local patches. We only need to analyse the patches that are not related by symmetry.

The Cox coordinates associated to the generators of the cone σ_1 in (2.9) are

$$\xi_1 \overset{\text{def}}{=} x\,, \qquad \xi_5 \overset{\text{def}}{=} y\,, \qquad \xi_{13} \overset{\text{def}}{=} z\,, \qquad \xi_{29} \overset{\text{def}}{=} w\,, \qquad \xi_{61} \overset{\text{def}}{=} v\,.$$

Using the leftover scalings to set the other 57 Cox coordinates ξ to unity, we can identify the invariants $\Xi_1, \ldots, \Xi_5$ as

$$\Xi_1 = xyzwv\,, \qquad \Xi_2 = yzwv\,, \qquad \Xi_3 = zwv\,, \qquad \Xi_4 = wv\,, \qquad \Xi_5 = v\,.$$

By further identifying these Ξ_i with the coordinates X_i on the torus, we can write the polynomials P^1 and P^2 as

$$\begin{aligned} P^1 &= 1 + v + wv + zwv + yzwv + xyzwv\,, \\ P^2 &= \varphi^0 + \frac{\varphi^1}{xyzwv} + \frac{\varphi^2}{yzwv} + \frac{\varphi^3}{zwv} + \frac{\varphi^4}{wv} + \frac{\varphi^5}{v}\,. \end{aligned}$$

The analogous relations for the remaining cones, σ_2 and σ_3, can be found in a similar manner.

By studying the equations $P^1 = P^2 = \mathrm{d}P^1 \wedge \mathrm{d}P^2 = 0$, it is not difficult to see that, generically, the variety HV does not have singularities. As in the original analysis of Hulek and Verrill [1], we find that there are singularities if and only if

$$\Delta \overset{\text{def}}{=} \prod_{\epsilon_i \in \{\pm 1\}} \left(\sqrt{\varphi^0} + \epsilon_1 \sqrt{\varphi^1} + \epsilon_2 \sqrt{\varphi^2} + \epsilon_3 \sqrt{\varphi^3} + \epsilon_4 \sqrt{\varphi^4} + \epsilon_5 \sqrt{\varphi^5} \right) = 0\,. \tag{2.15}$$

The algorithm in [19,20], implemented in PALP [21], gives the Hodge numbers of this variety as

$$h^{p,q} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 45 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 45 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\,.$$

When $\varphi^0 = 1$ and $\varphi^i = \varphi$ for $i \neq 0$, the manifold admits a $\mathbb{Z}_5 \times \mathbb{Z}_2 \subset S_5 \times \mathbb{Z}_2$ symmetry group, which acts freely outside of the singular locus $\Delta = 0$. The actions of $\mathbb{Z}_5$ and $\mathbb{Z}_2$ on the coordinates can be written as

$$\mathfrak{A} : X_i \mapsto X_{i+1}\,, \qquad \mathfrak{B} : X_i \mapsto \frac{1}{X_i}\,, \tag{2.16}$$

with the indices understood mod 5. The Hodge numbers of the varieties obtained by taking the quotients are given in Table 5.

Table 5: The Hodge numbers h^{11} and h^{12} for the free quotients of HV.

Manifold	HV	HV/$\mathbb{Z}_5$	HV/$\mathbb{Z}_{10}$
(h^{11},h^{12})	(45,5)	(9,1)	(5,1)

The mirror Hulek-Verrill manifold HΛ

The mirror Hulek-Verrill manifold can be defined as the vanishing locus of two polynomials corresponding to the polytopes ∇_1 and ∇_2 inside the ambient variety $\mathbb{P}_{\nabla^*}$ associated to the triangulated polytope ∇^*.

The monomials associated to the vertices of ∇_1 are

$$1\,,\qquad Y_i\,,\qquad Y_iY_j\,,\qquad Y_iY_jY_k\,,\qquad Y_iY_jY_kY_l\,,\qquad Y_iY_jY_kY_lY_m\,,$$

with distinct indices understood to take distinct values. The monomials associated to ∇_2 are simply the inverses of these.

$$1\,,\qquad \frac{1}{Y_i}\,,\qquad \frac{1}{Y_iY_j}\,,\qquad \frac{1}{Y_iY_jY_k}\,,\qquad \frac{1}{Y_iY_jY_kY_l}\,,\qquad \frac{1}{Y_iY_jY_kY_lY_m}\,.$$

Looking at the vertices of ∇^* listed in appendix A, we see that the ambient variety $\mathbb{P}_{\nabla^*}$ is nothing but the product $(\mathbb{P}^1)^5$. The Cox coordinates are the homogeneous coordinates on each $\mathbb{P}^1$, which we often denote by $\mathbb{P}^1_i$ with $i=0,\ldots,4$ if there is a need to distinguish between different factors in the product $(\mathbb{P}^1)^5$. The coordinates Y_i on the torus are identified with the affine coordinates

$$Y_i=\frac{Y_{i,1}}{Y_{i,0}}\,,$$

with $[Y_{i,0}:Y_{i,1}]$ giving the homogeneous coordinates on $\mathbb{P}^1_i$. It is convenient to intoduce the following monomials of homogeneous coordinates

$$M_{abcde}\;=\;Y_{1,a}Y_{2,b}Y_{3,c}Y_{4,d}Y_{5,e}\,,$$

where $a,b,c,d,e\in\{0,1\}$. Using these, the most general polynomials associated to ∇_1 and ∇_2 can be written as

$$Q^1=\sum_{a,b,c,d,e}A_{abcde}M_{abcde}\,,\qquad Q^2=\sum_{a,b,c,d,e}B_{abcde}M_{abcde}\,. \tag{2.17}$$

For a special choice of coefficients A and B, the simultaneous vanishing locus of Q^1 and Q^2 admits $\mathbb{Z}_5$, $\mathbb{Z}_5\times\mathbb{Z}_2$ or $\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry [9]. These act freely, and thus can be used to obtain smooth quotient manifolds. Denoting the generator of the $\mathbb{Z}_5$ as S, the generator of the first $\mathbb{Z}_2$ as U and the second $\mathbb{Z}_2$ as V, we can take the symmetry transformations to act on the coordinates as

$$S\,:\,Y_{i,a}\mapsto Y_{i+1,a}\,,\qquad U\,:\,Y_{i,a}\mapsto(-1)^aY_{i,a}\,,\qquad V\,:\,Y_{i,0}\longleftrightarrow Y_{i,1}\,, \tag{2.18}$$

where addition is again understood modulo 5. The symmetries S and V can be seen to descend from the $\mathbb{Z}_5$ and $\mathbb{Z}_2$ symmetries acting on the polytope ∇^*. To write down the polynomials invariant under there symmetries, it is convenient to introduce the $\mathbb{Z}_5$ invariant combinations of the monomials M_{abcde},

$$m_{abcde} = \sum_{i=1}^{5} Y_{i,a} Y_{i+1,b} Y_{i+2,c} Y_{i+3,d} Y_{i+4,e} \,.$$

The polynomials defining the $\mathbb{Z}_5$ symmetric manifolds can be found by specialising the coefficients A and B so that the vanishing locus $Q^1 = Q^2 = 0$ is invariant under $\mathbb{Z}_5$, or equivalently by finding the $\mathbb{Z}_5$ orbits of Q^1 and Q^2. In this manner, we find

$$\begin{aligned} Q^1 &= A_{00000}\, m_{00000} + A_{10000}\, m_{10000} + A_{11000}\, m_{11000} + A_{10100}\, m_{10100} + A_{11100}\, m_{11100} \\ &\qquad + A_{11010}\, m_{11010} + A_{11110}\, m_{11110} + A_{11111}\, m_{11111}\,, \\ Q^2 &= B_{00000}\, m_{00000} + B_{10000}\, m_{10000} + B_{11000}\, m_{11000} + B_{10100}\, m_{10100} + B_{11100}\, m_{11100} \\ &\qquad + B_{11010}\, m_{11010} + B_{11110}\, m_{11110} + B_{11111}\, m_{11111}\,. \end{aligned} \tag{2.19}$$

To find the defining polynomials in the $\mathbb{Z}_5 \times \mathbb{Z}_2$ symmetric case, we can further demand that the vanishing locus of the polynomials is invariant under the $\mathbb{Z}_2$ generated by V, which gives us two polynomials of the form

$$\begin{aligned} Q^1 &= A_0\, m_{00000} + A_1\, m_{10000} + A_2\, m_{11000} + A_3\, m_{10100} + A_4\, m_{11100} + A_5\, m_{11010} + A_6\, m_{11110} + A_7\, m_{11111}\,, \\ Q^2 &= A_0\, m_{11111} + A_1\, m_{11110} + A_2\, m_{11100} + A_3\, m_{11010} + A_4\, m_{11000} + A_5\, m_{10100} + A_6\, m_{10000} + A_7\, m_{00000}\,. \end{aligned}$$

Alternatively, we can demand that the vanishing locus is invariant under the second $\mathbb{Z}_2$ generated by U. In this case, the polynomials can be written as

$$\begin{aligned} Q^1 &= A_0\, m_{00000} + A_1\, m_{11000} + A_2\, m_{10100} + A_3\, m_{11110}\,, \\ Q^2 &= B_0\, m_{11111} + B_1\, m_{11100} + B_2\, m_{11010} + B_3\, m_{10010}\,. \end{aligned}$$

Note that the actions of U and V are exchanged under a suitable redefinition of coordinates, and therefore we can choose either of these two forms for the polynomials defining the $\mathbb{Z}_5 \times \mathbb{Z}_2$ invariant variety. Note also that in the latter case the polynomials Q^1 and Q^2 are not each $\mathbb{Z}_2$ invariant, but instead are mapped to each other under the action on $\mathbb{Z}_2$, thus keeping their mutual vanishing locus invariant.

Finally, we can consider the variety invariant under the full $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we can write the defining polynomials as

$$\begin{aligned} Q^1 &= \frac{A_0}{5}\, m_{00000} + A_1\, m_{11000} + A_2\, m_{10100} + A_3\, m_{11110}\,, \\ Q^2 &= \frac{A_0}{5}\, m_{11111} + A_1\, m_{11100} + A_2\, m_{11010} + A_3\, m_{10010}\,. \end{aligned} \tag{2.20}$$

It turns out that the varieties defined in this way and their quotients under their respective symmetry groups are smooth Calabi-Yau manifolds, which we can identify as mirror manifolds of the five-parameter family $\mathrm{HV}_{(a_0,\dots,a_5)}$. We call these *mirror Hulek-Verrill manifolds* $\mathrm{H}\Lambda$. The Hodge number of the corresponding quotient varieties were already found in [9]. We reproduce these in Table 6.

Counting the parameters in the polynomials seems naïvely to produce too many parameters compared to the Hodge numbers. However, on taking into account rescalings; remaining automorphisms of the ambient variety $(\mathbb{P}^1)^5$; and $\mathrm{SL}(2,\mathbb{C})$ transformation of the polynomials,

Table 6: The Hodge numbers h^{11} and h^{12} for the different free quotients that $\text{H}\Lambda$ allows.

Manifold	$\text{H}\Lambda$	$\text{H}\Lambda/\mathbb{Z}_2$	$\text{H}\Lambda/\mathbb{Z}_2\times\mathbb{Z}_2$	$\text{H}\Lambda/\mathbb{Z}_5$	$\text{H}\Lambda/\mathbb{Z}_5\times\mathbb{Z}_2$	$\text{H}\Lambda/\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$
(h^{11},h^{12})	(5,45)	(5,25)	(5,15)	(1,9)	(1,5)	(1,3)

we find that the number of free parameters in the defining polynomials agrees with the Hodge numbers. We leave the details to appendix C.

Finally, we note that this variety is birational to the singular $\widehat{\text{H}\Lambda}$. This is most easily seen by observing that the intersection $\widehat{\text{H}\Lambda}\cap\mathbb{T}^4$ can be obtained from $\text{H}\Lambda$ by blowing up a suitable set of degree-1 rational lines, as we will discuss in detail in §7.

3 The periods of Hulek-Verrill manifolds

The periods of the HV manifold are essential for understanding both the geometry and physics of the Hulek-Verrill manifolds as well as their mirrors. The series expansions of periods about large complex structure points allow for a mirror-symmetry computation of the instanton numbers for the manifold $\text{H}\Lambda$. In this section, we derive series expressions that we utilise to perform this computation in §4. Additionally, the periods as functions of the complex structure moduli of HV are instrumental in describing string theory compactifications on HV. We hope to return to this point in future work, to study flux vacua in type IIB string theory compactified on HV.

Our approach begins with investigating some differential equations satisfied by the fundamental period ϖ^0, which is long known to admit concise descriptions [1, 14]. We find a set of PDEs which, together with asymptotic data coming from mirror-symmetry considerations, allow us to find all periods within the large complex structure regions of moduli space. We go further by using the methods of [14] to study an ODE satisfied by the fundamental, and indeed all, periods. This latter equation is used to analytically continue the periods, and with the data we obtain from this, we can give expressions for the periods in all regions of moduli space.

We derive also formulae that express all periods using integrals of products of Bessel functions. To our knowledge, this is the first appearance of such expressions and we anticipate that these also have applications in the study of banana graph amplitudes. For instance, the expansion (4.16) of [2] expresses the full non-equal mass 4-loop banana integral in the large momentum region of parameter space, where the simplest available expression (their equation (2.10)) does not converge. The authors gave the first few terms of the series expansions of the functions that are used as a basis. The integral expressions that we use to describe the periods also fit this purpose after a change of basis. Appropriate generalisations of our expressions relevant to higher-dimensional Hulek-Verrill manifolds will perform the same task for higher-loop banana diagrams.

3.1 Moduli space

The parameters $\varphi^0, \cdots, \varphi^5$ in the equation (2.2) defining the manifold HV constitute a set of projective coordinates for $\mathbb{P}^5$. The parameters $\varphi^0, \cdots, \varphi^5$ appear symmetrically, which we can use to great effect to describe different regions in the moduli space. A convenient atlas for $\mathbb{P}^5$ is given by the six sets where one of the projective coordinates is nonvanishing. In the following sections, we mostly work in the patch where $\varphi^0 \neq 0$, but the arguments go through in the other five patches mutatis mutandis. Accordingly, the Latin subscripts $i, j, k, \ldots$ are always understood to run from 1 to 5, whereas the Greek subscripts $\mu, \nu, \lambda, \ldots$ are taken to run from 0 to 5.

It can be seen that the manifold HV is singular on the locus

$$E \;=\; \varphi^0 \varphi^1 \varphi^2 \varphi^3 \varphi^4 \varphi^5 \;=\; 0\,. \tag{3.1}$$

We denote the irreducible components in this locus by

$$E_\mu \;=\; \left\{ \left(\varphi^0, \varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^5\right) \in \mathbb{P}^5 \,\middle|\, \varphi^\mu = 0 \right\}. \tag{3.2}$$

The intersections of 5 of these hypersurfaces turn out to be large complex structure points, or points of maximal unipotent monodromy, as we will verify in §4 by computing the monodromies around these hypersurfaces explicitly.

As we have reviewed earlier in §2.4, the Hulek-Verrill manifold has conifold singularities on the locus

$$\Delta \overset{\text{def}}{=} \prod_{\epsilon_i \in \{\pm 1\}} \left(\sqrt{\varphi^0} + \epsilon_1 \sqrt{\varphi^1} + \epsilon_2 \sqrt{\varphi^2} + \epsilon_3 \sqrt{\varphi^3} + \epsilon_4 \sqrt{\varphi^4} + \epsilon_5 \sqrt{\varphi^5} \right) \;=\; 0\,. \tag{3.3}$$

It is often useful to consider the square roots $\sqrt{\varphi^i}$ as coordinates on the moduli space. This of course gives rise to a multiple cover. We can, to start, choose branches for the square roots with $\mathrm{Re}[\sqrt{\varphi^i}] > 0$. The functions that we study are related to those in other branches via monodromy transformations $\varphi^i \mapsto \mathrm{e}^{2\pi\mathrm{i}} \varphi^i$ around the large complex structure point.

In the coordinates $\sqrt{\varphi^i}$ it is convenient to study the vanishing loci of the individual factors in Δ. Let I be a subset of indices in $\{0, \ldots, 5\}$ and I^c be its complement in $\{0, \ldots, 5\}$. Then we define the following closed components D_I corresponding to each set I, sketched in Figure 4:

$$D_I \;=\; \left\{ (\varphi^0, \cdots, \varphi^5) \in \mathbb{P}^5 \,\middle|\, \sum_{i \in I} \mathrm{Re}\left[\sqrt{\varphi^i}\,\right] = \sum_{j \in I^c} \mathrm{Re}\left[\sqrt{\varphi^j}\,\right] \right\}. \tag{3.4}$$

3.2 The fundamental period

The holomorphic period for $\mathrm{HV}_{(\varphi^0, \ldots, \varphi^5)}$ can be found by integrating the holomorphic three form over the torus. We briefly review this procedure. As we consider the torus, we can use the equation (2.14) defining $\widehat{\mathrm{H}\Lambda}$ in order to obtain this period by the Dwork-Katz-Griffiths method [24].

$$\varpi^{(0);0}(\varphi) = -\frac{\varphi^0}{(2\pi\mathrm{i})^5} \int \prod_{i=1}^5 \frac{\mathrm{d}X_i}{X_i} \frac{1}{P(\mathbf{X};\varphi)} = -\frac{\varphi^0}{(2\pi\mathrm{i})^5} \int \prod_{i=1}^5 \frac{\mathrm{d}X_i}{X_i} \left[\sum_{i=1}^5 \frac{\varphi^i}{X_i} \sum_{j=1}^5 X_j - \varphi^0 \right]^{-1}.$$

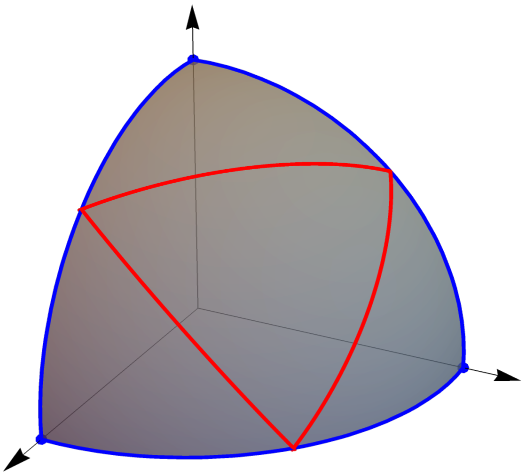

Figure 4: A heuristic sketch of the moduli space in coordinates $\sqrt{\varphi^\mu}$, with the branch choice of $\sqrt{\varphi^i} > 0$. The grey shell is the moduli space $\mathbb{P}^2$, the red lines represent the irreducible components $D_{\{\mu\}}$ of the discriminant locus $\Delta = 0$, the blue lines are the loci E_μ, and the blue points are the large complex structure points. The four triangular regions between these lines correspond to the sets $U_{\{\mu\}}$ and $U_{\{0,1\}} \cap U_{\{0,2\}} \cap U_{\{1,2\}}$, which we define in (3.21).

The first superscript (0) is to remind us that we are working in the coordinate patch of $\mathbb{P}^5$ where $\varphi^0 = 1$. The second superscript 0 indicates that this is the holomorphic period near the large complex structure point at $\varphi^1 = \varphi^2 = \cdots = \varphi^5 = 0$, where one finds the series expansion

$$\begin{aligned}
\varpi^{(0);0}(\varphi) &= \sum_{n=0}^{\infty} \frac{(\varphi^0)^{-n}}{(2\pi \mathrm{i})^5} \int \prod_{i=1}^{5} \frac{\mathrm{d}X_i}{X_i} \left(\sum_{i=1}^{5} \frac{\varphi^i}{X_i} \sum_{j=1}^{5} X_j \right)^n \\
&= \sum_{n=0}^{\infty} \frac{(\varphi^0)^{-n}}{(2\pi \mathrm{i})^5} \sum_{\deg(\boldsymbol{p})=n} \sum_{\deg(\boldsymbol{q})=n} \binom{n}{\boldsymbol{p}} \binom{n}{\boldsymbol{q}} \prod_{i=1}^{5} \int \frac{\mathrm{d}X_i}{X_i} \boldsymbol{X}^{\boldsymbol{p}-\boldsymbol{q}} \, \boldsymbol{\varphi}^{\boldsymbol{q}} \\
&= \sum_{n=0}^{\infty} (\varphi^0)^{-n} \sum_{\deg(\boldsymbol{p})=n} \binom{n}{\boldsymbol{p}}^2 \boldsymbol{\varphi}^{\boldsymbol{p}} \\
&\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} c_n(\boldsymbol{\varphi}) (\varphi^0)^{-n} .
\end{aligned} \tag{3.5}$$

We will next identify a set of differential operators that annihilate this fundamental period, the expectation being that the other periods should satisfy the same equations. Although this set of equations is demonstrably *not* the full Picard-Fuchs system, we can proceed using the high degree of symmetry and the asymptotics for the periods found from mirror symmetry considerations. In this way, we are able to find expressions for the periods using the 32 solutions to this partial Picard-Fuchs system. As a very non-trivial check, we are able to compute several genus-0 instanton numbers in §4, the first few of which match the numbers that we find from geometric arguments in §7.

In principle one could also obtain the periods and the full Picard-Fuchs system from the toric data using the methods described in [25, 26] to derive the Gel'fand-Kapranov-Zelevinski system satisfied by the fundamental period (3.5). This system could then be factored to obtain the Picard-Fuchs system. However, we have shown that it is possible to find the periods with a partial Picard-Fuchs system together with suitable boundary conditions, and therefore we do not need to use this more cumbersome procedure here.

3.3 The ordinary differential equation obeyed by the fundamental period

Consider the sequence of c_n which gives the coefficients in the series (3.5). In principle, one could use a recurrence relation that c_n satisfies in order to write an ODE — containing derivatives only with respect to φ^0, but coefficients functions of all φ^μ — which is satisfied by the fundamental period. Such recurrence relations which themselves depend on the φ^i were studied by Verrill in [14], wherein a method for determining such a recurrence was given. It was shown that c_n is a holonomic sequence, solving a linear recurrence with polynomial coefficients.

Unfortunately, the fully general case with all φ^i set to indeterminates is not amenable to a computer analysis as the rational functions of the φ^i that appear are prohibitively large. Nonetheless, finding this recurrence for fixed values of φ^i is possible with the methods of [14]. Although this recurrence can be used to obtain a differential equation annihilating ϖ^0, this is not generally of minimal order.[9] We get around this by using the recurrence relation to generate a large number of terms in the series efficiently, and then use these to fix a lower-degree differential equation. One example we will use later is given by the line $(\varphi^0, \ldots, \varphi^5) = (1, \varphi, \varphi/20, \ldots, \varphi/20)$, where the differential operator takes the form

$$\mathcal{L}^{(6)} = S_6\,\theta^6 + S_5\,\theta^5 + S_4\,\theta^4 + S_3\,\theta^3 + S_2\,\theta^2 + S_1\,\theta + S_0\,, \tag{3.6}$$

with coefficients polynomials S_n of degree 11.

$$\begin{aligned}
S_6 = {} & -4393216\,\varphi^{11}+367906816\,\varphi^{10}+2766668800\,\varphi^9-39077007900\,\varphi^8+206484873000\,\varphi^7-612252422500\,\varphi^6 \\
& +898848500000\,\varphi^5-698473812500\,\varphi^4+301613125000\,\varphi^3-63023437500\,\varphi^2+1968750000\,\varphi+781250000\,, \\
S_5 = {} & -52718592\,\varphi^{11}+2701502528\,\varphi^{10}+35940053200\,\varphi^9-311032483500\,\varphi^8+1552596065500\,\varphi^7-3847452445000\,\varphi^6 \\
& +3932465125000\,\varphi^5-1862764937500\,\varphi^4+296554687500\,\varphi^3+111468750000\,\varphi^2-33250000000\,\varphi-1562500000\,, \\
S_4 = {} & -254806528\,\varphi^{11}+7499038076\,\varphi^{10}+150742085265\,\varphi^9-1014941685775\,\varphi^8+5431523295000\,\varphi^7-11316503848750\,\varphi^6 \\
& +9307004090625\,\varphi^5-4235035421875\,\varphi^4+739773593750\,\varphi^3+87307812500\,\varphi^2+10281250000\,\varphi+781250000\,, \\
S_3 = {} & -632623104\,\varphi^{11}+9348961064\,\varphi^{10}+303965630550\,\varphi^9-1813508252350\,\varphi^8+10679775875000\,\varphi^7-18511281897500\,\varphi^6 \\
& +13297668268750\,\varphi^5-5361295718750\,\varphi^4+157421562500\,\varphi^3+235818750000\,\varphi^2-5250000000\,\varphi\,, \\
S_2 = {} & -847890688\,\varphi^{11}+4174802636\,\varphi^{10}+326073152765\,\varphi^9-1845417676975\,\varphi^8+11974702116500\,\varphi^7-17568183998750\,\varphi^6 \\
& +11730618440625\,\varphi^5-4223423609375\,\varphi^4-204717031250\,\varphi^3+170156250000\,\varphi^2-937500000\,\varphi\,, \\
S_1 = {} & -579904512\,\varphi^{11}-1001197360\,\varphi^{10}+180117501740\,\varphi^9-1000845945900\,\varphi^8+7134958504500\,\varphi^7-9085888402500\,\varphi^6 \\
& +5823431700000\,\varphi^5-1860707500000\,\varphi^4-250435000000\,\varphi^3+61875000000\,\varphi^2\,, \\
S_0 = {} & -158155776\,\varphi^{11}-992481296\,\varphi^{10}+40441278660\,\varphi^9-224468019900\,\varphi^8+1746333878500\,\varphi^7-1982812512500\,\varphi^6 \\
& +1243608875000\,\varphi^5-349708500000\,\varphi^4-79745000000\,\varphi^3+9250000000\,\varphi^2\,.
\end{aligned}$$

[9] The recurrence provided by this method is of lowest possible order, but without any constraint on the degree of the polynomial coefficients therein. This leads to extraneous factors in the differential equation provided.

3.4 Partial differential equations obeyed by the fundamental period

We adopt the following notation:

$$\partial_i = \frac{\partial}{\partial \varphi^i}, \qquad \theta_i = \varphi^i \partial_i, \qquad \Theta = \sum_{i=1}^{5} \theta_i. \tag{3.7}$$

Note that on a single term $\boldsymbol{\varphi}^{\boldsymbol{p}} (\varphi^0)^{-n}$, where $\deg(\boldsymbol{p}) = n$, the action of the operator Θ is the same as that of $-\varphi^0 \partial_0$. Using this fact, we find that the fundamental period ϖ^0 obeys the following five differential equations:

$$\mathcal{L}_i \varpi^0(\boldsymbol{\varphi}) \stackrel{\text{def}}{=} \left(\frac{1}{\varphi^0} (\Theta+1)^2 - \frac{1}{\varphi^i} \theta_i^2 \right) \varpi^0(\boldsymbol{\varphi}) = 0. \tag{3.8}$$

These equations are, after a change of variables, equivalent to the differential equations (4.8) of [2]. In addition, we have equations obtained by taking differences of the above equations, or by directly inspecting (1.10):

$$\mathcal{L}_{i,j} \varpi^0(\boldsymbol{\varphi}) \stackrel{\text{def}}{=} \left(\frac{1}{\varphi^i} \theta_i^2 - \frac{1}{\varphi^j} \theta_j^2 \right) \varpi^0(\boldsymbol{\varphi}) = 0. \tag{3.9}$$

These equations (3.9) are manifestly separable, which is suggestive of a route to the other periods.

3.5 The Frobenius method

We prove that the fundamental period $\varpi^0(\boldsymbol{\varphi})$ is (up to scale) the sole holomorphic power series solution to (3.8). Make the projective choice $\varphi^0 = 1$, and then suppose that one has a series

$$f(\boldsymbol{\varphi}) = \sum_{\boldsymbol{p}} f_{\boldsymbol{p}} \, \boldsymbol{\varphi}^{\boldsymbol{p}},$$

that solves (3.8). After comparing powers of φ^i in $\mathcal{L}_1 f(\boldsymbol{\varphi}) = 0$, one obtains

$$p_1^2 f_{(p_1,p_2,p_3,p_4,p_5)} = (p_1+p_2+p_3+p_4+p_5)^2 f_{(p_1-1,p_2,p_3,p_4,p_5)}, \qquad n \geq 1. \tag{3.10}$$

There is a similar relation obtained from the other four equations $\mathcal{L}_i f(\boldsymbol{\varphi}) = 0$. Together these five relations (3.10) require

$$f_{\boldsymbol{p}} = \left(\frac{(p_1+p_2+p_3+p_4+p_5)!}{p_1!p_2!p_3!p_4!p_5!} \right)^2 f_{\boldsymbol{0}}. \tag{3.11}$$

While the system of equations $\mathcal{L}_i F = 0$ has a unique holomorphic solution, it is shown below that the system has a solution-space of dimension greater than $\dim H^3(\text{HV}) = 12$. Therefore it cannot be the entire Picard-Fuchs system, since it is not sufficiently constrained. The solution space can be suitably restricted by the differential equation discussed in §3.3, but this is too difficult to write down in full generality. Perhaps more simply, one can appeal to the method of Frobenius and the homological interpretation discussed in §3.8. Equivalently, one can fix the system of 12 periods by imposing boundary conditions consistent with mirror symmetry.

The Frobenius method reveals that there are 32 functions, sums of power series multiplied by logarithms of the φ^i, which solve (3.8). To see this, one sets up an indicial equation. With $f_{\boldsymbol{p}}$ from (3.11) and $f_{\boldsymbol{0}} = 1$, take a solution ansatz

$$\varpi^{\boldsymbol{\epsilon}}(\boldsymbol{\varphi}) = \sum_{\boldsymbol{p}} \frac{f_{\boldsymbol{p}+\boldsymbol{\epsilon}}}{f_{\boldsymbol{\epsilon}}} \, \boldsymbol{\varphi}^{\boldsymbol{p}+\boldsymbol{\epsilon}}, \tag{3.12}$$

where the $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_5)$ is a five-component multi-index consisting of as-yet undetermined algebra elements, which we refer to here as Frobenius elements, and $f_{\mathbf{p}+\boldsymbol{\epsilon}}$ is defined by replacing $x!$ by $\Gamma(1+x)$ in (3.11). One can compute

$$\mathcal{L}_i\, \varpi^{\boldsymbol{\epsilon}}(\varphi) \;=\; -\frac{\epsilon_i^2}{\varphi^i}\Big(1+\mathcal{O}(\varphi)\Big)+\mathcal{O}(\epsilon_i^3)\,. \tag{3.13}$$

We recover the original series solution $\varpi^0(\varphi)$ by taking $\boldsymbol{\epsilon} \to \mathbf{0}$. Additionally, there are new solutions obtained by first differentiating $\varpi^{\boldsymbol{\epsilon}}(\varphi)$ once with respect to any number of the ϵ_i and then taking $\boldsymbol{\epsilon} \to \mathbf{0}$. Each such derivative introduces a logarithmic dependence on the φ^i. There are five ϵ_i with respect to which we can either differentiate zero or one times. In total all such choices give us $2^5 = 32$ independent solutions.

The Taylor expansion of $\varpi^{\boldsymbol{\epsilon}}$ in $\boldsymbol{\epsilon}$ is infinite, but only terms not containing the square of an ϵ_i have coefficients that are solutions of (3.8). Introduce the algebraic relation $\epsilon_i^2 = 0$ for the ϵ_i. The Taylor expansion of $\varpi^{\boldsymbol{\epsilon}}$ then truncates, and every coefficient left is a solution to (3.8).

$$\varpi^{\boldsymbol{\epsilon}} = \varpi^0 + \varpi^i\epsilon_i + \frac{1}{2!}\varpi^{ij}\epsilon_i\epsilon_j + \frac{1}{3!}\varpi^{ijk}\epsilon_i\epsilon_j\epsilon_k + \frac{1}{4!}\varpi^{ijkl}\epsilon_i\epsilon_j\epsilon_k\epsilon_l + \frac{1}{5!}\varpi^{ijklm}\epsilon_i\epsilon_j\epsilon_k\epsilon_l\epsilon_m\,. \tag{3.14}$$

In the above implicit sums, the terms with any two indices equal are absent. To obtain the 12 periods among these solutions, we will later in §3.8 impose that ϵ_i satisfy cohomology relations. Equivalently, we can impose boundary conditions consistent with mirror symmetry. Explicit expressions for the coefficients in the above series are obtained by differentiating (3.12) and then taking $\epsilon_i \to 0$.

3.6 Separation of variables

Upon expanding the operators θ_i, the differential equations $\mathcal{L}_{i,j}F = 0$ become

$$\Big[\partial_i - \partial_j + \varphi^i\partial_i^2 - \varphi^j\partial_j^2\Big]F \;=\; 0\,. \tag{3.15}$$

Making a separation-of-variables ansatz $F(\varphi) = \prod_{j=1}^5 G_j(\varphi^j)$ and simplifying $\mathcal{L}_{i,j}F = 0$, one obtains

$$\frac{\partial_i G_i(\varphi^i) + \varphi^i\,\partial_i^2 G_i(\varphi^i)}{G_i(\varphi^i)} \;=\; \frac{\partial_j G_j(\varphi^j) + \varphi^j\,\partial_j^2 G_j(\varphi^j)}{G_j(\varphi^j)}\,.$$

Employ the traditional separation of variables logic: both sides of this equation respectively depend only on φ^i and φ^j, and so both must equal a constant. With a certain prescience, we denote this constant by $z^2/4$. Attention should then be turned to the ordinary differential equation that the G_i satisfy:

$$x\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}G(x) + \frac{\mathrm{d}}{\mathrm{d}x}G(x) \;=\; \frac{z^2}{4}G(x)\,.$$

This has the following general solution:

$$G(x) \;=\; C_1(z)I_0(z\sqrt{x}) + C_2(z)K_0(z\sqrt{x})\,.$$

$C_1(z)$ and $C_2(z)$ are arbitrary functions of the parameter z. Therefore, for any choice of the functions $C_1(z)$, $C_2(z)$, the equations $\mathcal{L}_{i,j}F = 0$ for $i,j = 1,\ldots,5$ have solutions of the form

$$F(\varphi) \;=\; \int \mathrm{d}z\, C(z)\prod_{j=1}^{5} B_j\Big(z\sqrt{\varphi^j}\Big)\,, \tag{3.16}$$

where the five functions B_j are each taken to be modified Bessel functions I_0 or K_0. This brings us closer to the periods, but at this stage of our reasoning, only looking at the system $\mathcal{L}_{i,j}F = 0$, there is still a considerable degree of ignorance as to what the function C should be and which combinations of these solutions we should take to give the periods.

In the regime

$$\mathrm{Re}\left[\sum_{i=1}^{5}\sqrt{\varphi^i}\right] < \mathrm{Re}\left[\sqrt{\varphi^0}\right],$$

the following expression for the fundamental period is valid:

$$\varpi^{(0);0}(\varphi) \,=\, \varphi^0\int_0^\infty \mathrm{d}z\, z\, K_0\left(\sqrt{\varphi^0}z\right)\prod_{i=1}^{5} I_0\left(\sqrt{\varphi^i}z\right). \tag{3.17}$$

We give a proof of this claim in appendix B. The identity (3.17) suggests that $C(z)$ should be taken to be $K_0(\sqrt{\varphi^0}z)$. Indeed, by replacing the I_0 functions in the above integral with K_0 functions, we can form 32 functions f that obey the equations $\mathcal{L}_{i,j}f = 0$. These 32 functions can be seen to satisfy the system $\mathcal{L}_i f = 0$, and therefore must furnish a basis of series solutions of the system $\mathcal{L}_i f = \mathcal{L}_{i,j} f = 0$. To be sure, the 32 functions obtained in this way have powers series that form a basis for the linear span of the 32 Frobenius solutions given by the construction in §3.5.

On symmetry grounds, there will be a role for functions obtained by replacing the K_0 with an I_0 in patches $\varphi^i = 1$ in the moduli space. The reason for this is that, from the global perspective, φ^0 is not distinguished from the φ^i.

3.7 Determining closed form expressions for all periods

We have seen that the partial Picard-Fuchs system given by (3.9) and (3.8) should have exactly 32 solutions. Furthermore, we have seen that the integrals of Bessel functions of the form

$$\frac{\varphi^0}{\mathrm{i}\pi}\int_0^\infty \mathrm{d}z\, z\, B_0(\sqrt{\varphi^0}z)\prod_{i=1}^{5} B_i\left(\sqrt{\varphi^i}z\right) \tag{3.18}$$

furnish a set of solutions to our partial differential equations. The $B_\mu(\sqrt{\varphi^\nu}\; z)$ above are replaced by a conveniently normalised modified Bessel function: either $K_0(\sqrt{\varphi^\mu}\; z)$ or $\mathrm{i}\pi\; I_0(\sqrt{\varphi^\mu}\; z)$. Naïvely it seems that this would give us 64 solutions. However, not all of these converge simultaneously. Indeed, an integral of this form converges in the region of the moduli space where

$$\sum_{\mu=0}^{5}\epsilon_\mu \mathrm{Re}\left[\sqrt{\varphi^\mu}\right] < 0, \tag{3.19}$$

where $\epsilon_\mu = \pm 1$, and the negative sign is chosen when $B_0(z\sqrt{\varphi^\mu}) = K_0(z\sqrt{\varphi^\upsilon})$, and the positive sign when $B_0(z\sqrt{\varphi^\mu}) = \mathrm{i}\pi I_0(z\sqrt{\varphi^\mu})$. This follows from demanding that the product of Bessel functions decays exponentially in the limit $z \to \infty$ and recalling the asymptotics of the Bessel functions for large z:

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}, \qquad \mathrm{i}\pi I_0(z) \sim \mathrm{i}\sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{z}.$$

The boundary between the different regions of convergence is exactly the restriction of the conifold locus $\Delta = 0$ to the real plane.

On a generic[10] point in the moduli space corresponding to a non-singular manifold, there are exactly 32 convergent integrals of Bessel functions of the form (3.18). This is seen as follows: every curve of the form

$$\mathrm{Re}\Big[\sqrt{\varphi^0}\Big] + \sum_{i=1}^{5} \epsilon_i \mathrm{Re}\Big[\sqrt{\varphi^i}\Big] = 0 \tag{3.20}$$

divides the space into two regions, those 'above' and 'below'. The curve itself belongs to the discriminant locus. There is exactly one Bessel function integral of the form (3.18) that converges almost everywhere above the curve (3.20) and exactly one converging almost everywhere below the curve. As there are 32 such curves we find exactly 32 convergent integrals at any given point. We can find an almost[11] open covering, where every open subset of the covering corresponds to a different set of Bessel functions.

We can express these covering sets as intersections of suitably-defined sets U_I. Let I be a set of indices in $\{0,\ldots,5\}$ and I^c be its complement in $\{0,\ldots,5\}$. Then we define open sets in the moduli space corresponding to each set I:

$$U_I = \left\{ (\varphi^0, \cdots, \varphi^5) \in \mathbb{P}^5 \;\middle|\; \sum_{i\in I} \mathrm{Re}\Big[\sqrt{\varphi^i}\Big] > \sum_{j\in I^c} \mathrm{Re}\Big[\sqrt{\varphi^j}\Big] \right\}. \tag{3.21}$$

These have the following convenient properties

$$U_I \subset U_J \quad \text{if} \quad J \subset I \subset \{0,\ldots,5\}, \qquad\qquad U_I^c = U_{I^c} \setminus \mathrm{Re}\Delta,$$

where $\mathrm{Re}\Delta$ denotes the space of all points that satisfy any of the equations (3.20).

In the subset of each patch where they converge, these Bessel function integrals satisfy the partial differential equations (3.9) and (3.8). There are exactly 32 solutions to these equations, so it follows that the periods, which should solve the differential equations, can be expressed in terms of the convergent Bessel function integrals in any patch. In the next subsection we will present an argument, based on known asymptotics, to fix the periods as sums of these Bessel integrals in the regions $U_{\{i\}}$ and $U_{\{0\}}$. To find the correct linear combinations of these integrals to give the periods in other regions we study the ODE of §3.3. Choosing values $\varphi^i = s_i \varphi$ in this ODE gives a differential equation that the restrictions of the periods to these lines must satisfy. Given enough lines, we can always find enough equations to completely fix the periods in terms of the Bessel integrals.

To find the relation between the bases of periods in different patches, we analytically continue the Bessel integrals from one region to another. In practice, the easiest way to do this is to numerically integrate the Picard-Fuchs equation along a line crossing multiple regions, and then find the relations between each pair of bases. By the normalisation of the Bessel function integrals, these matrices relating different bases are integral. In what follows, we will not need most of these relations, hence we do not record them here. However, an important special case that we will be using relates the basis of periods near the large complex structure point in the patch $U_{\{0\}}$ to the basis in the patch $U_{\{i\}}$, where there is another large complex structure point.

[10] In addition to the restriction of the discriminant locus to the real plane, the Bessel function integrals also diverge on points whose real parts satisfy the equation (3.20).

[11] The open sets cover the moduli space apart from points which satisfy (3.20).

For instance, we can study the line $(1, \varphi, \frac{\varphi}{20}, \ldots, \frac{\varphi}{20})$ where the periods satisfy the Picard-Fuchs equation $\mathcal{L}^{(6)} f = 0$, with the operator $\mathcal{L}^{(6)}$ given by (3.6). The Bessel function integrals near $\varphi^i = 0$ that satisfy this equation are given by

$$\widehat{\pi}^{(0)} = \frac{1}{\mathrm{i}\pi} \int_0^\infty \mathrm{d}z \, \frac{z}{\varphi} \begin{pmatrix} \mathcal{A}_0 \mathcal{B}_1 \mathcal{B}^4 \\ \mathcal{B}_0 \mathcal{B}_1 \mathcal{A}^4 \\ 4\, \mathcal{B}_0 \mathcal{A}_1 \mathcal{B} \mathcal{A}^3 \\ 6\, \mathcal{B}_0 \mathcal{A}_1 \mathcal{B}^2 \mathcal{A}^2 \\ 12\, \mathcal{B}_0 \mathcal{B}_1 \mathcal{B} \mathcal{A}^3 + 12\, \mathcal{B}_0 \mathcal{A}_1 \mathcal{B}^2 \mathcal{A}^2 \\ 4 \mathcal{B}_0 \mathcal{A}_1 \mathcal{B}^3 \mathcal{A} + 6 \mathcal{B}_0 \mathcal{B}_1 \mathcal{B}^2 \mathcal{A}^2 \end{pmatrix},$$

where we have used the following shorthand for the Bessel functions appearing here

$$\mathcal{A}_0 = \mathrm{i}\pi I_0\left(\varphi^{-1/2} z\right), \quad \mathcal{B}_0 = K_0\left(\varphi^{-1/2} z\right), \qquad \mathcal{A}_1 = \mathrm{i}\pi I_0(z), \quad \mathcal{B}_1 = K_0(z),$$
$$\mathcal{A} = \mathrm{i}\pi I_0\left(20^{-1/2} z\right), \qquad \mathcal{B} = K_0\left(20^{-1/2} z\right).$$

On the line, the discriminant locus $\Delta = 0$ has singularities at five points:

$$\varphi \simeq 0.2786, \quad \varphi \simeq 0.4775, \quad \varphi = 1, \quad \varphi \simeq 3.2725, \quad \text{and} \quad \varphi \simeq 89.7214.$$

The region $|\varphi| > 89.7214$ lies in the region $U_{\{1\}}$, which contains the large complex structure point at $\varphi^0 = \varphi^2 = \varphi^3 = \varphi^4 = \varphi^5 = 0$. By symmetry, we can deduce that the Bessel function integrals giving a basis of solutions to the Picard-Fuchs equation $\mathcal{L}^{(6)} f = 0$ are

$$\widehat{\pi}^{(1)} = \frac{1}{\mathrm{i}\pi} \int_0^\infty \mathrm{d}z \, \frac{z}{\varphi} \begin{pmatrix} \mathcal{B}_1 \mathcal{A}_0 \mathcal{A}^4 \\ \mathcal{B}_1 \mathcal{B}_0 \mathcal{A}^4 \\ 4\, \mathcal{B}_1 \mathcal{A}_0 \mathcal{B} \mathcal{A}^3 \\ 6\, \mathcal{B}_1 \mathcal{A}_0 \mathcal{B}^2 \mathcal{A}^2 \\ 12\, \mathcal{B}_1 \mathcal{B}_0 \mathcal{B} \mathcal{A}^3 + 12\, \mathcal{B}_1 \mathcal{A}_0 \mathcal{B}^2 \mathcal{A}^2 \\ 4 \mathcal{B}_1 \mathcal{A}_0 \mathcal{B}^3 \mathcal{A} + 6 \mathcal{B}_1 \mathcal{B}_0 \mathcal{B}^2 \mathcal{A}^2 \end{pmatrix}.$$

Given the operator $\mathcal{L}^{(6)}$ it is indeed easy to check that these integrals satisfy the equation.

By integrating the Picard-Fuchs operator $\mathcal{L}^{(6)}$ numerically, we can find the continuation of the period vector $\widehat{\pi}^{(0)}$ to the region $|\varphi| > 89.7214$, giving the following relation between the vectors $\widehat{\pi}^{(0)}$ and $\widehat{\pi}^{(1)}$:

$$\widehat{\pi}^{(0)} = \widehat{\mathrm{T}}\, \widehat{\pi}^{(1)}, \qquad \text{with} \qquad \widehat{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -6 & -3 & -1 & -3 & 0 & -6 \\ -4 & 0 & 0 & -1 & 0 & -4 \\ -4 & -3 & -2 & -3 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.22}$$

We have written the Bessel function integrals in $\widehat{\pi}^0$ and $\widehat{\pi}^1$ in this particular way because these are natural restrictions of the 12 periods to the line $(\varphi^0, \ldots, \varphi^5) = (1, \varphi, \frac{\varphi}{20}, \ldots, \frac{\varphi}{20})$. The generic 12-component period vectors are given by

$$\pi^{(0)} = \frac{\varphi^0}{\mathrm{i}\pi} \int_0^\infty \mathrm{d}z\, z \left(\pi^{(0);0}, \pi^{(0);1}, \ldots, \pi^{(0);5}, \pi^{(0)}_1, \ldots, \pi^{(0)}_5, \pi^{(0)}_0\right)^T, \tag{3.23}$$

in which

$$\pi^{(0);0} = \mathcal{B}_0\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4\mathcal{A}_5\,, \qquad \pi^{(0);i} = \mathcal{B}_0\mathcal{B}_i\prod_{j\neq i}\mathcal{A}_j\,,$$

$$\pi^{(0)}_i = \sum_{\substack{m<n\\ m,n\neq i}}\mathcal{B}_0\mathcal{B}_m\mathcal{B}_n\prod_{j\neq m,n}\mathcal{A}_j\,, \qquad \pi^{(0)}_0 = \sum_{l<m<n}\mathcal{B}_0\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\prod_{j\neq l,m,n}\mathcal{A}_j\,,$$

$$\mathcal{A}_\mu = \mathrm{i}\pi\, I_0\left(\sqrt{\varphi^\mu}\,z\right), \qquad \mathcal{B}_\mu = K_0\left(\sqrt{\varphi^\mu}\,z\right).$$

The vector $\pi^{(1)}$ is given by interchanging the indices 0 and 1. In terms of these quantities, restricted to the line, we have a natural way of writing the relations (3.22) in a symmetric form. For example, the relation corresponding to the third row of the matrix can be written as

$$\pi^{(0);2}+\pi^{(0);3}+\pi^{(0);4}+\pi^{(0);5} = -4\pi^{(1);0}-4\pi^{(1);1}-\pi^{(1);2}-\pi^{(1);3}-\pi^{(1);4}-\pi^{(1);5}\,.$$

The coordinates φ^2, φ^3, φ^4, and φ^5 must appear symmetrically in all of these relations. Thus we are able to guess that the relations in the case where all of the coordinates are unequal are

$$\pi^{(0);j} = -\pi^{(1);0}-\pi^{(1);1}-\pi^{(1);j}\,.$$

We can verify this expectation by studying the line $(\varphi^0,\ldots,\varphi^5)=(1,\varphi,\frac{\varphi}{50},\frac{\varphi}{100},\ldots,\frac{\varphi}{100})$, which singles out the period $\pi^{(0);2}$, and allows us to verify the above relation in the case $j=2$. The other relations then follow by symmetry. Working in this way, we find that in general the period vectors $\pi^{(0)}$ and $\pi^{(1)}$ are related by

$$\pi^{(0)} = \mathrm{T}_{\pi^{(0)}\pi^{(1)}}\pi^{(1)}\,, \qquad \text{with} \qquad \mathrm{T}_{\pi^{(0)}\pi^{(1)}} = \begin{pmatrix} -1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 1&-1&-1&0&0&0&0&0&0&0&0&0\\ 1&-1&0&-1&0&0&0&0&0&0&0&0\\ 1&-1&0&0&-1&0&0&0&0&0&0&0\\ 1&-1&0&0&0&-1&0&0&0&0&0&0\\ -6&6&3&3&3&3&1&-1&-1&-1&-1&0\\ -3&3&0&2&2&2&0&-1&0&0&0&0\\ -3&3&2&0&2&2&0&0&-1&0&0&0\\ -3&3&2&2&0&2&0&0&0&-1&0&0\\ -3&3&2&2&2&0&0&0&0&0&-1&0\\ 4&-4&-3&-3&-3&-3&0&1&1&1&1&-1 \end{pmatrix}. \tag{3.24}$$

3.8 The Frobenius elements as cohomology

It has long been known that the Frobenius elements ϵ_i have an interpretation in the (co-)homology of the mirror manifold $\mathrm{H}\Lambda$ [26, 27]. These relations are instructive in the present case, and we work through this here. The Frobenius elements ϵ_i satisfy the relations

$$\epsilon_i\epsilon_j = \epsilon_j\epsilon_i\,, \qquad \epsilon_i^2 = 0\,, \qquad \epsilon_i\epsilon_j\epsilon_k = Y_{ijk}\eta\,, \qquad \eta\,\epsilon_i = 0\,. \tag{3.25}$$

The elements ϵ_i can be understood as cohomology two-forms, or as their duals, which are four-surfaces. If we think of ϵ_i as four-surfaces, then the products corresponds to intersections, the Y_{ijk} are the intersection numbers, and η is a point. If we think of ϵ_i as two-forms, the products correspond to the wedge product, and η is the volume form. As surfaces, the intersection $\epsilon_i\epsilon_j$ is a curve. These are not all independent, since we know that $b^2(\mathrm{H}\Lambda)=5$. We can choose a basis μ^j to be dual to the ϵ_i so that we have the relations

$$\epsilon_i\mu^j = \delta_i^j\eta\,, \qquad \epsilon_i\epsilon_j = Y_{ijk}\mu^k\,. \tag{3.26}$$

The form of the second relation is dictated by the intersection relation in (3.25). Now the Frobenius period is

$$\varpi = \varpi^0+\varpi^i\,\epsilon_i+\varpi_i\,\mu^i+\varpi_0\,\eta\,. \tag{3.27}$$

Comparing with (3.14), we obtain expressions for ϖ^i, ϖ_i, and ϖ_0. The ϵ_i have also explicit representations as 12×12 matrices, which follow from the monodromy matrices given in (6.1)

$$\mathrm{M}_{E_j} = \mathrm{I}_{12} + 2\pi\mathrm{i}\,\epsilon_j\,,$$

with I_{12} the identity matrix. It is a pleasure to check the indicial algebra above for these matrices.

3.9 The periods near large complex structure points

The set $U_{\{0\}}$ is a neighbourhood of the large complex structure point at $E_1\cap\cdots\cap E_5$, and the $U_{\{i\}}$ are neighbourhoods of other large complex structure points. In the region $U_{\{0\}}$, according to the discussion above, the convergent integrals are of the form.

$$\frac{\varphi^0}{\mathrm{i}\pi}\int_0^\infty \mathrm{d}z\, z\, K_0\Big(\sqrt{\varphi^0}z\Big)\prod_{i=1}^5 B_i\Big(\sqrt{\varphi^i}z\Big)\,. \tag{3.28}$$

A basis for the periods can be given as 12 linear combinations of these functions.[12] To fix the precise combinations, we compare the equation (3.27) with the result of applying the relations (3.26) to the expansion (3.14). It is only a matter of direct comparison to express the coefficients of the ϵ-series (3.27) in terms of Bessel integrals. The relation between the periods in the Bessel integral basis $\pi^{(\mu)}$ and the periods in the Frobenius basis $\varpi^{(\mu)}$ is

$$\varpi^{(\mu)} = \mathrm{T}_{\varpi\pi}\pi^{(\mu)}\,, \qquad \text{with} \qquad \mathrm{T}_{\varpi\pi} = \begin{pmatrix} \frac{1}{\pi^4} & \mathbf{0}^T & \mathbf{0}^T & 0 \\ \mathbf{0} & -\frac{2\mathrm{i}}{\pi^3}\mathbb{I} & \mathbb{O} & \mathbf{0} \\ -\frac{4}{\pi^2}\mathbf{1} & \mathbb{O} & -\frac{8}{\pi^2}\mathbb{I} & \mathbf{0} \\ 80\frac{\zeta(3)}{\pi^4} & -\frac{4\mathrm{i}}{\pi}\mathbf{1}^T & \mathbf{0}^T & \frac{16\mathrm{i}}{\pi} \end{pmatrix}. \tag{3.29}$$

Explicitly, this means that the single-logarithm periods near the large complex structure point at $\varphi^1 = \cdots = \varphi^5 = 0$ are given by

$$\varpi^{(0);j}(\varphi) = -2\varphi^0 \int \mathrm{d}z\, z\, K_0\Big(\sqrt{\varphi^0}z\Big)K_0\Big(\sqrt{\varphi^j}z\Big)\prod_{i\neq j} I_0\Big(\sqrt{\varphi^i}z\Big). \tag{3.30}$$

For the double-logarithm periods, we have

$$\varpi_j^{(0)}(\varphi) = 8\varphi^0 \int \mathrm{d}z\, z \sum_{\substack{m<n \\ m,n\neq j}} K_0\Big(\sqrt{\varphi^0}z\Big)K_0\Big(\sqrt{\varphi^m}z\Big)K_0\Big(\sqrt{\varphi^n}z\Big)\prod_{i\neq m,n} I_0\Big(\sqrt{\varphi^i}z\Big) - 4\pi^2\varpi^{(0)}(\varphi).$$

The period that is cubic in logarithms is

$$\begin{aligned}\varpi_0^{(0)}(\varphi) = -16 \sum_{l<m<n} \varphi^0 \int \mathrm{d}z\, z\, K_0\Big(\sqrt{\varphi^0}z\Big)K_0\Big(\sqrt{\varphi^l}z\Big)K_0\Big(\sqrt{\varphi^m}z\Big)K_0\Big(\sqrt{\varphi^n}z\Big)\prod_{i\neq l,m,n} I_0\Big(\sqrt{\varphi^i}z\Big) \\ -4\pi^2\sum_{k=1}^5 \varpi^{(0);k}(\varphi) + 80\zeta(3)\,\varpi^{(0);0}(\varphi)\,.\end{aligned} \tag{3.31}$$

[12] Recall that for a Calabi-Yau threefold X, $\dim H^3(X) = 2h^{2,1}+2$.

Series expansions

We collect some expressions below that are used to express the periods as series. Denote by $H_n^{(r)}$ the n^{th} harmonic number of order r:

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}, \qquad H_n = H_n^{(1)}.$$

We express the periods using the following intermediate series:

$$h_i^{(0)}(\varphi) = \sum_{n=0}^{\infty} \sum_{\deg(\boldsymbol{p})=n} 2\left(H_n - H_{p_i}\right)\binom{n}{\boldsymbol{p}}^2 \varphi^{\boldsymbol{p}}\,(\varphi^0)^{-n},$$

$$h_{ij}^{(0)}(\varphi) = \sum_{n=0}^{\infty} \sum_{\deg(\boldsymbol{p})=n} \left[4\left(H_n - H_{p_i}\right)\left(H_n - H_{p_j}\right) - 2H_n^{(2)}\right]\binom{n}{\boldsymbol{p}}^2 \varphi^{\boldsymbol{p}}\,(\varphi^0)^{-n},$$

$$h_{ijk}^{(0)}(\varphi) = \sum_{n=0}^{\infty} \sum_{\deg(\boldsymbol{p})=n} \left[8\left(H_n - H_{p_i}\right)\left(H_n - H_{p_j}\right)\left(H_n - H_{p_k}\right) - 4\left(3H_n - H_{p_i} - H_{p_j} - H_{p_k}\right)H_n^{(2)} \right.$$
$$\left. + 4H_n^{(3)}\right]\binom{n}{\boldsymbol{p}}^2 \varphi^{\boldsymbol{p}}\,(\varphi^0)^{-n}.$$

The periods (3.30)-(3.31) can be expressed near the point $\varphi^i = 0$ in terms of the above series. These results can be derived by considering the Frobenius expansion (3.14), but we also sketch a derivation from the Bessel function integrals in appendix B.

$$\begin{aligned}
\varpi^{(0);j}(\varphi) &= \varpi^{(0);0}(\varphi)\log\frac{\varphi^j}{\varphi^0} + h_j^{(0)}(\varphi), \\
\varpi_j^{(0)}(\varphi) &= 2\sum_{\substack{m<n \\ m,n\neq j}} \left[\varpi^{(0);0}(\varphi)\log\frac{\varphi^m}{\varphi^0}\,\log\frac{\varphi^n}{\varphi^0} + h_n^{(0)}(\varphi)\log\frac{\varphi^m}{\varphi^0} + h_m^{(0)}(\varphi)\log\frac{\varphi^n}{\varphi^0} + h_{mn}^{(0)}(\varphi)\right], \\
\varpi_0^{(0)}(\varphi) &= 2\sum_{l<m<n}\left[\varpi^{(0);0}(\varphi)\log\frac{\varphi^l}{\varphi^0}\,\log\frac{\varphi^m}{\varphi^0}\,\log\frac{\varphi^n}{\varphi^0}\right. \\
&\quad + h_n^{(0)}(\varphi)\log\frac{\varphi^l}{\varphi^0}\,\log\frac{\varphi^m}{\varphi^0} + h_l^{(0)}(\varphi)\log\frac{\varphi^m}{\varphi^0}\,\log\frac{\varphi^n}{\varphi^0} + h_m^{(0)}(\varphi)\log\frac{\varphi^n}{\varphi^0}\,\log\frac{\varphi^l}{\varphi^0} \\
&\quad \left. + h_{mn}^{(0)}(\varphi)\log\frac{\varphi^l}{\varphi^0} + h_{lm}^{(0)}(\varphi)\log\frac{\varphi^n}{\varphi^0} + h_{nl}^{(0)}(\varphi)\log\frac{\varphi^m}{\varphi^0} + h_{lmn}^{(0)}(\varphi)\right].
\end{aligned} \tag{3.32}$$

4 Mirror map and large complex structure

To determine the mirror map, we recall that near the large complex structure limit the period vector takes the form [28]

$$\boldsymbol{\Pi} = \begin{pmatrix} \mathcal{F}_0 \\ \mathcal{F}_i \\ z^0 \\ z^i \end{pmatrix}, \qquad i = 1,\ldots,5, \qquad \mathcal{F}_\mu = \frac{\partial \mathcal{F}}{\partial z^\mu}. \tag{4.1}$$

Here z^i are the projective coordinates on the Kähler moduli space of $\mathrm{H}\Lambda_{(\varphi^1,\dots\varphi^5)}$. We often use the corresponding affine coordinates $t^i \stackrel{\text{def}}{=} z^i/z^0$, so that for example the complexified Kähler class of $\mathrm{H}\Lambda_{(\varphi^1,\dots\varphi^5)}$ is given by

$$B + \mathrm{i}J \;=\; \sum_{i=1}^{5} t^i\, \mathrm{e}_i\,,$$

where e_i generate the second integral cohomology $H^2(\mathrm{H}\Lambda, \mathbb{Z})$. The quantities $\mathcal{F}_0$ and $\mathcal{F}_i$ are derivatives of the prepotential $\mathcal{F}$, which near the large complex structure point is given in terms of the genus-0 instanton numbers $n_{\boldsymbol{p}}$ by

$$\mathcal{F}(z^0,\dots,z^5) = -\frac{1}{3!}\sum_{a,b,c=0}^{5} Y_{abc}\frac{z^a z^b z^c}{z^0} - \frac{\left(z^0\right)^2}{(2\pi\mathrm{i})^3}\sum_{\boldsymbol{p}\neq\boldsymbol{0}} n_{\mathrm{p}}\,\mathrm{Li}_3(\boldsymbol{q}^{\boldsymbol{p}})\,, \qquad q_i \stackrel{\text{def}}{=} \exp(2\pi\mathrm{i}\, t^i)\,.$$

The Y_{abc} are given by topological quantities related to $\mathrm{H}\Lambda$:

$$Y_{ijk} \;=\; \int_{\mathrm{H}\Lambda} \mathrm{e}_i \wedge \mathrm{e}_j \wedge \mathrm{e}_k\,, \qquad\qquad Y_{ij0} \;\in\; \left\{0, \frac{1}{2}\right\},$$

$$Y_{i00} \;=\! -\frac{1}{12}\int_{\mathrm{H}\Lambda} c_2(\mathrm{H}\Lambda)\wedge \mathrm{e}_i\,, \qquad Y_{000} \;= -3\chi(\mathrm{H}\Lambda)\frac{\zeta(3)}{(2\pi\mathrm{i})^3}\,.$$

To find the triple intersection numbers Y_{ijk}, we first note that $\mathrm{e}_i \wedge \mathrm{e}_i = 0$ for every i. Therefore the only non-vanishing triple intersection numbers are those with all indices different. To find these numbers, we recall that e_i is dual to a hypersurface $\{Y_i - y_i = 0\} \subset \mathrm{H}\Lambda$, where y_i is a constant. The intersection of two of these hyperplanes gives an elliptic curve, which in turn intersects a third hyperplane generically in two points. Therefore the Y_{ijk} are given by

$$Y_{ijk} \;=\; \int_{\mathrm{H}\Lambda} \mathrm{e}_i \wedge \mathrm{e}_j \wedge \mathrm{e}_k \;=\; \begin{cases} 2\,, & \text{for } i,\, j,\, k \text{ distinct,} \\ 0\;, & \text{otherwise.} \end{cases} \tag{4.2}$$

For the quantities Y_{i00}, we need to compute the second Chern class of $\mathrm{H}\Lambda$. Applying the adjunction formula gives the total Chern class as

$$c(\mathrm{H}\Lambda) \;=\; \frac{\prod_{r=1}^{5}(1+\mathrm{e}_r)^2}{(1+\sum_{r=1}^{5}\mathrm{e}_r)^2}\,.$$

From this we can verify the Calabi-Yau condition $c_1(\mathrm{H}\Lambda) = 0$, and find that the second Chern class $c_2(\mathrm{H}\Lambda)$ can be written as

$$c_2(\mathrm{H}\Lambda) \;=\; 2\sum_{r<s}\mathrm{e}_s\wedge \mathrm{e}_r\,.$$

Integrating this against e_i and using the integral computed in (4.2) gives

$$Y_{i00} \;=\; -\frac{1}{12}\int_{\mathrm{H}\Lambda} c_2(\mathrm{H}\Lambda)\wedge \mathrm{e}_i \;= -2\,.$$

Naïvely, the numbers Y_{ij0} would equal $\int_{\mathrm{H}\Lambda} c_1(\mathrm{H}\Lambda)\wedge \mathrm{e}_i \wedge \mathrm{e}_j$ and so vanish. This argument is not correct, and in fact Y_{ij0} can in some cases take the value $1/2$. Based on the gamma class [29], it is expected that in the one-parameter case one can take $Y_{110} = 0$ exactly when Y_{111} is even. On the quotient $\mathrm{H}\Lambda/\mathbb{Z}_5$ the triple intersection number Y_{111} is 24, so $Y_{110} = 0$. The five-parameter prepotential is related to the prepotential for one-parameter manifolds essentially by setting $t^1 = \ldots = t^5 = t$ and dividing by 5. It follows that the quantities Y_{ij0} can in fact be taken to vanish.

$$Y_{ij0} = 0\,.$$

As we know the Hodge numbers $h^{11} = 5$ and $h^{12} = 45$ of HΛ, the Euler characteristic is immediately given by $\chi(\mathrm{H}\Lambda) = 2(h^{11} - h^{12}) = -80$. So the last quantity Y_{000} is given by

$$Y_{000} = 240\frac{\zeta(3)}{(2\pi\mathrm{i})^3}\,.$$

Note that as a consequence of the highly symmetric nature of the manifold HΛ, none of the couplings depend on the particular values of the indices i, j, k, apart from the fact that they need to be distinct for the triple intersection numbers to be nonvanishing. It is then convenient to write the non-vanishing quantities Y_{abc} as

$$Y_{ijk} \stackrel{\text{def}}{=} Y\,, \qquad\qquad Y_{i00} \stackrel{\text{def}}{=} Y_{00}\,.$$

The large complex structure points are located on loci where all but one of the parameters φ^i vanish. For concreteness, we are going to concentrate on the large complex structure point at $\varphi^1 = \ldots = \varphi^5 = 0$ in the affine patch $\varphi^0 = 1$. We denote the integral period vector in this patch by $\Pi^{(0)}$. The other cases are related to this one by the permutation symmetry.

The affine coordinates t^i of the Kähler moduli space are related to the periods ϖ by

$$t^i = \frac{1}{2\pi\mathrm{i}}\frac{\varpi^i}{\varpi^0} = \frac{1}{2\pi\mathrm{i}}\log\varphi^i + \mathcal{O}(\boldsymbol{\varphi})\,.$$

The second relation gives the asymptotic form in the limit $\varphi^1, \ldots, \varphi^5 \to 0$, and $\mathcal{O}(\boldsymbol{\varphi})$ denotes terms that are of order 1 or higher in any φ^i. Inverting this map order-by-order one finds the coordinates φ^i in terms of t^i. It is useful to write the resulting map in terms of the elementary symmetric polynomials[13] $\sigma_i(\boldsymbol{q})$:

$$\begin{aligned}\varphi^i = q_i\Big[&1-\big(2\sigma_1+2q_i\big)+\big(\sigma_1^2+2\sigma_2-2\sigma_1 q_i+q_i^2\big)-\big(2\sigma_1\sigma_2+14\sigma_3-\big(16\sigma_2+2\sigma_1^2\big)q_i+10\sigma_1 q_i^2-12q_i^3\big)\\ &+\big(\sigma_2^2+26\sigma_1\sigma_3-174\sigma_4+\big(2\sigma_1^3-22\sigma_2\sigma_1+130\sigma_3\big)q_i+\big(18\sigma_1^2-136\sigma_2\big)q_i^2+116\sigma_1 q_i^3-136q_i^4\big)\\ &+\big(-12\sigma_3\sigma_1^2+192\sigma_4\sigma_1-28\sigma_2\sigma_3+\big(4\sigma_2\sigma_1^2-132\sigma_3\sigma_1+28\sigma_2^2-1376\sigma_4\big)q_i\\ &+\big(-10\sigma_1^3+122\sigma_2\sigma_1+1346\sigma_3\big)q_i^2+\big(-128\sigma_1^2-1328\sigma_2\big)q_i^3+1488\sigma_1 q_i^4-1350q_i^5\big)\Big]+\mathcal{O}(\boldsymbol{q})^7\,.\end{aligned}$$

Near this large complex structure point the periods in the Frobenius basis have the asymptotic form

$$\begin{pmatrix}\varpi^{(0);0}\\ \varpi^{(0);i}\\ \varpi^{(0)}_i\\ \varpi^{(0)}_0\end{pmatrix} = \begin{pmatrix}1\\ \log\varphi^i\\ \frac{1}{2!}Y_{imn}\log\varphi^m\log\varphi^n\\ \frac{1}{3!}Y_{lmn}\log\varphi^l\log\varphi^m\log\varphi^n\end{pmatrix} + \mathcal{O}(\varphi\log^3\varphi) = \begin{pmatrix}1\\ 2\pi\mathrm{i}\,t^i\\ \frac{(2\pi\mathrm{i})^2}{2}\,Y_{imn}t^m t^n\\ \frac{(2\pi\mathrm{i})^3}{6}\,Y_{lmn}t^l t^m t^n\end{pmatrix} + \mathcal{O}(t^3\boldsymbol{q})\,, \tag{4.3}$$

[13] Due to the identity $q_1^5 - q_1^4\sigma_1 + q_1^3\sigma_2 - q_1^2\sigma_3 + q_1\sigma_4 - \sigma_5 = 0$, this expression is not unique. Unique expressions are obtained, for example, by using this identity to eliminate occurrences of σ_5, or explicit appearances of powers of q_1 higher than four.

where repeated indices are summed over. On the other hand, the asymptotics of $\mathbf{\Pi}^0$ can be read directly from the prepotential and are given by

$$\Pi^{(0)} = \begin{pmatrix} \mathcal{F}_0 \\ \mathcal{F}_i \\ z^0 \\ z^i \end{pmatrix} = z^0 \begin{pmatrix} Y \sum\limits_{l<m<n} t^l t^m t^n - \frac{1}{2} Y_{00} \sum\limits_{n} t^n - \frac{1}{3} Y_{000} \\ -Y \sum\limits_{\substack{m<n \\ m,n\neq i}} t^m t^n - \frac{1}{2} Y_{00} \\ 1 \\ t^i \end{pmatrix} + \mathcal{O}(\boldsymbol{q})\,.$$

By requiring that the asymptotic forms match,[14] we find that the period vectors must be related by

$$\Pi^{(0)} = \mathrm{T}_{\Pi^{(0)}\varpi^{(0)}}\, \varpi^{(0)} = \rho\, \nu^{-1} \varpi^{(0)}\,, \tag{4.4}$$

with matrices

$$\rho = \begin{pmatrix} -\frac{1}{3}Y_{000} & \mathbf{1}^T & \mathbf{0}^T & 1 \\ \mathbf{1} & \mathbb{0} & -\mathbb{I} & \mathbf{0} \\ 1 & \mathbf{0}^T & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I} & \mathbb{0} & \mathbf{0} \end{pmatrix}, \qquad \text{and} \qquad \nu = \mathrm{diag}(1, (2\pi \mathrm{i})\mathbf{1}, (2\pi\mathrm{i})^2\mathbf{1}, (2\pi\mathrm{i})^3)\,.$$

4.1 Yukawa couplings and instanton numbers

To find the instanton numbers, we compute the Yukawa couplings

$$y_{IJK} = -\int_{\mathrm{H}\Lambda} \Omega \wedge \frac{\partial^3 \Omega}{\partial \varphi^I \partial \varphi^J \partial \varphi^K}\,,$$

where the indices I, J, K run from 1 to 5. The couplings can be computed using the relation between forms on the manifold HΛ and the ring of defining polynomials modulo the Jacobian ideal [30]. Alternatively, one can find y_{ijk} as a series in $\boldsymbol{q}$ by a direct computation. As we are mostly interested in finding the instanton numbers, the latter method is sufficient. We express the Yukawa couplings in terms of the period vectors as

$$y_{IJK} = -\Pi^{(0)\,T}\, \Sigma\, \partial_{IJK} \Pi^{(0)} = -\varpi^{(0)\,T}\, \nu^{-1} \rho^T \Sigma \rho\, \nu\, \partial_{IJK}\, \varpi^{(0)}\,,$$

where Σ is the matrix giving the standard symplectic inner product

$$\Sigma = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}.$$

Using the period formulae, we start by computing y_{IJK} to order 20 in the variables φ^i. It is expected that the Yukawa couplings are rational functions of the moduli a. A natural first guess for the denominator is the discriminant Δ as given in (3.3), which is a polynomial of order 16. In fact, we find that

$$y_{IJK} = \frac{P_{IJK}}{\varphi^I \varphi^J \varphi^K \Delta}\,,$$

[14] Note that we have identified $z^0 = \varpi^{(0);0}$, which has asymptotics $1 + O(\varphi)$.

where the P_{IJK} are degree-14 polynomials in the moduli. Fully expanded, P_{123} is a sum of 11628 monomials while P_{111} and P_{112} each have 8568. Given this size, and the fact that we have no essential need of them in what follows, we omit their display. Simpler expressions can be arrived at by exploiting symmetries. For example, P_{111} is symmetric in φ^2, φ^3, φ^4, φ^5. This allows one to write P_{111} as a degree-4 polynomial in φ^1 with coefficients that are polynomials in the five elementary symmetric polynomials in the five φ^i. The resulting expression contains 730 monomials when fully expanded. Further simplification is obtained by reintroducing the sixth variable φ^0. This reduces P_{111} to an expression comprising of 212 monomials. Similar simplifications are possible for P_{112} and P_{123}.

We then express the Yukawa coupling in terms of the quantities q_i. The y_{ijk} above is computed in the gauge $z^0 = \varpi^{(0);0}$. To be able to compare this to the expression (4.6) we need to transform to the gauge $z^0 = 1$ in addition to the tensor transformation:

$$y_{ijk} \;=\; \frac{(2\pi\mathrm{i})^3}{(\varpi^{(0);0})^2}\, q_i q_j q_k \sum_{I,J,K} \frac{\partial\varphi^I}{\partial q_i}\frac{\partial\varphi^J}{\partial q_j}\frac{\partial\varphi^K}{\partial q_k}\, y_{IJK}\,. \tag{4.5}$$

Owing to the symmetries, there are only three independent Yukawa couplings up to permutation of coordinates. These can be taken to be y_{111}, y_{112} and y_{123}. For the purposes of finding the instanton numbers, we need only one of these, say y_{111}, this being somewhat simpler to compute. Expressing it as series in $\boldsymbol{q}$, we find

$$\begin{aligned} y_{111} \;=\; 24q_1\Big[&1+\sigma_1+\frac{1}{3}\big(-14q_1\sigma_1+17q_1^2+14\sigma_2\big)+\big(-36q_1^3+37q_1^2\sigma_1-38q_1\sigma_2+\sigma_1\sigma_2+43\sigma_3\big)\\ &+\Big(-36q_1^3\sigma_1+37q_1^2\sigma_1^2-2q_1\left(19\sigma_1\sigma_2+3\sigma_3\right)+\sigma_2^2+44\sigma_1\sigma_3+306\sigma_4+\frac{312\sigma_5}{q_1}\Big)+\mathcal{O}(\boldsymbol{q}^5)\Big]. \end{aligned}$$

Similar expressions hold for y_{112} and y_{123}. The series expansions for the Yukawa couplings can be written in terms of the instanton numbers as

$$y_{ijk} \;=\; Y_{ijk} + \sum_{\boldsymbol{p}} n_{\boldsymbol{p}}\, p_i p_j p_k\, \frac{\boldsymbol{q}^{\boldsymbol{p}}}{1-\boldsymbol{q}^{\boldsymbol{p}}}\,. \tag{4.6}$$

By comparing this to the series expansion (4.5), we can identify the instanton numbers up to degree 29 as listed in appendix E.

4.2 Genus-1 instanton numbers

It is possible [15] to define a genus-1 prepotential, which effectively counts the genus-1 curves. In the topological limit it can be expressed as

$$F_1 \;=\; \log\left[\left(\varpi^{(0);0}\right)^{-\left(3+h^{11}(\mathrm{H}\Lambda)-\chi(\mathrm{H}\Lambda)/12\right)} \det\left(\frac{\partial\varphi}{\partial\boldsymbol{t}}\right) f\,\right] + \text{const.}\,,$$

where f is a holomorphic function which can be fixed by imposing appropriate boundary conditions. In particular, the prepotential F_1 must be regular at the points inside the Kähler moduli space corresponding to nonsingular manifolds. In the large complex structure limit, F_1 has an expansion [31]

$$F_1 \;=\; 2\pi\mathrm{i}\sum_{i=1}^{5} Y_{i00}\, t^i - 2\sum_{\boldsymbol{p}}\left(d_{\boldsymbol{p}} + \frac{1}{12}n_{\boldsymbol{p}}\right)\mathrm{Li}_1\left(\boldsymbol{q}^{\boldsymbol{p}}\right) + \text{const.}\,, \tag{4.7}$$

where d_p are the genus-1 instanton numbers.[15]

To get the correct growth in the large complex structure limit, f must contain a factor of $\left(\prod_i \varphi^i\right)^{-3}$. Outside the locus $\varphi^\mu = 0$, F_1 can be singular only on the discriminant locus (3.3). These considerations fix the form of the holomorphic ambiguity f, up to a multiplicative constant, as

$$f \;=\; \frac{\Delta^c}{\left(\prod_{i=1}^5 \varphi^i\right)^3}\,.$$

In the one-parameter cases, where the singularities appear as points φ_* in the moduli space, conifold singularities produce a factor of $(\varphi-\varphi_*)^{-1/6}$. We assume that a straightforward generalisation of this holds in the multiparameter case, and thus we take $c=-\frac{1}{6}$. With this choice we find the genus-1 instanton numbers up to degree 29. These are recorded in appendix E.2.

4.3 Relations between the instanton numbers: from S_5 to S_6 and beyond

Inspecting the tables of appendix E, it is striking that there are many recurrences of the instanton numbers. For example, 24, 80, and many other numbers repeat. In fact, if

$$\boldsymbol{I} \;=\; (i,\,j,\,k,\,l,\,m)\,, \quad \text{and} \quad \widetilde{\boldsymbol{I}} \;=\; (\deg(\boldsymbol{I})-2i,\,j,\,k,\,l,\,m)\,, \tag{4.8}$$

and all the components of I and $\widetilde{I}$ are nonnegative, where

$$\deg(\boldsymbol{I}) \;=\; i+j+k+l+m \tag{4.9}$$

is the *degree* of $\boldsymbol{I}$, then

$$n_{\boldsymbol{I}} \;=\; n_{\widetilde{\boldsymbol{I}}}\,. \tag{4.10}$$

This identity has many ramifications, and a proper understanding of these devolves upon a discussion of what we term the *web of indices* related to $\boldsymbol{I}$, $W[\boldsymbol{I}]$, obtained by composing the operations $\boldsymbol{I}\mapsto\widetilde{\boldsymbol{I}}$ with permutations, in all possible ways. We note in passing that the only $\boldsymbol{I}$ in our tables such that $\widetilde{\boldsymbol{I}}$ has a negative entry is $(1,0,0,0,0)$, and its permutations. We discuss these matters is detail in §5. Our immediate aim however is to discuss the origins of this group of symmetries.

The symmetry (4.8), which we will here refer to as a duality, has its origin in the presentation (1.2) of the Hulek-Verrill manifold. In this presentation the six coordinates φ^μ, $\mu=0,1,\ldots,5$ are projective coordinates that enter on an equal footing. We subsequently passed to five affine coordinates φ^i by setting $\varphi^0=1$. Clearly we could have chosen coordinates by setting any of the φ^μ to unity and using the other φ^μ as coordinates. Now we have taken the large complex structure point to be the point $\varphi^0 = 1$, $\boldsymbol{\varphi}=(1,0,0,0,0,0)$, and a consequence of the symmetry of the presentation (1.2) is that there are in fact six large complex structure points, corresponding to taking one of the φ^μ to be unity and the other φ^μ to vanish. We could equally expand the Yukawa coupling about any of these points. We will study the effect of this below, but it is this choice, coupled with the fact that the form of the instanton expansion must remain invariant, that enforces (4.10).

[15] The reader should be aware that we are computing F_1 as in [31], which generates counts of BPS wrappings of M2-branes. This prescription differs from that of [15], which was used in [3], which instead counts primitive elliptic curves.

After making such a change in choice of large complex structure point, the new period vector is obtained using the transition matrix $\mathrm{T}_{\Pi^{(0)}\Pi^{(1)}}$ given in (6.7). This matrix can be seen to act on the complex structure moduli space coordinates as

$$z^0 \mapsto -z^0\,, \qquad z^1 \mapsto z^1\,, \qquad z^i \mapsto -z^i - z^1 - z^0\,, \quad i \neq 0,1\,.$$

This can be stated equivalently in terms of the quantities t^i or q_i:

$$t^1 \mapsto -t^1\,, \quad t^i \mapsto t^i + t^1 + 1\,; \qquad \text{or equivalently} \qquad q_1 \mapsto q_1^{-1}\,, \quad q_i \mapsto q_i q_1\,; \quad i \neq 1\,. \tag{4.11}$$

We can combine these transformations with monodromies around the loci E_i to find a simpler form for them. Consider the following matrix

$$\mathrm{M}_{E_5}^{-1}\mathrm{M}_{E_4}^{-1}\mathrm{M}_{E_3}^{-1}\mathrm{M}_{E_2}^{-1}\mathrm{T}_{\Pi^{(0)}\Pi^{(1)}} = -\begin{pmatrix} 1 & 0 & \mathbf{0}_4^T & 0 & 0 & \mathbf{0}_4^T \\ 0 & -1 & \mathbf{1}_4^T & 0 & 12 & \mathbf{0}_4^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbb{I}_4 & \mathbf{0}_4 & \mathbf{0}_4 & 0 \\ 0 & 0 & \mathbf{0}_4^T & 1 & 0 & \mathbf{0}_4^T \\ 0 & 0 & \mathbf{0}_4^T & 0 & -1 & \mathbf{0}_4^T \\ \mathbf{0} & \mathbf{0} & \mathbb{O}_4 & \mathbf{0}_4 & \mathbf{1}_4 & \mathbb{I}_4 \end{pmatrix},$$

where $\mathbf{0}_4 = (0,0,0,0)^T$, $\mathbf{1} = (1,1,1,1)^T$, $\mathbb{O}_4$ is a 4×4 zero matrix, and $\mathbb{I}_4$ a 4×4 identity matrix. and the other matrices obtained by swapping the second and $(i+2)$'th column and row and the seventh and $(i+7)$'th column and row with each other.

Rather than discussing the Yukawa coupling to see the consequences of these symmetries on instanton numbers, it is simpler to consider the prepotential, bearing in mind the symplectic symmetries. The genus-0 prepotential is

$$\mathcal{F}^{(0)}(\boldsymbol{t}) = -\left(z^0\right)^2\left(\frac{1}{6}Y_{ijk}\,t^i t^j t^k + \frac{1}{2}Y_{i00}\,t^i - \frac{\zeta(3)}{2(2\pi\mathrm{i})^3}\,\chi(\mathrm{H}\Lambda) + \frac{1}{(2\pi\mathrm{i})^3}\sum_{\boldsymbol{p}} n_{\boldsymbol{p}}\,\mathrm{Li}_3(q^{\boldsymbol{p}})\right). \tag{4.12}$$

The sum is over *positive* multi-indices $\boldsymbol{p}$, where by *positive* we mean all entries are nonnegative and at least one is positive. We now make a change of coordinates,

$$t^i \mapsto \widetilde{t}^i\,, \qquad \widetilde{t}^1 = -t^1\,, \quad \widetilde{t}^i = t^i + t^1\,, \quad i > 1\,, \tag{4.13}$$

which differs from the transformation (4.11) by a constant term $\widetilde{t}^i \mapsto t^i - 1$, amounting to an action of monodromy transformation, which simplifies the following analysis.

Writing $\widetilde{q}_i = \mathrm{e}^{2\pi\mathrm{i}\widetilde{t}^i}$, $\boldsymbol{p} = (i,j,k,l,m)$, and $q^{\boldsymbol{p}} = q_1^i q_2^j q_2^k q_4^l q_5^m$, we have $q^{\boldsymbol{p}} = \widetilde{q}^{\widetilde{\boldsymbol{p}}}$, with $\widetilde{I}$ obtained from I by (4.8). Writing the prepotential $\mathcal{F}^{(0)}(\boldsymbol{t})$ in terms of the coordinates $\widetilde{t}^i$, we find

$$\mathcal{F}^{(0)}(\boldsymbol{t}) = \mathcal{F}^{(0)}(\widetilde{\boldsymbol{t}}) - 4\left(\widetilde{t}^1\right)^3 - 2\widetilde{t}^1\,. \tag{4.14}$$

The cubic and linear terms account for the difference in the perturbative parts of the prepotentials.

In making the change of coordinates (4.13), we have given the expansion of the prepotential around a different large complex structure point to that of (4.12). Due to the S_6 symmetry, the functional form of the prepotential should be, up to an effect of a symplectic transformation, invariant under the change of coordinates from t^i to $\widetilde{t}^i$. In other words, the functional dependence of the left-hand side of (4.14) on $\boldsymbol{t}$ should be the same as the functional dependence of the right-hand side of (4.14) on $\widetilde{\boldsymbol{t}}$. Requiring this, we obtain

$$4t_1^3+2t_1-\frac{1}{(2\pi \mathrm{i})^3}\sum_{\boldsymbol{p}} n_{\boldsymbol{p}}\mathrm{Li}_3\big(q^{\widetilde{\boldsymbol{p}}}\big)\simeq -\frac{1}{(2\pi \mathrm{i})^3}\sum_{\boldsymbol{p}} n_{\boldsymbol{p}}\mathrm{Li}_3(q^{\boldsymbol{p}})\,, \tag{4.15}$$

with the equality holding up to a symplectic transformation. One now needs to rewrite the sum on the left-hand side. The map (4.8) is an involution, i.e. $\widetilde{\widetilde{\boldsymbol{p}}}=\boldsymbol{p}$, but the sums above are over positive $\boldsymbol{p}$. The only $\boldsymbol{p}$ such that $\widetilde{\boldsymbol{p}}$ has a negative entry is $(1,0,0,0,0)$. Therefore, we have an equality

$$\sum_{\boldsymbol{p}>0} n_{\boldsymbol{p}}\mathrm{Li}_3\big(\boldsymbol{q}^{\widetilde{\boldsymbol{p}}}\big)=\left[\sum_{\substack{\boldsymbol{p}\text{ for which}\\ \boldsymbol{p},\widetilde{\boldsymbol{p}}>0}} n_{\widetilde{\boldsymbol{p}}}\,\mathrm{Li}_3(\boldsymbol{q}^{\boldsymbol{p}})+n_{(1,0,0,0,0)}\,\mathrm{Li}_3(q_1)\right]+n_{(1,0,0,0,0)}\big(\mathrm{Li}_3\big(q_1^{-1}\big)-\mathrm{Li}_3(q_1)\big)\,. \tag{4.16}$$

Substituting (4.16) into (4.15) one obtains

$$4t_1^3+2t_1\simeq\frac{1}{(2\pi \mathrm{i})^3}n_{(1,0,0,0,0)}\big(\mathrm{Li}_3\big(q_1^{-1}\big)-\mathrm{Li}_3(q_1)\big)-\frac{1}{(2\pi \mathrm{i})^3}\sum_{\substack{\boldsymbol{p}\text{ for which}\\ \boldsymbol{p},\widetilde{\boldsymbol{p}}>0}}(n_{\boldsymbol{p}}-n_{\widetilde{\boldsymbol{p}}})\mathrm{Li}_3(\boldsymbol{q}^{\boldsymbol{p}})\,. \tag{4.17}$$

This imposes that $n_{\boldsymbol{p}}=n_{\widetilde{\boldsymbol{p}}}$ for positive $\boldsymbol{p}$ for which $\widetilde{\boldsymbol{p}}$ is also positive. The left hand side can then be uniquely cancelled by taking $n_{(1,0,0,0,0)}=24$, in light of the identity

$$\frac{24}{(2\pi \mathrm{i})^3}\left(\mathrm{Li}_3\big(\mathrm{e}^{-2\pi \mathrm{i}t_1}\big)-\mathrm{Li}_3\big(\mathrm{e}^{2\pi \mathrm{i}t_1}\big)\right)\;=\;2t_1-6t_1^2+4t_1^3\,,\qquad \mathrm{Im}[t_1]>0\,,$$

and the fact that the quadratic term $-6t_1^2$ can be removed by a change of symplectic integral basis of $H^3(\mathrm{HV},\mathbb{Z})$. It is interesting that we have 'derived' the instanton number $n_{(1,0,0,0,0)}=24$ purely on the grounds of symmetry, without performing a mirror computation or curve-counting. Equation (4.17) then imposes (4.10), which applies for all positive $\boldsymbol{p}$. This includes all $\boldsymbol{p}$ in our tables besides $(1,0,0,0,0)$ and its permutations.

Analogous relations can be seen to hold for the genus-1 instanton numbers that we compute. Indeed, the above argument hold *mutatis mutandis* for higher-genus prepotentials implying that this duality continues to hold for all higher-genus numbers.

4.4 Recovering the results on quotient manifolds

Using these results, instanton numbers $\widehat{n}_s$ and $\widehat{d}_s$ on the quotient manifolds $\mathrm{H}\Lambda_{\mathbb{Z}_5}$ and $\mathrm{H}\Lambda_{\mathbb{Z}_{10}}$ can be recovered. The first few instanton numbers for the one-parameter manifolds are reproduced from [3] in Table 7.

We fix attention here on the $\mathbb{Z}_5$ quotient. The action of the $\mathbb{Z}_5$ on the cohomology $H^2(\mathrm{H}\Lambda,\mathbb{Z})$ is given by

$$\mathrm{e}_i\mapsto \mathrm{e}_{i+1}\,,$$

where the index is understood to take values in $\mathbb{Z}_5$, and we have taken this action to be consistent with the choice (2.16) for the action of $\mathbb{Z}_5$ on HV. This also induces an action on $H^4(\mathrm{H}\Lambda,\mathbb{Z})$ via Hodge duality. The $\mathbb{Z}_5$ action on the periods of HV is

$$\varpi^0\mapsto\varpi^0\,,\qquad \varpi^i\mapsto\varpi^{i+1}\,,\qquad \varpi_i\mapsto\varpi_{i+1}\,,\qquad \varpi_0\mapsto\varpi_0\,.$$

The locus of $\mathbb{Z}_5$ symmetric Hulek-Verrill manifolds is $\varphi^1 = \cdots = \varphi^5 \stackrel{\text{def}}{=} \varphi$, and the corresponding mirror manifolds are found on the locus $t^1 = \cdots = t^5 \stackrel{\text{def}}{=} t$. Thus one identifies the generator of the second cohomology of the one-parameter manifold with

$$\mathrm{e} \,=\, \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_4 + \mathrm{e}_5\,.$$

The prepotential $\widehat{\mathcal{F}}$ on the one-parameter family is identified with that of the five-parameter family by

$$\widehat{\mathcal{F}}(t) \,=\, \frac{1}{5}\mathcal{F}(t,t,t,t,t)\,.$$

Indeed, this agrees with the following simple computation of the classical Yukawa coupling for the $\mathbb{Z}_5$ quotient:

$$\widehat{Y}_{111} \,=\, \int_{\mathrm{H}\Lambda/\mathbb{Z}_5} \mathrm{e}\wedge\mathrm{e}\wedge\mathrm{e} \,=\, \frac{1}{5}\sum_{i,j,k=1}^{5}\int_{\mathrm{H}\Lambda} \mathrm{e}_i\wedge\mathrm{e}_j\wedge\mathrm{e}_k \,=\, \frac{1}{5}\sum_{i,j,k=1}^{5} Y_{ijk} \,=\, 24\,.$$

We can identify the other topological numbers Y_{abc} and the instanton numbers in a similar fashion. Since the group $\mathbb{Z}_5$ has no proper subgroups, curves on the manifold must either belong to an orbit of 5 curves or be mapped to themselves. If a curve with Euler character χ is mapped to itself by $\mathbb{Z}_5$ then the quotient map will take said curve to a curve with Euler character $\chi/5$. In particular, the Euler character of a genus-0 curve is 2, and so there cannot be any genus-0 curves fixed by the $\mathbb{Z}_5$ action. Similar considerations applied to $\mathbb{Z}_{10}$ action show that every curve lies in an orbit of length 10.

$$\widehat{n}_s \,=\, \frac{\kappa}{10}\sum_{\deg(\boldsymbol{p})=s} n_{\boldsymbol{p}}\,, \tag{4.18}$$

where, again, $\kappa = 1$ for $\mathbb{Z}_{10}$ and $\kappa = 2$ for $\mathbb{Z}_5$.

For the genus-1 numbers $d_{\boldsymbol{p}}$ the relation is more complex since a genus-1 curve has $\chi = 0$, so there can exist genus-1 curves, invariant under the symmetry group, whose quotient is again a genus-1 curve. The formula analogous to (4.18) is now

$$\widehat{d}_s \,=\, \frac{1}{5}\sum_{\deg\boldsymbol{p}=s} d_{\boldsymbol{p}}^{\,\text{non-inv}} + d_{s,s,s,s,s}^{\,\text{inv}}\,, \tag{4.19}$$

and serves to compute the numbers $d_{s,s,s,s,s}^{\text{inv}}$ of $\mathbb{Z}_5$ invariant genus-1 curves of degree k. A small check is that the numbers $d_{s,s,s,s,s} - d_{s,s,s,s,s}^{\text{inv}}$ should be divisible by 5, which they are, to the extent of the tables.

The fact that all the instanton numbers we have computed agree with those computed on the one-parameter families through increasingly intricate relations provides a non-trivial consistency check of the results of sections §3 and §4.

Table 7: The constants $\widehat{n}_k$ and $\widehat{d}_k$ are respectively the genus-0 and genus-1 degree-k instanton numbers for the quotient manifolds. The quantity κ is taken to equal 1 or 2 depending on whether one is working on the $\mathbb{Z}_{10}$ or $\mathbb{Z}_5$ quotient. Note that this differs from the table in [3], as here we are using the conventions of [31], which differ from the conventions of [15] used in [3].

k	$\widehat{n}_k$	$\widehat{d}_k$
1	12κ	$20-10\kappa$
2	24κ	$122-40\kappa$
3	112κ	$1200-448\kappa$
4	624κ	$12218-4468\kappa$
5	4200κ	$133800-48948\kappa$
6	31408κ	$1513032-550744\kappa$
7	258168κ	$17647096-6407540\kappa$
8	2269848κ	$210213862-76165868\kappa$
9	21011260κ	$2545256772-920643890\kappa$
10	202527600κ	$31212555028-11273167424\kappa$
11	2017537884κ	$386727907556-139494386722\kappa$
12	20654747200κ	$4832557014112-1741106595848\kappa$
13	216372489804κ	$60820504439316-21890039477898\kappa$
14	2311525544064κ	$770126009447308-276916199510504\kappa$
15	25115533695300κ	$9802710122684812-3521744606430982\kappa$
16	276942939016224κ	$125345359041305658-44996106493639596\kappa$
17	3093639869100240κ	$1609189343845395984-577237489764357432\kappa$
18	34957447938066952κ	$20732103880969324866-7431797272240376304\kappa$
19	399082284262216044κ	$267947664660167267380-95989385991015664466\kappa$
20	$4598143339631725920\kappa$	$3472847998706120977380-1243366526906482828392\kappa$
21	$53420849666489458232\kappa$	$45126364143189924137384-16147280867335074115108\kappa$
22	$625334338772563692216\kappa$	$587733058797585235078306-210193232419243602788840\kappa$
23	$7370491340262022774308\kappa$	$7670883739613425230865660-2741991368623024758775 54\kappa$
24	$87419782094909562148112\kappa$	$100310865002094048953427112-35839510423715766917658440\kappa$
25	$1042868408542514775921540\kappa$	$1314072243953354318776636044-469285414203290732598797814\kappa$
26	$12507178017340321543927896\kappa$	$17242438907892929716931810362-6155022515235842521035603944\kappa$
27	$150738741242255934466584688\kappa$	$226585807117189893207597411984-80851004064783102896144565240\kappa$
28	$1825033540198187573067367200\kappa$	$2981765590671191125416200860324-1063545103192576581182654608448\kappa$
29	$22190047278214311145705359228\kappa$	$39289668166799514883939622674020-14008698625940299577598837703530\kappa$

5 Duality webs

We refer to the operation in (4.8), together with permutations of indices, as duality operations. We denote the set of all multi-indices related to $\boldsymbol{I}$ by dualities as $W[\boldsymbol{I}]$, the *duality web* containing $\boldsymbol{I}$. The duality operations form a group that we will denote by $\mathcal{W}$. The web $W[\boldsymbol{I}]$ is the group orbit $\mathcal{W}\cdot\boldsymbol{I} = \{w\boldsymbol{I} \,|\, w \in \mathcal{W}\}$ of $\boldsymbol{I}$.

We begin with some elementary properties of the webs, and will observe later that $\mathcal{W}$ is a Coxeter group.

- These duality relations are equivalence relations, so if $\boldsymbol{J} \in W[\boldsymbol{I}]$, then $W[\boldsymbol{J}] = W[\boldsymbol{I}]$.
- If an integer r divides $\boldsymbol{I}$, that is r divides each component of $\boldsymbol{I}$, then r divides any dual of $\boldsymbol{I}$. It follows that the greatest common divisor of the elements of $\boldsymbol{I}$, $\gcd \boldsymbol{I}$, is preserved by the duality transformation and that $\gcd \boldsymbol{J} = \gcd \boldsymbol{I}$, for all $\boldsymbol{J} \in W[\boldsymbol{I}]$. Moreover, for an integer r, the web of $r\boldsymbol{I}$ is obtained by multiplying each multi-index in $W[\boldsymbol{I}]$ by r. We can say that $W[r\boldsymbol{I}] = r\,W[\boldsymbol{I}]$. This observation shows, for example, that $(1,1,1,0,0)$ and $(3,3,3,0,0)$ belong to different duality webs.

The webs have interesting properties mod 3. As in (4.9), we denote the sum of elements of a multi-index $\boldsymbol{I}$ by $\deg(\boldsymbol{I})$. If the operation g_1 takes $\boldsymbol{I}$ to $\widetilde{\boldsymbol{I}}$, as in (4.8), then

$$\deg\big(\widetilde{\boldsymbol{I}}\big) ~=~ 2\deg(\boldsymbol{I}) - 3I_1 ~=-\deg(\boldsymbol{I}) \mod 3\,.$$

- It follows that $\deg(\boldsymbol{I})^2 \mod 3$ is an invariant of the web, so either $\deg(\boldsymbol{J}) = 0 \mod 3$, for all $\boldsymbol{J} \in W[\boldsymbol{I}]$, or $\deg(\boldsymbol{J})^2 = 1 \mod 3$ for all $\boldsymbol{J} \in W[\boldsymbol{I}]$. Moreover, if $\deg(\boldsymbol{I}) = 0 \mod 3$, we see from (4.8) that $\boldsymbol{J} = \boldsymbol{I} \mod 3$, for all $\boldsymbol{J} \in W[\boldsymbol{I}]$.

Recall that $\boldsymbol{I}$ is *positive* if all the components of $\boldsymbol{I}$ are nonnegative and at least one is strictly positive. We say that $\boldsymbol{I}$ is *negative* if $-\boldsymbol{I}$ is positive. A nonzero multi-index that is neither positive nor negative has both strictly positive and strictly negative components and we refer to such a multi-index as *mixed*.

We say that a *web* W *is positive* if all its multi-indices are positive. A *web* W *is negative* if $-W$ is positive. A web that is neither positive nor negative is said to be *mixed*.

- The only multi-index in appendix E that has a duality transform that is not positive is $(1,0,0,0,0)$ which has among its transforms the multi-index $(-1,0,0,0,0)$. We have

  $$W[(1,0,0,0,0)] ~=~ \mathbb{W}_+ \cup \mathbb{W}_-\,,$$

  where

  $$\mathbb{W}_+ ~=~ \{(1,0,0,0,0),\ (1,1,0,0,0)\,,\ (2,1,1,0,0)\,,\dots\}$$

  consists entirely of positive multi-indices, and $\mathbb{W}_- = -\mathbb{W}_+$.
- It follows from the discussion of §4.3, that the instanton numbers $n_{\boldsymbol{I}}$ can be nonzero precisely for multi-indices that lie in the positive webs and in $\mathbb{W}_+$. For $\boldsymbol{J} \in \mathbb{W}_+$ we have $n_{\boldsymbol{J}} = 24$, and for $\boldsymbol{J}$ in a positive web $W[\boldsymbol{I}]$ we have $n_{\boldsymbol{J}} = n_{\boldsymbol{I}}$. To the extent of the tables, all the genus-zero instanton numbers that are permitted in this way are in fact nonzero.

There are some intriguing identities that are explained by the duality operations. For example:

$$n_{(i,j,k,0,0)} ~=~ n_{(2-i,2-j,2-k,0,0)}\,, \quad \text{for} \quad 0 \leqslant i,j,k \leqslant 2\,, \quad \text{and} \quad (i,j,k) \neq (0,0,0)\,,\ \text{or}\ (2,2,2)\,.$$

For $(i,j,k) = (1,1,1)$, the identity is trivial, and for the other values of i,j,k this is explained by the fact that all these multi-indices are in the web $\mathbb{W}_+$. This equality can also be explained as a consequence of the elliptic fibration structure of the HΛ manifold as we will see in §7.

In fact, as is evident from the fact that both permutations and duality operations keep instanton numbers invariant: if $\boldsymbol{J} \in W[\boldsymbol{I}]$, then the genus-0 and genus-1 instanton numbers associated to $\boldsymbol{I}$ and $\boldsymbol{J}$ agree, $n_{\boldsymbol{J}} = n_{\boldsymbol{I}}$ and $d_{\boldsymbol{J}} = d_{\boldsymbol{I}}$. Thus it is possible to associate to each web W unique instanton numbers n_W and d_W by $n_W = n_{\boldsymbol{I}}$ and $d_W = d_{\boldsymbol{I}}$ for any $\boldsymbol{I} \in W$. This explains most of the repetitions in the tables in appendix E.

Table 8, however, sets out an intriguing relation among the instanton numbers that are not explained in this way. There are distinct positive webs W and W' such that $n_W = n_{W'}$. Moreover, precisely when this happens (to the extent of the tables) we have also $d_W = d_{W'}$, where d_W denotes the genus-one instanton numbers. There exists also a quadratic invariant $h(W)$, which we will define in (5.1), associated to each web. It is an interesting observation that the webs with equal instanton numbers, corresponding to a single row of Table 8 have equal invariants $h(W)$. It is also worth noting that the vectors listed in the table all contain a zero component. The instanton numbers that are involved, in these relations, are highly nontrivial.

While it is possible that some of the webs that are stated to be distinct could in fact coincide, we should state that, in all cases, we have checked the webs up to total degree 250, and these partial webs are distinct. Moreover, in many cases, the webs are demonstrably distinct. The webs in the first two rows of the table, for example, are all distinct since each index vector $\boldsymbol{I}$ has a distinct greatest common divisor. Row 3 of the table refers to two webs that have degrees that are both zero mod 3. The index vectors of the first web are equal to $(2,2,2,0,0)$ mod 3, up to permutation, while the second web consists of index vectors that, up to permutation, are $(1,1,1,0,0)$ mod 3. Many other rows of the table correspond to webs that can be shown to be distinct, in a similar way.

5.1 The Coxeter group of dualities

Denote the operation analogous to (4.8) acting on the r'th coordinate by g_r. These operations, combined with permutations of the indices, generate $\mathcal{W}$. The S_5 subgroup, corresponding to the permutations, can be generated by the transpositions (r,s). In fact, it can be generated by just the four transpositions $s_r = (r, r+1)$, $r = 1,2,3,4$. Once we admit the permutations, then the dualities g_r can all be obtained from g_1, say; for example $g_2 = s_1 g_1 s_1$.

We have

$$g_1^2 \,=\, s_1^2 \,=\, s_2^2 \,=\, s_3^2 \,=\, s_4^2 \,=\, 1\,,$$

and it is clear that the elements of the generating set $\mathfrak{S} = \{g_1, s_1, s_2, s_3, s_4\}$ commute, apart from g_1 with s_1, and s_r with s_{r+1}, $r = 1,2,3$. The operation $s_r s_{r+1}$ amounts to a cyclic permutation of the three indices $(I_r I_{r+1} I_{r+2}) \to (I_{r+2} I_r I_{r+1})$, so $(s_r s_{r+1})^3 = 1$ and one checks also that $(g_1 s_1)^6 = 1$. We have not been able to find any other relations between the elements of $\mathfrak{S}$ so we assume that in fact there are no further relations. If there were additional relations, the webs would break into smaller webs, giving fewer relations between the instanton numbers. Then the instanton numbers would be expected to take different values — one corresponding to each of the smaller webs. However, we do not observe this to the extent of the tables.

In this way, we observe that $\mathcal{W}$ is a Coxeter group corresponding to the graph in Figure 5.

Table 8: Pairs (n_W, d_W) that arise for the distinct webs that are shown on the right. The webs indicated on the right have the same quadratic invariant h that is defined in (5.1).

#	n	d	h	Webs
1	112	0	0	$(2k+1)\,W[(1,1,1,0,0)]\,;\ \ k=1,2,\ldots$
2	80	4	0	$k\,W[(2,2,2,0,0)]\,;\ \ k=1,2,\ldots$
3	234048	-5600	-9	(3,2,2,2,0) (4,4,3,1,0)
4	795936	-29136	-11	(3,3,2,2,0) (4,4,4,1,0)
5	4326048	-251520	-14	(3,3,3,2,0) (5,5,5,1,0)
6	33777312	-3031872	-18	(3,3,3,3,0) (5,5,3,2,0) (7,7,6,1,0)
7	7371792	-484896	-15	(4,3,3,2,0) (6,6,5,1,0)
8	88179456	-9395616	-20	(4,3,3,3,0) (7,7,7,1,0)
9	20578560	-1679040	-17	(4,4,3,2,0) (6,6,6,1,0)
10	347078520	-46049040	-23	(4,4,3,3,0) (5,5,4,2,0) (8,8,8,1,0)
11	1935300720	-327015680	-27	(4,4,4,3,0) (6,5,5,2,0) (6,6,3,3,0)
12	14386855920	-3110590260	-32	(4,4,4,4,0) (6,6,6,2,0)
13	539120544	-76342880	-24	(5,4,3,3,0) (9,9,8,1,0)
14	140436672	-16170272	-21	(5,4,4,2,0) (8,8,7,1,0)
15	4392333792	-824199120	-29	(5,4,4,3,0) (6,6,5,2,0)
16	45007048752	-11043084816	-35	(5,4,4,4,0) (6,6,4,3,0) (7,7,6,2,0)
17	1272585120	-203310240	-26	(5,5,3,3,0) (5,5,5,2,0) (9,9,9,1,0)
18	193411225936	-55127514240	-39	(5,5,4,4,0) (8,7,7,2,0)
19	65215603200	-16642969280	-36	(5,5,5,3,0) (7,7,4,3,0)
20	1096632180480	-368134832160	-44	(5,5,5,4,0) (7,7,5,3,0)
21	7888589144400	-3138370134624	-50	(5,5,5,5,0) (7,7,6,3,0) (9,9,9,2,0)
22	65215569408	-16642956928	-36	(6,4,4,4,0) (8,8,6,2,0)
23	21143067840	-4775506080	-33	(6,5,4,3,0) (7,6,6,2,0)
24	391409808576	-119442727776	-41	(6,5,4,4,0) (6,6,5,3,0) (8,8,7,2,0)
25	135171775392	-37176746592	-38	(6,5,5,3,0) (7,7,7,2,0)
26	2981800050480	-1093125957120	-47	(6,5,5,4,0) (7,6,6,3,0) (7,7,4,4,0) (9,9,8,2,0)
27	27765085214112	-12215408263200	-54	(6,5,5,5,0) (7,7,7,3,0) (9,9,6,3,0)
28	1096632086784	-368134868160	-44	(6,6,4,4,0) (8,8,8,2,0)
29	10848408360480	-4429601736480	-51	(6,6,5,4,0) (8,7,6,3,0)
30	126532108859856	-62415555336480	-59	(6,6,5,5,0) (7,6,6,4,0) (8,8,5,4,0) (8,8,7,3,0)
31	1535514818112	-531223501536	-45	(6,6,6,3,0) (7,6,4,4,0) (8,8,5,3,0) (9,8,8,2,0)
32	725912434085952	-405156007308576	-65	(6,6,6,5,0) (8,7,6,4,0) (9,8,8,3,0)
33	51294957112992	-23657221999872	-56	(7,5,5,5,0) (7,7,5,4,0) (8,7,7,3,0)
34	553728279360	-174588053440	-42	(7,6,5,3,0) (9,9,7,2,0)
35	20350993239840	-8738280013680	-53	(7,6,5,4,0) (8,8,6,3,0)
36	305922925426848	-160791639748800	-62	(7,6,5,5,0) (9,9,7,3,0)
37	2235977596096128	-1345692401785920	-69	(7,6,6,5,0) (9,8,6,4,0)
38	19503820669876800	-13461969999093600	-77	(7,6,6,6,0) (8,7,6,5,0) (9,8,7,4,0)
39	408865565088240	-219322647849280	-63	(7,7,6,4,0) (8,6,5,5,0) (8,8,8,3,0)
40	8765016259161504	-5758034709276000	-74	(7,7,6,5,0) (9,9,5,5,0)
41	92700939550359360	-70199768003592720	-83	(7,7,6,6,0) (9,8,6,5,0) (9,8,8,4,0)
42	1692511362069504	-1000067627051904	-68	(7,7,7,4,0) (8,7,5,5,0) (9,9,8,3,0)
43	42801528146793216	-30974226462689184	-80	(7,7,7,5,0) (9,9,7,4,0)
44	3704581973944705776	-3435204329200397376	-98	(7,7,7,7,0) (9,7,7,6,0)
45	3881643757375656	-2421429008571216	-71	(8,6,6,5,0) (8,7,7,4,0) (8,8,5,5,0) (9,9,6,4,0)
46	42801528135993600	-30974226462442944	-80	(8,6,6,6,0) (8,8,8,4,0)
47	1810611871504105272	-1616941505273075616	-95	(8,7,7,6,0) (8,8,8,5,0) (9,8,6,6,0)
48	55456767284050560	-40752668020556480	-81	(8,8,6,5,0) (9,6,6,6,0)
49	419093788958668992	-345708325307878560	-89	(8,8,7,5,0) (9,7,6,6,0)
50	5101035246140064	-3238317072282240	-72	(9,7,7,4,0) (9,8,5,5,0)
51	198280729061595552	-156843054827785632	-86	(9,7,7,5,0) (9,9,6,5,0)
52	170193515484672	-85781801925696	-60	(9,8,7,3,0) (9,9,5,4,0)

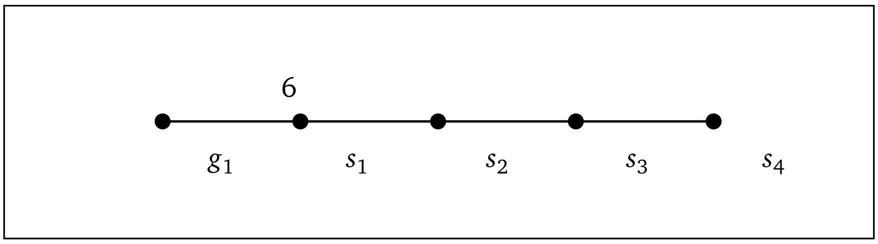

Figure 5: The Coxeter graph corresponding to $\mathcal{W}$.

For the following discussion it is convenient to think of multi-indices as integral elements of the vector space $\mathbb{R}^5$. We shall refer to the elements of these vector spaces as *index vectors*. For two index vectors $\boldsymbol{J}, \boldsymbol{J}'$ let $H(\boldsymbol{J}, \boldsymbol{J}')$ denote the bilinear form

$$H(\boldsymbol{J},\boldsymbol{J}') = \frac{3}{2}\boldsymbol{J}\cdot\boldsymbol{J}' - \frac{1}{2}\deg(\boldsymbol{J})\deg(\boldsymbol{J}'),$$

where $\boldsymbol{J}\cdot\boldsymbol{J}'$ denotes the standard inner product. This bilinear form is Lorentzian with signature $(-,+,+,+,+)$. The squared length of an index vector $\boldsymbol{J}$ with respect to H is denoted by $h(\boldsymbol{J})$

$$h(\boldsymbol{J}) = H(\boldsymbol{J},\boldsymbol{J}). \tag{5.1}$$

In addition, let us define root vectors α^r, $r = 0, \ldots, 4$,

$$\begin{aligned}
\alpha^0 &= \quad (1, \;\; 0,0,0,0), \\
\alpha^1 &= \frac{1}{\sqrt{3}}(-1,1,0,0,0), \\
\alpha^2 &= \frac{1}{\sqrt{3}}(0,-1,1,0,0), \\
\alpha^3 &= \frac{1}{\sqrt{3}}(0,0,-1,1,0), \\
\alpha^4 &= \frac{1}{\sqrt{3}}(0,0,0,-1,1).
\end{aligned}$$

These are all spacelike and normalised to unity, with respect to H. The action of the generators on the index vectors are realised as reflections relative to the metric H

$$\begin{aligned}
g_1\boldsymbol{J} &= \boldsymbol{J} - 2H(\alpha^0,\boldsymbol{J})\,\alpha^0, \\
s_r\boldsymbol{J} &= \boldsymbol{J} - 2H(\alpha^r,\boldsymbol{J})\,\alpha^r, \quad r = 1,2,3,4.
\end{aligned}$$

Clearly H is invariant under permutations of the indices and so under the actions of the s_r. It is also an immediate check, from the first of the relations above, that

$$H\big(g_1\boldsymbol{J},\, g_1\boldsymbol{J}'\big) = H(\boldsymbol{J},\boldsymbol{J}').$$

Thus $h(\boldsymbol{J})$ is an invariant of a web and $h(\boldsymbol{J}) = h(\boldsymbol{I})$ for all $\boldsymbol{J} \in W[\boldsymbol{I}]$.

5.2 Positive webs

We wish now to describe the structure of positive webs in some detail. In particular, we show that they admit unique characterisation in terms of simple index vectors we term *source vectors*.

To define these, consider now a positive index vector $\boldsymbol{I} = (I_1, I_2, I_3, I_4, I_5)$, with its indices in *standard order*, that is with $I_1 \geqslant I_2 \geqslant I_3 \geqslant I_4 \geqslant I_5$. We shall refer to such a vector as an *ordered source*, if

$$I_1 \leqslant \frac{\deg(\boldsymbol{I})}{3}\,.$$

More generally, a *source* is an ordered source up to permutation of indices. For a source, the condition on the indices becomes $\max(\boldsymbol{I}) \leqslant \deg(\boldsymbol{I})/3$. Examples of sources are the vectors $(1,1,1,0,0)$, which is null, and $(1,1,1,1,0)$, which has $h=-2$. Note that the special vector $(1,0,0,0,0)$ is *not* a source, and has $h=1$.

A first observation is that *positive webs exist*. We will see in the following section that the dualities are generated by flops on the genus zero curves whose index vectors are permutations of $(1,0,0,0,0)$. Given these symmetries, the instanton numbers $n_{\boldsymbol{I}}$ can only be nonzero if $\boldsymbol{I}$ belongs to a positive web. We have found index vectors for which the $n_{\boldsymbol{I}}$ are nonzero by explicit computation. This shows that positive webs exist.

At root, however, this observation concerns the representations and orbits of a Coxeter group and it should be possible to prove the existence of positive webs, and indeed of the conjecture that follows below, purely from the properties of the group and its representations. We gather some observations along these lines in the appendix D.

Based on extensive numerical computation we make:

Conjecture 1 *The positive webs are in one-one correspondence with ordered source vectors, and each positive web contains an ordered source vector as a vector of minimal degree.*

This conjecture has been checked by computer calculation for index vectors up to and including degree 100. Among these vectors there are 144,223 positive webs and there is a one-one correspondence between these and ordered sources. Assuming the conjecture, this characterisation of positive webs as being in one-one correspondence with ordered sources is surprisingly simple, since the ordered source vectors are easily listed.

The fact that every positive web contains an ordered source as a vector of minimal degree is easily shown. A positive web has an index vector $\boldsymbol{I}$ of lowest degree, which we can take to be ordered. This being so, we have

$$\deg(g_1\boldsymbol{I}) \geqslant \deg(\boldsymbol{I})\,.$$

However

$$\deg(g_1\boldsymbol{I}) - \deg(\boldsymbol{I}) \;=\; \deg(\boldsymbol{I}) - 3I_1\,.$$

Since the right hand side of this last relation is positive, it follows that $\boldsymbol{I}$ is an ordered source.

Several properties follow from the conjecture, some of which admit simple proofs independent from the conjecture:

1. *The degree of a source $\boldsymbol{I}$, which we may take to be ordered, is not lowered by the action of any $w \in \mathcal{W}$.*

   This follows immediately from the conjecture. Since, if there were vectors of degree lower than $\deg(\boldsymbol{I})$, then there would have to be an ordered vector of minimal degree, whose degree is lower than $\deg(\boldsymbol{I})$. Such a vector would be another ordered source, contrary to the conjecture.

2. *Positive webs are infinite. In particular, it is always possible to increase the degree of an index vector $\boldsymbol{I}$ by acting on it with an element of the Coxeter group.*

   To see this, take $\boldsymbol{I}$ to be ordered. Then clearly $I_5 \leqslant \deg(\boldsymbol{I})/5$ and

$$\deg(g_5\boldsymbol{I}) \;=\; 2\deg(\boldsymbol{I}) - 3I_5 \;\geqslant\; \frac{7}{5}\deg(\boldsymbol{I}) \;>\; \deg(\boldsymbol{I})\,.$$

3. *A positive web W has $h \leqslant 0$, so every vector in W is future pointing and timelike, or future pointing and null.*

   To see this, note that h is constant across a web, and for the unique ordered source $\boldsymbol{I}$ in W we have

$$h(\boldsymbol{I}) \;=\; H(\boldsymbol{I},\boldsymbol{I}) \;=\! -\frac{1}{2}\left(\deg(\boldsymbol{I})^2 - 3\boldsymbol{I}\cdot\boldsymbol{I}\right) \;\leqslant\! -\frac{1}{2}\left(\deg(\boldsymbol{I})^2 - 3I_1\deg(\boldsymbol{I})\right),$$

   where, in passing to the last expression we have used the definition of a source, and the right-hand side is non-positive in virtue of the properties of an ordered source.

5.3 Cone of positive webs

The index vectors belonging to positive webs form a convex cone which is specified by identifying its generators. This identification proceeds in two stages where the first is to note that the source vectors themselves form a convex cone.

Note first that the condition for an ordered source, $i \leqslant \deg(\boldsymbol{I})/3$, is linear. Thus a linear combination with positive coefficients of ordered sources is again an ordered source. The converse is also true: every ordered source can be written as a linear combination with positive coefficients of basis elements. To see this, we can proceed by adding sources to the basis as necessary as we increase the degree, say, although it is not clear a priori that such a process will lead to a finite basis. However, on the basis of explicit calculation up to the extent of the tables have the following conjecture:

Conjecture 2 *Each source may be written as a positive linear combination with integer coefficients of index vectors taken from the 16 sources*

$$\begin{aligned}&(1,1,1,0,0) + \text{permutations},\\ &(1,1,1,1,0) + \text{permutations},\\ &(1,1,1,1,1)\,.\end{aligned} \tag{5.2}$$

These elements are certainly necessary. At degree three it is clear that none of the permutations of (1,1,1,0,0) can be written as a positive linear combination of the other permutations. Similarly at degree four the permutations of (1,1,1,1,0) cannot be written as positive linear combinations of the other permutations and the degree three sources. At degree five it is clear that a degree five vector cannot be written as a positive linear combination of vectors of degree three and four.

At degree six, we meet the sources $(2,2,2,0,0)$, $(2,2,1,1,0)$, $(2,1,1,1,1)$, together with their permutations, and now it *is* possible to write these in terms of earlier sources:

$$\begin{aligned}(2,2,2,0,0) &= 2(1,1,1,0,0)\,,\\ (2,2,1,1,0) &= \;\;(1,1,1,0,0)+(1,1,0,1,0)\,,\\ (2,1,1,1,1) &= \;\;(1,1,1,0,0)+(1,0,0,1,1)\,.\end{aligned}$$

We do not have a proof that all sources of degree greater than five can be written as positive linear combinations of the index vectors (5.2), but we have checked that up to degree 80 all the sources can be so written.

If we accept the conjecture 2, then the cone subtended by the vectors in positive webs can be described in simple terms.

Corollary 1 *The index vectors that belong to the positive webs form a cone whose generators are the index vectors that lie in the union of the webs determined by the generators of the web of sources, given above, that is by*

$$W[(1,1,1,0,0)] \cup W[(1,1,1,1,0)] \cup W[(1,1,1,1,1)] .$$

To see this, note first that all the index vectors in $W[(1,1,1,0,0)]$ are necessary. We give two arguments; the first is immediately clear, the second is an argument that can be applied to other webs also. For the first argument note that all the vectors of $W[(1,1,1,0,0)]$ are future pointing and null, and no two are linearly dependent. This being so, any nontrivial positive linear combination of vectors in this web yields a vector that is timelike so cannot be a member of this web. For the alternative argument suppose that some vector $\boldsymbol{J}$ can be written as a positive integral combination of other vectors $\boldsymbol{K}_a$ so that

$$\boldsymbol{J} \;=\; \sum_a \alpha^a \boldsymbol{K}_a\,, \qquad \boldsymbol{K}_a \in W[(1,1,1,0,0)]\,,$$

with positive integral coefficients α^a. By acting with some element $g \in \mathcal{W}$ we can transform $\boldsymbol{J}$ to the source $(1,1,1,0,0)$. So now the relation reads

$$(1,1,1,0,0) \;=\; \sum_a \alpha^a \widetilde{\boldsymbol{K}}_a\,,$$

where $\widetilde{\boldsymbol{K}}_a = g\boldsymbol{K}_a$. Now the source has degree three and each $\deg(\widetilde{\boldsymbol{K}}_a) \geqslant 3$ for each a, and this relation is impossible for nontrivial combinations.

To show that all the vectors of $W[(1,1,1,1,0)]$ are needed as generators of the cone, suppose that some vector $\boldsymbol{J} \in W[(1,1,1,1,0)]$ can be expressed as a nontrivial integral linear combination of the vectors in $W[(1,1,1,0,0)] \cup W[(1,1,1,1,0)]$. Then it is possible to write

$$\boldsymbol{J} \;=\; \sum_a \alpha^a \boldsymbol{K}_a + \sum_b \beta^b \boldsymbol{L}_b\,, \qquad \boldsymbol{K}_a \in W[(1,1,1,0,0)]\,, \quad \boldsymbol{L}_b \in W[(1,1,1,1,0)]\,.$$

Again, we transform $\boldsymbol{J}$ to the source, which is now $(1,1,1,1,0)$ and we see that now

$$(1,1,1,1,0) \;=\; \sum_a \alpha^a \widetilde{\boldsymbol{K}}_a + \sum_b \beta^b \widetilde{\boldsymbol{L}}_b\,.$$

The degree of the source is four while $\deg(\widetilde{\boldsymbol{L}}_b) \geqslant 4$ and the $\deg(\widetilde{\boldsymbol{K}}_a)$ are multiples of three, so this relation is also impossible, for nontrivial combinations.

We see in a similar way that all the vectors in $W[(1,1,1,1,1)]$ are also generators.

For the remaining positive webs, we reverse the argument. Let $\mathring{\boldsymbol{I}}$ be the source of such a web, so with $\deg \mathring{\boldsymbol{I}} \geqslant 6$; we know that it may be written in terms of the 16 generators of the cone of sources

$$\mathring{\boldsymbol{I}} \;=\; \sum_a \alpha^a \mathring{\boldsymbol{K}}_a + \sum_b \beta^b \mathring{\boldsymbol{L}}_b + \gamma\,(1,1,1,1,1)\,,$$

where, in this relation, the $\mathring{\boldsymbol{K}}_a$ are the permutations of $(1,1,1,0,0)$ and the $\mathring{\boldsymbol{L}}_b$ are the permutations of $(1,1,1,1,0)$. For $\boldsymbol{J}$ any multi-index in $W\big[\,\mathring{\boldsymbol{I}}\,\big]$ we choose $g_{\boldsymbol{J}} \in \mathcal{W}$ that transforms $\mathring{\boldsymbol{I}}$ to $\boldsymbol{J}$. Then by applying $g_{\boldsymbol{J}}$ to the relation above, we express $\boldsymbol{J}$ as a sum of vectors in the union of the webs $W[(1,1,1,0,0)] \cup W[(1,1,1,1,0)] \cup W[(1,1,1,1,1)]$.

5.4 Birational varieties and the linear sigma model

We are not the first to observe the action of an infinite group on the instanton numbers. Many aspects of the situation with respect to the action of the Coxeter group were explained in terms of birational models by Hosono and Takagi [32] in the context of certain two-parameter families of Calabi-Yau manifolds. We note in addition the recent paper by Brodie, Constantin, Lukas, and Ruehle [33], which also observes infinite groups generated by flopping curves.

We pause to briefly recall aspects of the analysis of Hosono and Takagi, adapted to the mirror HV manifold. Note that we may write the two polynomials Q^f, $f=0,1$, in terms of the coordinates $Y_{i,b}$, $i=0,\ldots,4$ of the five $\mathbb{P}^1$'s, in the form

$$Q^f \;=\; A_{abcdef}\, Y_{0,a}Y_{1,b}Y_{2,c}Y_{3,d}Y_{4,e}\,,$$

where, in the general case, the coefficients A_{abcdef} have no particular symmetry.

Let us introduce Lagrange multipliers $Y_{5,f}$ and a superpotential

$$\mathcal{W} \;=\; Y_{5,f}\,Q^f \;=\; A_{abcdef}\, Y_{0,a}Y_{1,b}Y_{2,c}Y_{3,d}Y_{4,e}Y_{5,f}\,,$$

and consider the linear sigma model for the B-model of the mirror HV manifold [34]. This corresponds to a path integral

$$\int D\mathcal{Y}\, \mathrm{e}^{\mathrm{i}\mathcal{W}(\mathcal{Y})}\,,$$

where $D\mathcal{Y}$ denotes integration over superfields $\mathcal{Y}_{j,a}$ corresponding to the coordinates $Y_{j,a}$ and $\mathcal{W}(\mathcal{Y})$ denotes $\mathcal{W}$ with $Y_{j,a}$ replaced by $\mathcal{Y}_{j,a}$. In this form, the coordinates $Y_{j,a}$ enter symmetrically. Any of the six $Y_{j,a}$, $j=0,\ldots,5$, for given a, can be regarded as Lagrange multipliers. In this way we see that the linear sigma model makes reference to the six manifolds $\mathcal{X}_j$, with our original manifold $\mathcal{X}$ identified with $\mathcal{X}_5$.

$$\begin{aligned}
\mathcal{X}_0:&\quad A_{abcdef}\, Y_{1,b}Y_{2,c}Y_{3,d}Y_{4,e}Y_{5,f} \;=\; 0\,,\\
\mathcal{X}_1:&\quad A_{abcdef}\, Y_{0,a}Y_{2,c}Y_{3,d}Y_{4,e}Y_{5,f} \;=\; 0\,,\\
\vdots\;&\qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
\mathcal{X}_5:&\quad A_{abcdef}\, Y_{0,a}Y_{1,b}Y_{2,c}Y_{3,d}Y_{4,e} \;=\; 0\,.
\end{aligned}$$

The polynomials corresponding to $\mathcal{X}_j$ are $\partial\mathcal{W}/\partial Y_{j,a}$. For generic coefficients, these are different.

The equations for $\mathcal{X}_j$ are multilinear in the remaining coordinates, so linear in $Y_{i,a}$, for given $i\neq j$. We have two equations that are linear in the two coordinates $Y_{i,a}$, $a=0,1$. So the determinant of the coefficient matrix

$$\Delta_{ij} \stackrel{\text{def}}{=} \det\left(\frac{\partial^2\mathcal{W}}{\partial Y_{i,a}\partial Y_{j,b}}\right)_{a,b=0,1}$$

must vanish. Denote the variety that corresponds to the vanishing of Δ_{ij} by $\mathcal{Z}_{ij}$. The determinant is symmetric in the labels i and j, so $\mathcal{Z}_{ij}=\mathcal{Z}_{ji}$. If instead we were to start with the equations for the manifold $\mathcal{X}_i$ and eliminate the coordinates $Y_{i,a}$, $a=0,1$, we would arrive at the same determinant Δ_{ij}. The variety $\mathcal{Z}_{ij}$ is singular with a certain number of conifold singularities. The situation is as in Figure 6. The projection π_{ij} blows down copies of $\mathbb{P}^1_j$ while π_{ji} blows down copies of $\mathbb{P}^1_i$. The inverses of these projections are defined almost everywhere, so proceeding around the diagram from, say, $\mathcal{X}_i$ to $\mathcal{X}_j$ via $\mathcal{Z}_{ij}$ furnishes a birational map. This

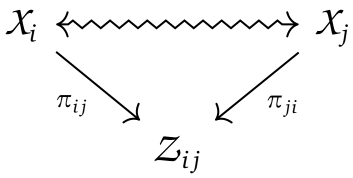

Figure 6: The birational maps between $\mathcal{X}_i$ and $\mathcal{X}_j$ derive from the projections π_{ij} and π_{ji}.

amounts to the flopping of curves, blowing down curves parallel to $\mathbb{P}^1_j$ and blowing up the singular points by copies of curves parallel to $\mathbb{P}^1_i$.

The birational map $\mathcal{X}_i \rightsquigarrow \mathcal{X}_j$ amounts to the familiar flopping of curves. A number of copies of $\mathbb{P}^1_i$ are contracted to points and these are then blown up to copies of $\mathbb{P}^1_j$.

It is instructive to see explicitly the action of the birational map on the instanton numbers. Consider generic coefficients A_{abcdef}, and the map $\mathcal{X}_5 \rightsquigarrow \mathcal{X}_0$, and let $\mathbb{A}$ denote the $2{\times}2$ matrix with the following components

$$\mathbb{A}_{af} \;=\; \frac{\partial^2 \mathcal{W}}{\partial Y_{0,a} \partial Y_{5,f}} \;=\; A_{abcdef}\, Y_{1,b} Y_{2,c} Y_{3,d} Y_{4,e}\,. \tag{5.3}$$

Now we have the familiar conifold story: the equations

$$\mathbb{A}_{af}\, Y_{5,f} \;=\; 0\,, \qquad \text{and} \qquad Y_{0,a}\, \mathbb{A}_{af} \;=\; 0 \tag{5.4}$$

describe $\mathcal{X}_5$ $(=\mathcal{X})$ and $\mathcal{X}_0$, respectively. In each case we have two equations in two unknowns and the $Y_{j,a}$, $a = 0, 1$, being the coordinates of a $\mathbb{P}^1$ cannot both vanish. Thus we recover the condition

$$\det(\mathbb{A}) \;=\; 0\,.$$

This equation describes $\mathcal{Z}_{5,0}$, which is a singular member of the family of tetraquadrics

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \!\! \left[\begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array}\right].$$

Since the rank of the matrix $\mathbb{A}$ is not 2, the generic case is that it is 1. When this is so, for each point in $\mathcal{Z}_{5,0}$ the equations (5.4) determine unique points in $\mathcal{X}_0$ and $\mathcal{X}_5$, respectively.

The rank of $\mathbb{A}$ vanishes when the four elements of the matrix vanish simultaneously. When this happens the $Y_{j,a}$ are completely undetermined and so correspond to copies of $\mathbb{P}^1_5$ and $\mathbb{P}^1_0$, respectively. In this way, we see that, in passing from $\mathcal{X}_5$ to $\mathcal{X}_0$, copies of $\mathbb{P}^1_5$ are replaced by copies of $\mathbb{P}^1_0$. The points where $\mathbb{A} = 0$ are the 24 nodes of $\mathcal{Z}_{5,0}$, so 24 copies of $\mathbb{P}^1_5$ are replaced by 24 copies of $\mathbb{P}^1_0$ as we pass from $\mathcal{X}_5$ to $\mathcal{X}_0$. We will review this in somewhat more detail in §7.

Consider now a curve $\gamma \subset \mathcal{X}_5$, of genus zero. Such a curve may project under under π_{50} to either a node of $\mathcal{Z}_{50}$, or to a curve of $\mathcal{Z}_{50}$. The curves that project to a node are the 24 curves with index vector $(1, 0, 0, 0, 0)$. To start, suppose that γ projects to a curve. The projection $\pi_{50}(\gamma)$ lifts to a curve $\tilde{\gamma}$ in $\mathcal{X}_0$. We want to understand the relation between the index vector I of γ and the index vector $\tilde{I}$ of $\tilde{\gamma} \subset \mathcal{X}_0$. The reason that $\tilde{I} \neq I$ is that, owing to the fact that some curves are flopped, the homology group H_2 changes, and so the dual group H_4 must also change. The index vector is a vector of intersection numbers $\gamma \cdot D_i$ with a basis of divisors $D_i \in H_4$, so this also changes.

Let us take an affine coordinate z for γ. We may take the coordinates $Y_{j,a}$ to be polynomials in z which are of minimal degree in the sense that $Y_{j,1}$ and $Y_{j,0}$ have no common factors. Let us also simplify notation by writing $Y_{5,a} = Y_a$ and $Y_{0,a} = \widetilde{Y}_a$. In virtue of (5.4) we have that

$$\begin{aligned} \frac{Y_1}{Y_0} &= -\frac{\mathbb{A}_{00}}{\mathbb{A}_{01}} = -\frac{\mathbb{A}_{10}}{\mathbb{A}_{11}}\,, \\ \frac{\widetilde{Y}_1}{\widetilde{Y}_0} &= -\frac{\mathbb{A}_{00}}{\mathbb{A}_{10}} = -\frac{\mathbb{A}_{01}}{\mathbb{A}_{11}}\,. \end{aligned} \tag{5.5}$$

We see from (5.3) that the $\mathbb{A}_{af}$ each have degree $I_2 + I_3 + I_4 + I_5$. We want to calculate the degrees I_1 and I_0 of the Y_a and the $\widetilde{Y_f}$, respectively. The various ratios of the $\mathbb{A}_{af}$ in (5.5) may have common factors. However, since we have assumed the Y_a and $\widetilde{Y}_f$ to have minimal degree, we may write

$$\begin{aligned} \mathbb{A}_{00} &= -Y_1 u\,, & \mathbb{A}_{10} &= -Y_1 v\,, \\ \mathbb{A}_{00} &= Y_0 u\,, & \mathbb{A}_{11} &= Y_0 v\,, \end{aligned}$$

where u and v are polynomials in z. From the second row in (5.5) we see that we may take $u = -\widetilde{Y}_1$ and $v = \widetilde{Y}_0$. In this way, we see that

$$\mathbb{A} = \begin{pmatrix} Y_1\widetilde{Y}_1 & -Y_1\widetilde{Y}_0 \\ -Y_0\widetilde{Y}_1 & Y_0\widetilde{Y}_0 \end{pmatrix}.$$

We have already observed that the degree of $\mathbb{A}$ is $I_2 + I_3 + I_4 + I_5$, and this degree is seen to be equal to that of the right hand side of the above relation, which is $I_0 + I_1$. It follows that

$$I_0 = -I_1 + I_2 + I_3 + I_4 + I_5\,. \tag{5.6}$$

Taken together with the fact that $\tilde{I}_j = I_j$, for $j = 2, 3, 4, 5$, we see that we have recovered the duality transformation $\tilde{I} = g_1 I$. Since this transformation law is a consequence of the transformation acting on H_4, it follows that the index vectors of the curves that project to the nodes are subject to the same transformation, so for example

$$(1, 0, 0, 0, 0) \longrightarrow (-1, 0, 0, 0, 0)\,.$$

In order to find the relation (5.6), we have identified curves in $\mathcal{X}_0$ and $\mathcal{X}_5$. Equally, we could have identified the homology groups and compared two curves γ_0 and γ_5 with the same index vectors, which would have given different curves.

6 Monodromies

We wish to find the monodromies around the loci E_μ and D_I defined in (3.2) and (3.4). In the next subsection, we will compute the monodromy around the varieties E_i using the series expansions for the periods around the large complex structure point. For the loci D_I, we use numerical integration of the Picard-Fuchs equation to find the monodromies. As we do not have the general five-parameter Picard-Fuchs equation and such an equation would in any case be impractical for this purpose, we use the Picard-Fuchs equations for one-parameter subfamilies as discussed in §3.3. Finally, using the relation between the natural basis of periods in the patch $\varphi^0 \neq 0$ and $\varphi^i \neq 0$, we are able to compute the monodromies around E_0 in §6.3.

6.1 Monodromies around the large complex structure points E_i

The monodromy matrices around the loci E_i can be read directly from the asymptotics of the period vector $\Pi^{(0)}$ in the integral basis. These correspond to coordinate transformations $\varphi^i \to e^{2\pi i}\varphi^i$, or alternatively $t^i \to t^i + 1$. These transformations give the following monodromies.

$$\mathrm{M}_{E_1} = \begin{pmatrix} 1 & -1 & \mathbf{0}_4^T & 2 & 0 & \mathbf{0}_4^T \\ 0 & 1 & \mathbf{0}_4^T & 0 & 0 & \mathbf{0}_4^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbb{I}_4 & \mathbf{0}_4 & \mathbf{0}_4 & 2\mathbb{I}_4 - \mathbb{2}_4 \\ 0 & 0 & \mathbf{0}_4^T & 1 & 0 & \mathbf{0}_4^T \\ 0 & 0 & \mathbf{0}_4^T & 1 & 1 & \mathbf{0}_4^T \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbb{0}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbb{I}_4 \end{pmatrix}. \tag{6.1}$$

The monodromies around other loci E_i are obtained by swapping the second and $(i+2)$'th column and row and the seventh and $(i+7)$'th column and row with each other.

6.2 Monodromies around the loci D_I

We now set $\varphi^i = s^i\varphi$, $\varphi^0 = 1$ with s^i complex constants. Δ becomes a polynomial of degree 16 in φ. This has 16 roots, which are the intersections of the singular locus $\Delta = 0$ with the plane $\varphi^i = s^i\varphi$. We will find particularly simple Picard-Fuchs operators when some of the s^i are equal. In these cases some of the periods become equal, hence there exists an operator of degree < 12, whose independent solutions are exactly the distinct periods. These differential equations can be integrated numerically, yielding the monodromy matrices for the independent periods.

Of course the matrices found this way do not give the complete monodromy, as not all of the 12 periods are independent on the lines that we study. However, there is a natural relation between these 'reduced' matrices and the full monodromy matrices, which can be used, together with the S_5 symmetry, to find the full monodromy. To exemplify this process, let us consider the case where $s^1 \neq s^2 = s^3 = s^4 = s^5$. This leaves a set of 6 independent periods, as

$$\begin{aligned} \varpi^{(0);2}(\varphi) &= \varpi^{(0);3}(\varphi) = \varpi^{(0);4}(\varphi) = \varpi^{(0);5}(\varphi), \\ \varpi_2^{(0)}(\varphi) &= \varpi_3^{(0)}(\varphi) = \varpi_4^{(0)}(\varphi) = \varpi_5^{(0)}(\varphi). \end{aligned}$$

The general monodromy matrix, giving the monodromy transformation of the periods around a singularity φ_*, can be written as

$$\mathrm{M}_{\varphi_*} = (\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{10}, \boldsymbol{u}_{11}), \tag{6.2}$$

where $\boldsymbol{u}_i$ are 12-component column vectors

$$\boldsymbol{u}_i = (u_i^0, u_i^1, \ldots, u_i^{10}, u_i^{11})^T.$$

Since some of the periods are equal, we cannot find their individual contributions to this matrix from the reduced monodromy matrix. Instead, the reduced matrix takes the form

$$\widehat{\mathrm{M}}_{\varphi^*} = (\hat{\boldsymbol{u}}_0, \hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2 + \hat{\boldsymbol{u}}_3 + \hat{\boldsymbol{u}}_4 + \hat{\boldsymbol{u}}_5, \hat{\boldsymbol{u}}_6, \hat{\boldsymbol{u}}_7, \hat{\boldsymbol{u}}_8 + \hat{\boldsymbol{u}}_9 + \hat{\boldsymbol{u}}_{10} + \hat{\boldsymbol{u}}_{11}), \tag{6.3}$$

where now $\hat{u}_i$ are 6 component column vectors

$$\hat{\boldsymbol{u}}_i \;=\; (u_i^0, u_i^1, u_i^2, u_i^6, u_i^7, u_i^8)^T\,.$$

Relations like this constrain the full 12×12 monodromy matrices. We can construct the full matrices from this data by numerically integrating the Picard-Fuchs equation along several paths in the complex line.

Finally, to make the computation slightly simpler, we use the fact that the singularities at $\Delta = 0$ correspond to conifolds. It is expected that the monodromies around the conifold loci take the form

$$\mathrm{M} \;=\; \mathrm{I}_{12} - \boldsymbol{w}(\Sigma \boldsymbol{w})^T\,, \tag{6.4}$$

where $\boldsymbol{w}$ is a 12-component vector that gives the cycle vanishing at the conifold point. Thus we can reduce the problem to finding 16 vectors corresponding to the different components D_I of the singular locus.

To get an idea of how the computation proceeds, we briefly explain the computation of some monodromies in a relatively simple example. To be precise, we study the case

$$s^1 = 1\,, \qquad s^2 \;=\; s^3 \;=\; s^4 \;=\; s^5 = \frac{95}{100}\,.$$

We have 6 independent periods and so can find, using the procedure outlined in §3.3, a Picard-Fuchs operator of degree 6. This operator has solutions $\varpi^{(0);0}(\varphi)$, $\varpi^{(0);1}(\varphi)$, $\varpi^{(0);2}(\varphi)$, $\varpi_1^{(0)}(\varphi)$, $\varpi_2^{(0)}(\varphi)$, and $\varpi_3^{(0)}(\varphi)$. In the ensuing discussion, we shall find use for the short-hands

$$\mu \;=\; 5\,\frac{81-4\sqrt{95}}{5041}\,, \quad \overline{\mu} \;=\; 5\,\frac{81+4\sqrt{95}}{5041}\,, \quad \nu \;=\; 5\,\frac{12-\sqrt{95}}{98}\,, \quad \overline{\nu} \;=\; 5\,\frac{12+\sqrt{95}}{98}\,. \tag{6.5}$$

The discriminant expressed in terms of φ is in this case, up to a multiplicative constant,

$$\Delta \;=\; (\varphi-1)^6(\varphi-\mu)(\varphi-\overline{\mu})(\varphi-\nu)^4(\varphi-\overline{\nu})^4\,.$$

Each of these factors corresponds to an intersection of a component D_I with the line. In this way, we can associate each factor with such a component:

$$\begin{aligned}
D_{\{0\}} \;&=\; \{\varphi \;=\; \mu\}\,,\\
D_{\{0,2\}} \;&=\; D_{\{0,3\}} \;=\; D_{\{0,4\}} \;=\; D_{\{0,5\}} \;=\; \{\varphi \;=\; \nu\}\,,\\
D_{\{0,1\}} \;&=\; \{\varphi \;=\; \overline{\mu}\}\,,\\
D_{\{0,2,3\}} \;&=\; D_{\{0,2,4\}} \;=\; D_{\{0,2,5\}} \;=\; D_{\{0,3,4\}} \;=\; D_{\{0,3,5\}} \;=\; D_{\{0,4,5\}} \;=\; \{\varphi = 1\}\,,\\
D_{\{0,1,2\}} \;&=\; D_{\{0,1,3\}} \;=\; D_{\{0,1,4\}} \;=\; D_{\{0,1,5\}} \;=\; \{\varphi \;=\; \overline{\nu}\}\,.
\end{aligned}$$

The monodromy matrices around these points are given by

$$\widehat{\mathrm{M}}_{\mathbb{\mu}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \widehat{\mathrm{M}}_{\mathbb{\nu}} = \begin{pmatrix} 9 & 0 & -8 & 16 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 4 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathrm{M}}_{\overline{\mathbb{\mu}}} = \begin{pmatrix} 3 & -2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \widehat{\mathrm{M}}_{1} = \begin{pmatrix} 25 & 0 & -48 & 96 & 48 & 96 \\ 12 & 1 & -24 & 48 & 24 & 48 \\ 6 & 0 & -11 & 24 & 12 & 24 \\ -6 & 0 & 12 & -23 & -12 & -24 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & -6 & 12 & 6 & 13 \end{pmatrix},$$

$$\widehat{\mathrm{M}}_{\overline{\mathbb{\nu}}} = \begin{pmatrix} 17 & -16 & -16 & 64 & 0 & 96 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & -6 & -5 & 24 & 0 & 36 \\ -4 & 4 & 4 & -15 & 0 & -24 \\ 4 & -4 & -4 & 16 & 1 & 24 \\ 1 & -1 & -1 & 4 & 0 & 7 \end{pmatrix}.$$

To find the full monodromy matrix corresponding to the monodromy around $D_{\{0\}}$, we use

$$\widehat{\mathrm{M}}_{\mathbb{\mu}} = \widehat{\mathrm{M}}_{\{0\}}, \tag{6.6}$$

where $\mathrm{M}_{\{0\}}$ is of the form (6.4), and $\widehat{\mathrm{M}}_{\{0\}}$ of the form (6.3). This equation allows us to partially fix the vector $\boldsymbol{w}$, which we denote by $\boldsymbol{w}_{\{0\}}$, giving conditions which can be solved by

$$\boldsymbol{w}_{\{0\}} = (0, 0, 0, w^4, w^5, -w^4, -w^5, 1, 0, 0, w^{10}, w^{11}, -w^{10} - w^{11}).$$

To proceed, we can compute the monodromies on other similar lines, such as $s^1 = s^3 = s^4 = s^5 = \frac{95}{100}$, $s^2 = 1$. Alternatively, we could impose the S_5 symmetry, whereby all the periods related by a permutations of the indices 2,3,4 and 5 must contribute equally. In this way we see that the cycle vanishing at $D_{\{0\}}$ has sixth component 1, and all other components zero:

$$\boldsymbol{w}_{\{0\}} = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0).$$

Next, we concentrate on the singularities at $\varphi = \overline{\mathbb{\mu}}$ and $\varphi = \mathbb{\nu}$. The latter lies on four singular loci, $D_{\{0,2\}}$, $D_{\{0,3\}}$, $D_{\{0,4\}}$, and $D_{\{0,5\}}$, while the former lies solely in $D_{\{0,1\}}$. Therefore we can use an expression of the form (6.4) for the monodromy matrix around the singularity at $\mathbb{\nu}$, while around $\overline{\mathbb{\mu}}$ the monodromy is a product of four similar matrices. By comparing to $\mathrm{M}_{\mathbb{\nu}}$, we find

$$\boldsymbol{w}_{\{0,1\}} = (-2, 0, 0, w^4, w^5, -w^4 - w^5, 1, -1, 0, w^{10}, w^{11}, -w^{10} - w^{11}).$$

By either computing monodromies with different values of s^i, or by a symmetry argument, we find that the vector is given by

$$\boldsymbol{w}_{\{0,1\}} = (-2, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0),$$

which allows us to compute the monodromy matrix $\mathrm{M}_{\{0,1\}}$. Again, by symmetry or considering different values of weights, it can be shown that the vectors giving the monodromy matrices $\mathrm{M}_{\{0,2\}}$, $\mathrm{M}_{\{0,3\}}$, $\mathrm{M}_{\{0,4\}}$ and $\mathrm{M}_{\{0,5\}}$ are given by permuting the components of the vector $\boldsymbol{w}_{\{0,1\}}$:

$$\begin{aligned} \boldsymbol{w}_{\{0,2\}} &= (-2, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0), \\ \boldsymbol{w}_{\{0,3\}} &= (-2, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0), \\ \boldsymbol{w}_{\{0,4\}} &= (-2, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 0), \\ \boldsymbol{w}_{\{0,5\}} &= (-2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1). \end{aligned}$$

As a consistency check, it can be seen that the matrix around $\mathbb{D}$ is given by a product of reduced monodromy matrices:

$$\mathrm{M}_{\mathbb{D}} \;=\; \widehat{\mathrm{M}}_{\{0,2\}}\,\widehat{\mathrm{M}}_{\{0,3\}}\,\widehat{\mathrm{M}}_{\{0,4\}}\,\widehat{\mathrm{M}}_{\{0,5\}}\,.$$

The matrices corresponding to the remaining loci can be found using similar techniques. This is made slightly more complicated by the fact that paths on the lines $s^1 \neq s^2 = s^3 = s^4 = s^5$ only circle intersections of multiple components. Perhaps the easiest way to circumvent this is to consider a new case where $s^1 \neq s^2 \neq s^3 = s^4 = s^5 \neq s^1$, and permutations thereof. In the case $s_1 \neq s_2 \neq s_3$, $D_{\{0,1,2\}}$ intersects the plane $\varphi^i = s^i \varphi$ in a point that is distinct from the other components. This computation, together with symmetry considerations, leads us to a form for the monodromy matrix where the vanishing cycle is given by

$$\boldsymbol{w}_{\{0,1,2\}} \;=\; (4,\, 0,\, 0,\, 2,\, 2,\, 2,\, -1,\, 1,\, 1,\, 0,\, 0,\, 0)\,.$$

The vectors in other cases are given by permuting the components of this vector. Again, one can check that the matrices $\widehat{\mathrm{M}}_1$ and $\widehat{\mathrm{M}}_{\overline{\mathbb{D}}}$ can be written in terms of the reduced matrices associated to these vectors:

$$\begin{aligned}
\widehat{\mathrm{M}}_1 \;&=\; \widehat{\mathrm{M}}_{\{0,2,3\}}\,\widehat{\mathrm{M}}_{\{0,2,4\}}\,\widehat{\mathrm{M}}_{\{0,2,5\}}\,\widehat{\mathrm{M}}_{\{0,3,4\}}\,\widehat{\mathrm{M}}_{\{0,3,5\}}\,\widehat{\mathrm{M}}_{\{0,4,5\}}\,,\\
\widehat{\mathrm{M}}_{\overline{\mathbb{D}}} \;&=\; \widehat{\mathrm{M}}_{\{0,1,2\}}\,\widehat{\mathrm{M}}_{\{0,1,3\}}\,\widehat{\mathrm{M}}_{\{0,1,4\}}\,\widehat{\mathrm{M}}_{\{0,1,5\}}\,.
\end{aligned}$$

We have found 16 matrices $M_{\{0\}}$, $M_{\{0,i\}}$, and $M_{\{0,i,j\}}$, and there remain 16 still unaccounted for. These, however, can be constructed from the known 16 by a change of indices $0 \longleftrightarrow i$. By symmetry, the matrices that are related to each other by such a permutation must be equal. We must, however, take into account that the monodromy transformations obtained in this way are expressed in different bases. Changing all to a common basis, which we take to be the symplectic basis where $\Pi^{(0)}$ is given by (4.1), gives matrices with different entries. Thus, for example

$$\mathrm{M}_{\{1\}} \;=\; \mathrm{T}^{-1}_{\Pi^{(1)}\Pi^{(0)}}\,\mathrm{M}_{\{0\}}\mathrm{T}_{\Pi^{(1)}\,\Pi^{(0)}}\,,$$

where $\mathrm{T}_{\Pi^{(1)}\Pi^{(0)}}$, given explicitly in (6.7), is a change of basis matrix from the canonical integral basis in the patch $\varphi^0 = 1$ to the canonical integral basis in the patch $\varphi^1 = 1$. We will see another explicit example of this in the next subsection where we use this observation to compute the monodromy around the locus E_0 'at infinity'.

6.3 Monodromy around infinity, E_0

The remaining singular locus is $\varphi^0 = 0$, which, seen from the patch $\varphi^0 = 1$ corresponds to the monodromy around infinity. Due to the S_5 symmetry, we know that the locus $\varphi^0 = 0$ is on a par with the other loci $\varphi^i = 0$. The only essential difference to the earlier computation is the use of a different basis for the periods.

To find the appropriate change of basis, we use the matrix $\mathrm{T}_{\pi^{(0)}\pi^{(1)}}$ from (3.24), which gives the relation between the period vectors $\pi^{(1)}$ and $\pi^{(0)}$, whose components give the periods as combinations of Bessel function integrals. Using the matrices $\mathrm{T}_{\varpi^{(i)}\pi^{(i)}}$ and $\mathrm{T}_{\Pi^{(i)}\varpi^{(i)}}$, we can change from this basis to the integral basis. Note that due to the symmetry, the relation of the vectors $\pi^{(1)}$ to the integral period vector $\Pi^{(1)}$ is same as that of $\pi^{(0)}$ to $\Pi^{(0)}$, so that

$\mathrm{T}_{\Pi^{(1)}\pi^{(1)}} = \mathrm{T}_{\Pi^{(0)}\pi^{(0)}}$. The transformation from $\Pi^{(1)}$ to $\Pi^{(0)}$ is thus given by

$$\begin{aligned}
\mathrm{T}_{\Pi^{(0)}\Pi^{(1)}} &= \mathrm{T}_{\Pi^{(0)}\varpi^{(0)}}\mathrm{T}_{\varpi^{(0)}\pi^{(0)}}\mathrm{T}_{\pi^{(0)}\pi^{(1)}}\left(\mathrm{T}_{\Pi^{(1)}\varpi^{(1)}}\mathrm{T}_{\varpi^{(1)}\pi^{(1)}}\right)^{-1} \\
&= \begin{pmatrix}
-1 & 0 & 1 & 1 & 1 & 1 & -16 & -12 & -6 & -6 & -6 & -6 \\
0 & 1 & -1 & -1 & -1 & -1 & 12 & 12 & 6 & 6 & 6 & 6 \\
0 & 0 & -1 & 0 & 0 & 0 & 6 & 6 & 0 & 4 & 4 & 4 \\
0 & 0 & 0 & -1 & 0 & 0 & 6 & 6 & 4 & 0 & 4 & 4 \\
0 & 0 & 0 & 0 & -1 & 0 & 6 & 6 & 4 & 4 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & -1 & 6 & 6 & 4 & 4 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1
\end{pmatrix}.
\end{aligned} \tag{6.7}$$

The monodromy of $\Pi^{(0)}$ around $\varphi^0 = 0$ is, by symmetry, equal to the mondromy of $\Pi^{(1)}$ around $\varphi^1 = 0$, which directly allows us to find the monodromy of $\Pi^{(1)}$ around the locus $\varphi^0 = 0$:

$$\mathrm{M}_{E_1} = \left(\mathrm{T}_{\Pi^{(1)}\Pi^{(0)}}\right)^{-1}\left(\mathrm{M}_{E_0}\right)^{-1}\mathrm{T}_{\Pi^{(1)}\Pi^{(0)}} = \begin{pmatrix}
1 & -1 & 1 & 1 & 1 & 1 & -2 & -12 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -12 & 24 & 6 & 6 & 6 & 6 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 6 & 2 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 & 2 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

We have used the inverse of the matrix M_{E_0} because the direction of the contour is reversed when changing patches.

6.4 Recovering monodromies for the quotient manifolds

Finally, let us briefly comment on the relation of the results presented here to those found for the quotient manifolds in [3]. Specialising to the locus $\varphi^i = \varphi$, $\varphi^0 = 1$, the discriminant vanishes for $\varphi \in \{\frac{1}{25}, \frac{1}{9}, 1\}$. The locus $D_{\{0\}}$ is associated to the first of these points, the loci $D_{\{0,i\}}$ to the second, and the loci $D_{\{0,i,j\}}$ to the last.

On the locus $\varphi^i = \varphi$, $\varphi^0 = 1$, only four of the elements of $\Pi^{(0)}$ are independent. We collect these into the reduced period vector $\widehat{\Pi}^{(0)}$. This is related to the integral period vector $\Pi_{\mathbb{Z}_{10/\kappa}}$ of the quotient manifold $\mathrm{HV}/\mathbb{Z}_{10/\kappa}$ by a matrix T_κ.

$$\Pi^{(0)} \overset{\text{def}}{=} \begin{pmatrix} \Pi_0^{(0);0} \\ \Pi^{(0);1} \\ \Pi_0^{(0)} \\ \Pi_1^{(0)} \end{pmatrix}, \qquad \Pi_{\mathbb{Z}_{10/\kappa}} = \mathrm{T}_\kappa \widehat{\Pi}^{(0)}, \qquad \mathrm{T}_\kappa = \begin{pmatrix} 10/\kappa & 0 & 0 & 0 \\ 0 & 2/\kappa & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can now give the monodromies M_1, $\mathrm{M}_{\frac{1}{9}}$, and $\mathrm{M}_{\frac{1}{25}}$ of $\Pi_{\mathbb{Z}_{10/\kappa}}$. First, take the product of the relevant matrices $\widehat{\mathrm{M}}_s$ which give the monodromies of $\widehat{\Pi}^0$, and then conjugate by T_κ to obtain the monodromies of $\Pi_{\mathbb{Z}_{10/\kappa}}$. For instance, for the $\mathbb{Z}_{10}$ quotient

$$\mathrm{M}_{\frac{1}{25}} = \mathrm{T}_1^{-1}\widehat{\mathrm{M}}_{\{0\}}\mathrm{T}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -10 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$
\begin{array}{ccccc}
 & & L_i & \hookrightarrow & \mathrm{H}\Lambda \\
 & & & & \downarrow \pi_j \\
\mathbb{P}^1 & \overset{\varphi}{\hookrightarrow} & F_j & \hookrightarrow & \widehat{\mathrm{H}\Lambda}_j \\
 & & \downarrow \pi_{m,n} & & \downarrow \pi_{m,n} \\
 & & B & \hookrightarrow & \mathbb{P}^1_m \times \mathbb{P}^1_n
\end{array}
$$

Figure 7: Structure of the fibrations relevant to counting some rational and elliptic curves. L_i denote the lines on HΛ that are blown down to obtain the singular mirror Hulek-Verrill manifold $\widehat{\mathrm{H}\Lambda}_j$ with the birational map denoted by π_j. $\widehat{\mathrm{H}\Lambda}_j$ is an elliptically fibred manifold with base $\mathbb{P}^1 \times \mathbb{P}^1$, and a generic fibre F_j. On the discriminant locus $\Delta = 0$ of the elliptic fibration, the fibre becomes singular. On a special set of points B, corresponding to nodes of the discriminant locus, the degenerate fibre is a union of two rational curves.

7 Counting curves on the mirror Hulek-Verrill manifold

There is an interesting problem in directly counting the numbers of various curves of different degrees on the Hulek-Verrill manifold and its quotients. This serves multiple purposes, such as confirming the predictions of mirror symmetry and counting microstates for some configurations of branes wrapped on various cycles on the manifold. In this section, we will find the rational curves up to degree five, and verify that their number agrees with the instanton numbers of §4.

It is good to recognise that the manifolds in HΛ can be realised as blowups of the five singular tetraquadrics $\widehat{\mathrm{H}\Lambda}_i$ $i = 0, \dots, 4$, with 24 nodes, using the procedure of [11]. $\widehat{\mathrm{H}\Lambda}_i$ are the spaces $\mathcal{Z}_{5i}$ of the section §5 and are singular limits of the family corresponding to the configuration

$$
\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix}
\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}_{\chi=-128} .
$$

Members of the family HΛ are elliptically fibred manifolds, and we are able to compute the discriminant of the fibration using standard methods [35]. It turns out that the the first few low-degree rational curves appear as irreducible components of singular fibres of the elliptic fibration as in Figure 7. The explicit embeddings of curves depend non-trivially on the coefficients in the defining polynomials, but the curve counts for generic members of the family of mirror manifolds agree. For this reason we will, in place of explicit expressions, discuss properties of a generic member of the family HΛ.

Parts of this discussion are best framed in terms of various embedding maps with different degrees. Among these appear numerous context-specific rational functions. For this reason we will often use the symbols $r_k(z)$, $\widetilde{r}_k(z)$, to denote a ratio of two *situation-dependent* polynomials of degree k. Two instances of these symbols in this section should not automatically be understood as referring to the same function. In this section the Latin indices generically run from 0 to 4. When two different indices appear in an expression, they are understood to refer to distinct numerical values.

7.1 Blow-down and elliptic fibration

The configuration matrix of HΛ is of the form considered in [11], which means that we can use the contraction procedure to obtain a quadric manifold $\widehat{\text{H}\Lambda}_i$ defined by one equation:

$$\begin{array}{c}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{array}\begin{bmatrix}1 & 1\\1 & 1\\1 & 1\\1 & 1\\1 & 1\end{bmatrix}_{\chi=-80} \qquad\longleftrightarrow\qquad \begin{array}{c}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{array}\begin{bmatrix}2\\2\\2\\2\end{bmatrix}_{\chi=-128} . \tag{7.1}$$

We frequently distinguish the five $\mathbb{P}^1$ factors in the product $(\mathbb{P}^1)^5$ by subscripts. For example, $\mathbb{P}^1_i$ denotes the i'th such $\mathbb{P}^1$, and has projective coordinates $Y_{i,0}, Y_{i,1}$. Throughout this section, we use affine coordinates $Y_i = \frac{Y_{i,1}}{Y_{i,0}}$. This makes the equations simpler, and the instances where projective coordinates are needed for statements to be strictly correct are few. Nonetheless, all polynomials in this section can be homogenised using projective coordinates, and in this way any minor ambiguities relating to points at infinity are cleared up.

To see in detail how the process depicted in (7.1) works, let us consider the contraction with respect to $\mathbb{P}^1_i$. The equations defining the manifold HΛ can be written as

$$\begin{aligned} Q^1(\boldsymbol{Y}) &= \alpha_i Y_i + \beta_i\,,\\ Q^2(\boldsymbol{Y}) &= \gamma_i Y_i + \delta_i\,, \end{aligned} \tag{7.2}$$

with $\alpha_i, \beta_i, \gamma_i, \delta_i$ multilinear functions of the four coordinates Y_j, $j \neq i$, and no sum over the repeated i is implied. The pair of conditions (7.2) is equivalent to the single matrix equation

$$\begin{pmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{pmatrix}\begin{pmatrix}Y_i\\ 1\end{pmatrix} = \boldsymbol{0}\,.$$

Existence of a solution is equivalent to the vanishing of the determinant of the matrix, that is

$$\widehat{Q}^i \stackrel{\text{def}}{=} \alpha_i\,\delta_i - \beta_i\,\gamma_i = 0\,. \tag{7.3}$$

We denote the variety defined by $\{\widehat{Q}^i = 0\} \subset (\mathbb{P}^1)^4$ as $\widehat{\text{H}\Lambda}_i$. One can see from (7.3) that $\widehat{\text{H}\Lambda}_i$ is a conifold. Since the functions $\alpha_i, \beta_i, \gamma_i, \delta_i$ are multilinear, the corresponding configuration matrix is indeed of the form (7.1).

Note that the varieties $\widehat{\text{H}\Lambda}_i$ are birational to HΛ. The projection $\pi_i : \text{H}\Lambda \to \widehat{\text{H}\Lambda}_i$ defined by

$$\pi_i(Y_i, Y_j, Y_k, Y_m, Y_n) = (Y_j, Y_k, Y_m, Y_n)$$

gives a birational map between the varieties. Given a point $(Y_j, Y_k, Y_m, Y_n) \in \widehat{\text{H}\Lambda}_i$, with $\alpha_i \neq 0$ or $\gamma_i \neq 0$, the equations $Q^1 = Q^2 = 0$ are solved by the unique point $Y_i = -\frac{\beta_i}{\alpha_i}$ or $Y_i = -\frac{\delta_i}{\gamma_i}$, respectively (when $\alpha_i, \gamma_i \neq 0$, these agree), and the inverse π_i^{-1} is well-defined. However, when $\alpha_i = \gamma_i = 0$, the conditions $Q^1 = Q^2 = 0$ are satisfied if and only if $\beta_i = \delta_i = 0$. If this is the case, the equation $Q^1 = Q^2 = 0$ is true for all values of Y_i, and the inverse image of the

$$\begin{array}{ccc} & & \mathrm{H}\Lambda \\ & & \big\downarrow{\scriptstyle \pi_i} \\ E_{i;m,n} & \hookrightarrow & \widehat{\mathrm{H}\Lambda}_i \\ & & \big\downarrow{\scriptstyle \pi_{m,n}} \\ & & \mathbb{P}^1_m \times \mathbb{P}^1_n . \end{array}$$

Figure 8: The Elliptic Fibration on $\mathrm{H}\Lambda_i$ with base $\mathbb{P}^1_m \times \mathbb{P}^1_n$.

point in $\widehat{\mathrm{H}\Lambda}_i$ is a line $\alpha_i = \beta_i = \gamma_i = \delta_i = 0$ on $\mathrm{H}\Lambda$. For generic values of parameters, including generic $\mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_2$ and $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric cases, these equations have 24 solutions. From the definition of $\widehat{Q}^i$, (7.3), it is clear that the points satisfying this condition are exactly the singularities of $\widehat{\mathrm{H}\Lambda}_i$.

The manifold $\mathrm{H}\Lambda$ is generically a smooth elliptic threefold, while $\widehat{\mathrm{H}\Lambda}_i$ is an elliptically fibred singular variety (see Figure 8). To see this explicitly, let us choose the base of the fibration to be $\mathbb{P}^1_m \times \mathbb{P}^1_n$. We can view the polynomial $\widehat{Q}^i$ as a biquadratic whose coefficients depend on Y_m and Y_n.

$$\widehat{Q}^i(Y_j, Y_k) \;=\; \sum_{a,b=0}^{2} A_{a,b}(Y_m, Y_n)\, Y_j^a\, Y_k^b\,, \tag{7.4}$$

where $A_{a,b}$ are functions of the base coordinates Y_m, Y_n. The exact form of these functions depends on the choice of the Calabi-Yau manifold $\mathrm{H}\Lambda$. This defines a biquadric subvariety $E_{i;m,n}$ of $\mathbb{P}^1_j \times \mathbb{P}^1_k$, which is a Calabi-Yau variety of dimension one, and thus an elliptic curve. Any biquadratic in $\mathbb{P}^1_m \times \mathbb{P}^1_n$ can be transformed into the Weierstrass form [35]. To this end, one first computes the quadratic discriminant of (7.4) with respect to Y_j.

$$\mathcal{D}_n(Y_k) \;=\; \left(\sum_{a=0}^{2} A_{a,1} Y_k^i\right)^2 - 4\left(\sum_{a=0}^{2} A_{i,2} Y_k^i\right)\left(\sum_{i=0}^{2} A_{i,0} Y_k^i\right) \overset{\text{def}}{=} b_4 Y_k^4 + 4 b_3 Y_k^3 + 6 b_2 Y_k^2 + 4 b_1 Y_k + b_0\,.$$

One computes the two “Eisenstein invariants of plane quartics” defined in [35] for this polynomial:

$$\begin{aligned} D_{m,n} \;&=\; b_4 b_0 + 3 b_2^2 - 4 b_3 b_1\,, \\ E_{m,n} \;&=\; b_4 b_1^2 + b_3^2 b_0 - b_4 b_2 b_0 - 2 b_3 b_2 b_1 + b_2^3\,, \end{aligned} \tag{7.5}$$

where each b_a is a function of Y_m and Y_n These can be used to write the Weierstrass form of the elliptic fibre as

$$y^2 = x^3 - D_{m,n} x + 2 E_{m,n}\,.$$

The discriminant of this elliptic curve is

$$\Delta_{i;\,m,n} = -D_{m,n}^3 + 27 E_{m,n}^2\,,$$

where the index i refers to the coordinate with respect to which we have contracted $\mathrm{H}\Lambda$ in order to obtain the singular manifold $\widehat{\mathrm{H}\Lambda}_i$, and the indices m and n refer to the choice of the base of the fibration. It is useful to observe that the discriminants satisfy the relations

$$\Delta_{i;\,m,n} = \Delta_{j;\,m,n} = \Delta_{k;\,m,n}\,,$$

Table 9: The Kodaira classification of singular fibres that appear in the elliptic fibration over the base $\mathbb{P}^1_m \times \mathbb{P}^1_n$.

Type	Ord(D)	Ord(E)	Ord(Δ)	Dynkin Label	Fibre	Number
I_1	0	0	1	A_1	1 nodal curve	Continuum
I_2	0	0	2	A_2	2 curves meeting at 2 points	200
II	1	1	2	A_1	1 cuspidal curve	192

In other words, for the purposes of computing the discriminant on the base $\mathbb{P}_m \times \mathbb{P}_n$, it does not matter which contraction we choose. We plot the zero loci for three $\Delta_{i;m,n}$ in Figure 10.

In the generic case, $\Delta_{i;m,n}$ is an irreducible bidegree $(24,24)$ polynomial.

$$\Delta_{i;m,n}(Y_m, Y_n) = \sum_{a,b=0}^{24} \alpha_{a,b} Y_m^a Y_n^b \,.$$

In case the manifold is symmetric under $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, the discriminant satisfies one or both of the following symmetry relations:

$$Y_m^{24} Y_n^{24} \Delta_{i;m,n}\left(\frac{1}{Y_m}, \frac{1}{Y_n}\right) = \Delta_{i;m,n}(Y_m, Y_n), \qquad \Delta_{i;m,n}(-Y_m, -Y_n) = \Delta_{i;m,n}(Y_m, Y_n).$$

A sketch of Δ for such a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric case is given in Figure 9. The vanishing locus of Δ corresponds to the singular locus of elliptic fibres. The types of singular fibres on elliptic surfaces have been classified by Kodaira [16, 17]. Table 9 below contains the cases relevant for us. Generically $\Delta_{i;m,n}$ is irreducible, so a generic point on the curve $\Delta_{i;m,n} = 0$ corresponds to a singularity of the type I_1. In other words, the fibre over a generic point over $\{\Delta_{i;m,n} = 0\} \subset \mathbb{P}^1_m \times \mathbb{P}^1_n$ is a nodal curve. This is related to the fibration structure of the manifold. Namely, the generic fibre over the projection $\text{H}\Lambda \to \mathbb{P}^1_n$ is a K3 surface. Furthermore, a K3 surface can be realised as an elliptic fibration over $\mathbb{P}^1_m$ with exactly 24 nodal curves. As $\Delta_{i;m,n}$ is a bidegree 24 polynomial, a generic fibre over $\mathbb{P}^1_n$ is an elliptically fibred $\mathbb{P}^1_m$ with 24 nodal fibres.

In addition to these generic points, the discriminant curve $\Delta_{i;m,n} = 0$ itself has singularities. We find that on $\widehat{\text{H}\Lambda}_i$ these fall into two categories, corresponding to cases I_2 and II in the Kodaira classification. In the generic case, there are 200 points of type I_2 and 192 of type II. These account for all 392 singularities on a generic curve. In accordance with the Kodaira classification, on singularities of type I_2, the polynomials $\widehat{Q}^i(Y_m, Y_n)$ factorise, with each factor corresponding to an irreducible rational curve. The two components meet at two points, which are the singularities of the fibre. The only exceptions to this are fibres which contain degree-5 rational curves on $\text{H}\Lambda$ — the second component of such a fibre is a degree-1 rational curve. When this curve is parallel to $\mathbb{P}_i$, it is exactly the line which has been blown down to obtain $\widehat{\text{H}\Lambda}_i$, and thus does not appear in the fibres on $\widehat{\text{H}\Lambda}_i$. In what follows, we mostly study the fibres on the singular varieties $\widehat{\text{H}\Lambda}_i$. However, using the birational map between $\widehat{\text{H}\Lambda}_i$ and $\text{H}\Lambda$ we can lift the curves on $\widehat{\text{H}\Lambda}_i$ found this way to curves on $\text{H}\Lambda$. Outside of the exceptional divisors the lift preserves the structure of the fibres. The two-component fibres of Kodaira type I_2 are unions of degree 1,2,3,4, and 5 rational curves. In particular, the singular fibres include

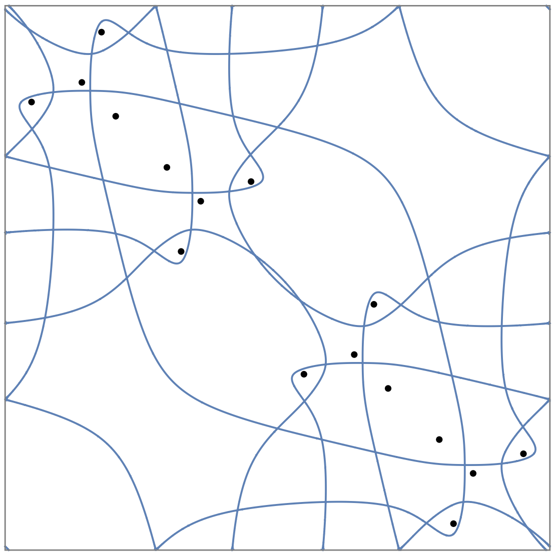

Figure 9: A sketch of the discriminant locus in $\mathbb{P}^1 \times \mathbb{P}^1$. Opposite edges of the figure are understood to be identified. The real section is drawn. The isolated dots that do not appear to lie on the discriminant locus are 'space invaders' that lie on suppressed complex branches of the curve. The sketch is made for a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric variety as in (2.20), and so the figure is invariant under two reflections. For the values of the parameters for which the sketch is drawn, none of the 192 cusps lie in the real section.

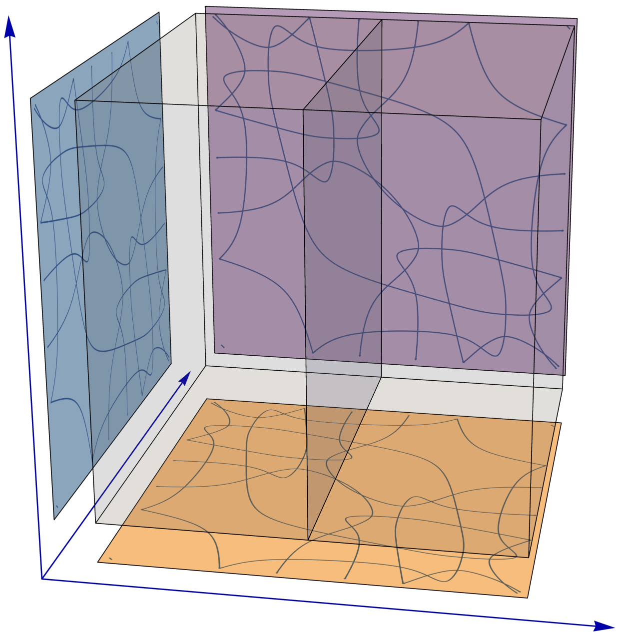

Figure 10: A heuristic sketch of the multiple fibrations. Each axis corresponds to a $\mathbb{P}^1$, and the indicated plane corresponds to a K3 fibre of a projection to one of the $\mathbb{P}^1$'s.

all lines, quadrics and cubics. We discuss each of these cases in detail in the following subsections §7.3, §7.4, and §7.5. First, however, it is convenient to briefly review some general aspects of curves on $(\mathbb{P}^1)^5$.

7.2 Complete intersection curves on $(\mathbb{P}^1)^5$

It turns out that the curves we consider in the following can be expressed as complete intersections of four polynomials in $(\mathbb{P}^1)^5$. The degrees and Euler characteristics of such curves are calculable by elementary techniques. Complete intersections on $(\mathbb{P}^1)^5$ can be systematically searched for, and doing this we obtain some evidence, consistent with the prediction of mirror symmetry, that there are no more curves than those we find here. We consider one-dimensional varieties defined by four equations

$$p^1 = p^2 = p^3 = p^4 = 0\,, \qquad \text{with} \qquad \deg_i(p^\alpha) = n_i^\alpha\,. \tag{7.6}$$

The two-form dual to the subvariety $p^\alpha = 0$ is given by

$$\mathcal{P}^\alpha = \sum_{i=0}^{4} n_i^\alpha J_i\,,$$

where J_i is the Kähler, or equivalently volume, form of $\mathbb{P}^1_i$. Then the dual form of the curve $p^1 = p^2 = p^3 = p^4 = 0$ is

$$\mathcal{C} \stackrel{\text{def}}{=} \mathcal{P}^1 \wedge \mathcal{P}^2 \wedge \mathcal{P}^3 \wedge \mathcal{P}^4 = \sum_{\varsigma\in\mathcal{S}_5} n^1_{\varsigma(1)} n^2_{\varsigma(2)} n^3_{\varsigma(3)} n^4_{\varsigma(4)} J_{\varsigma(1)} \wedge J_{\varsigma(2)} \wedge J_{\varsigma(3)} \wedge J_{\varsigma(4)}\,,$$

where the sum runs over all permutations of $\{0,\ldots,4\}$. The i'th degree of a curve dual to $\mathcal{C}$ is

$$\deg_i(\mathcal{C}) = \int_{(\mathbb{P}^1)^5} J_i \wedge \mathcal{C} = \sum_{\substack{\varsigma\in\mathcal{S}_4\\ a,b,c,d\neq i}} n^1_{\varsigma(a)} n^2_{\varsigma(b)} n^3_{\varsigma(c)} n^4_{\varsigma(d)}\,. \tag{7.7}$$

The total Chern class of the curve (7.6) is given by

$$c(\mathcal{C}) = \frac{\prod_{i=0}^4 (1+J_i)^2}{\prod_{\alpha=1}^4 (1+\sum_{i=0}^4 n_i^\alpha J_i)}\,.$$

It is straightforward to compute the Euler characteristic from the third Chern class:

$$\chi(\mathcal{C}) = \sum_{\varsigma\in\mathcal{S}_5} \left(2 - \sum_{\alpha=1}^{4} n^\alpha_{\varsigma(0)}\right) n^1_{\varsigma(1)} n^2_{\varsigma(2)} n^3_{\varsigma(3)} n^4_{\varsigma(4)} = 2\deg(\mathcal{C}) - \sum_{i=0}^{4}\sum_{\alpha=1}^{4} n_i^\alpha \deg_i(\mathcal{C})\,. \tag{7.8}$$

These formulae give the degrees and genera of various curves in the following sections. The degrees defined in this way will also agree with the degrees of isomorphisms $\varphi : \mathbb{P}^1 \to \mathcal{C}$.

As we are interested in curves in the Calabi-Yau manifold $\text{H}\Lambda$, we need to make sure that the curve $\mathcal{C}$ lies completely within this manifold. In the language of algebraic geometry, this is equivalent to requiring that the radical of the ideal generated by the polynomials p_i contains the polynomials Q^1 and Q^2 which define the $\text{H}\Lambda$ manifold.

It is often convenient to study the lines and other curves on the singular spaces $\widehat{\text{H}\Lambda}_i$, where their connection to the elliptic fibration can be immediately appreciated. Given a curve $\mathcal{C}$, and a projection π to a base B, then $\mathcal{C}$ may project to a curve of B, or project to a point. If $\mathcal{C}$ projects to a curve, it is said to be horizontal in the projection π, and if $\mathcal{C}$ projects to a point it is said to be vertical with respect to π.

In the following we will study each projection π_j, and we will sometimes say that a curve that projects to a point on the base is *vertical* with respect to the projection while one that projects to a curve on the base is *horizontal*. We will study each case in turn, and finally show that the lines can be associated to a unique degree-5 line and to a node in the discriminant $\Delta_{i;m,n}$.

7.3 Lines

Every degree-one rational curve in $(\mathbb{P}^1)^5$ is given by a set of four linear equations, each in a single variable. These read, for some $j \in \{0,1,2,3,4\}$ and each $s \in \{0,1,2,3,4\} \setminus j$,

$$Y_s - y_s = 0\,. \tag{7.9}$$

In this way $\boldsymbol{y} = (y_i, y_k, y_m, y_n)$ defines a line L_j, which is necessarily parallel to $\mathbb{P}^1_j$. Using the data of equations (7.9), the formulae (7.7) and (7.8) tell us that

$$\deg_i(L_j) \;=\; \delta_{ij}\,, \qquad \chi(L_j) \;=\; 2\,,$$

which is exactly as expected for a line. For a line L_j to lie on $\text{H}\Lambda$, the solutions to (7.9) must additionally satisfy $Q^1 = Q^2 = 0$. A substitution reveals that this condition amounts to

$$\alpha_j(\boldsymbol{y}) + \beta_j(\boldsymbol{y})Y_j \;=\; 0\,, \qquad \gamma_j(\boldsymbol{y}) + \delta_j(\boldsymbol{y})Y_j \;=\; 0\,.$$

Therefore the $\boldsymbol{y}$ must solve $\alpha_j = \beta_j = \gamma_j = \delta_j = 0$, and so gives a singularity on $\widehat{\text{H}\Lambda}_j$. As has already been mentioned, these equations have 24 solutions for each j. There are therefore $5 \times 24 = 120$ lines. In the $\mathbb{Z}_5$ symmetric case, the permissible values of $\boldsymbol{y}$ group into $\mathbb{Z}_5$ orbits and taking the quotient leaves 24 distinct lines. Similarly, in the $\mathbb{Z}_2$ symmetric cases, the involution $Y_i \mapsto -Y_i$ (or equivalently $Y_{i,0} \leftrightarrow Y_{i,1}$) identifies two lines. On $\text{H}\Lambda/\mathbb{Z}_5 \times \mathbb{Z}_2$ there are therefore 12 lines, each descending from a family of 10 lines on the covering space. Finally, the generic $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ quotient contains exactly 6 lines.

Horizontal lines

For definiteness, let us consider the projection π_4, the lines L_2, and the elliptic fibration $E_{4;0,1}$ with base $\mathbb{P}^1_0 \times \mathbb{P}^1_1$. The lines L_2 on $\text{H}\Lambda$ can be understood to arise as blow-ups of singular points $\boldsymbol{y}$ on $\widehat{\text{H}\Lambda}_2$, and can be given by the embedding

$$z \mapsto (y_0, y_1, z, y_3, y_4)\,.$$

The projection π_4 then takes this line to a line in $\widehat{\text{H}\Lambda}_4$, given by the embedding

$$z \mapsto (y_0, y_1, z, y_3)\,.$$

Thus L_2 forms a part of the fibre of $E_{4;0,1}$ lying over the basepoint (y_0, y_1). Tautologically, this fibre can be realised as the curve defined by the equation

$$\widehat{Q}^4(y_0, y_1, Y_2, Y_3) \;=\; 0\,.$$

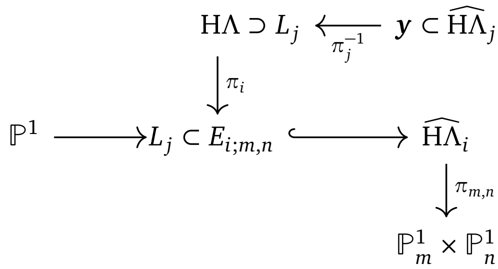

Figure 11: A chain of birational maps allows us to see lines L_i, corresponding to a singularity of $\widehat{\mathrm{H}\Lambda}_j$ at $\boldsymbol{y}$ explicitly as singular fibres on $\widehat{\mathrm{H}\Lambda}_i$ viewed as a fibration over $\mathbb{P}^1_m \times \mathbb{P}^1_n$. The polynomial $\widehat{Q}^i(Y_m, Y_n)$ factorises into two factors, one of degree $(0,1)$, corresponding to the line, and the other of degree $(2,1)$. This latter factor corresponds to a projection of a degree-5 curve down to $\widehat{\mathrm{H}\Lambda}_i$.

Reflecting the fact that this fibre contains a line and hence is reducible, the above polynomial factorises into degree-one and degree-three pieces (in homogeneous coordinates). The first factor is of course the equation of the image of the line L_2 on $\widehat{\mathrm{H}\Lambda}_4$.

The second factor of $\widehat{Q}^4$ has degree (1,2) with respect to Y_2, Y_3 and thus is a multidegree $(0,0,1,2)$ curve $\widehat{\mathcal{C}}_{(0,0,1,2)}$, which meets the line at two points. The map

$$z \mapsto (y_0, y_1, z, r_2(z)) \tag{7.10}$$

is a degree $(0,0,1,2)$ isomorphism taking $\mathbb{P}^1$ to $\widehat{\mathcal{C}}_{(0,0,1,2)}$.

These curves lift to degree-5 curves $\mathcal{C}_{(0,0,1,2,2)}$ on HΛ. The equations $Q^1 = Q^2 = 0$ are solved by setting $Y_4 = -\frac{\beta_4}{\alpha_4} = -\frac{\delta_4}{\gamma_4}$. Note that α_4 and β_4 are both linear in Y_0 and Y_1, so substituting in the values of Y_0 and Y_1 in terms of z from (7.10) into the ratio $\frac{\beta_4(z)}{\alpha_4(z)}$ gives a rational function $\widetilde{r}_2(z)$ of degree 2, as it can be shown that the quantities $\alpha_4(z)$ and $\beta_4(z)$ have exactly one linear factor in common. We arrive at a curve $\mathcal{C}_{(0,0,1,2,2)}$ with an isomorphism $\psi_{4;0,1;2} : \mathbb{P}^1 \to \mathcal{C}_{(0,0,1,2,2)}$ given by

$$\psi_{4;0,1;2}(z) = (y_0, y_1, z, r_2(z), \widetilde{r}_2(z)) \,. \tag{7.11}$$

Therefore, on HΛ the fibre over basepoint (y_0, y_1) consists of two rational curves that meet in two points. According to Kodaira's classification, the point (y_0, y_1) must be a node on the discriminant of this elliptic fibration. Upon projection to $\widehat{\mathrm{H}\Lambda}_4$, this becomes a node of $\Delta_{4;0,1}$, which is indeed what we find in the examples we have studied.

Other maps $\psi_{i;m,n;j}$ are defined similarly, with the privileged role of Y_4, Y_0, Y_1, Y_2 in this construction replaced by Y_i, Y_m, Y_n, Y_j. We display the interplay between these maps and projections in Figure 11.

Vertical lines

Let us now focus our attention to the line L_4, which is mapped to a point[16] $\boldsymbol{y}$ by π_4. By symmetry, over the point (y_0, y_1) on the base $\mathbb{P}^1_0 \times \mathbb{P}^1_1$ in HΛ, the fibre is given by the union of the line L_4 together with a degree-5 curve $\mathcal{C}_{(0,0,2,2,1)}$, which meets the line in two points.

[16] This point is not necessarily the same as the $\boldsymbol{y}$ in the previous subsection.

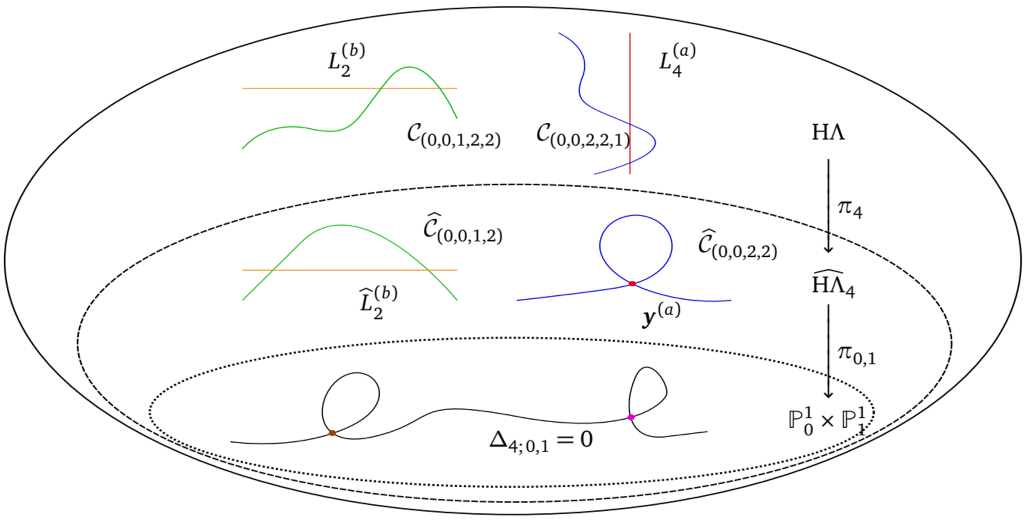

Figure 12: Schematic representation of elliptic fibres containing lines on HΛ. For concreteness, we have chosen here $i = 4$, $m = 0$, and $n = 1$. The largest oval represents the smooth manifold HΛ, on which the elliptic fibration over $\mathbb{P}_0 \times \mathbb{P}_1$ contains unions of two rational curves. Here we have pictured the fibres which consist of one line and a degree-5 curve.

Projecting this fibre down to $\widehat{\text{H}\Lambda}_4$ maps the line to a point $\boldsymbol{y}$, and the degree-5 curve to a degree-4 curve $\widehat{\mathcal{C}}_{(0,0,2,2)}$, which intersects itself at the point $\boldsymbol{y}$. So there exists a birational map $\mathbb{P}^1 \to \mathcal{C}_{(0,0,2,2)}$

$$z \mapsto \left(y_0^{(a)}, y_1^{(a)}, r_2(z), \widetilde{r}_2(z)\right), \tag{7.12}$$

which is not, however, an isomorphism due to the self-intersection. Such a curve will not fit Kodaira's classification, which can be traced back to the fact that $\widehat{\text{H}\Lambda}_4$ is singular. Indeed, the lift of the fibre is a union of two rational curves meeting at two points, and thus corresponds to a node in the discriminant locus of the fibration HΛ. Upon projecting down to $\widehat{\text{H}\Lambda}_4$, this becomes a node of the locus $\Delta_{4;0,1} = 0$. An alternative way of arriving at the same conclusion is by noting that, as we have remarked previously, $\Delta_{4;0,1} = \Delta_{2;0,1}$, and by the previous subsection, L_4 corresponds to a node of $\Delta_{2;0,1} = 0$.

A straightforward generalisation of the the results of the last two subsections reveals that the 72 lines L_i, L_j, and L_k, together with the degree-5 curves, account for 72 of the nodes of the discriminant locus $\Delta_{i;m,n} = 0$. The locus has in total 200 nodes, the rest of which turn out to correspond to curves of degrees 2, 3, and 4, as we show shortly.

Figure 12 sketches the lifts of singular fibres in $\widehat{\text{H}\Lambda}_4$ to HΛ.

7.4 Quadrics

The analysis of irreducible degree-2 curves proceeds largely along the same lines. Algebraic quadrics on HΛ can be expressed, for a triple k, m, n with constants q_k, q_m, q_n, as the complete intersections

$$Y_k - q_k = 0\,, \quad Y_m - q_m = 0\,, \quad Y_n - q_n = 0\,, \quad p\left(Y_0, Y_1, Y_2, Y_3, Y_4\right) = 0\,. \tag{7.13}$$

Here, p is an irreducible multidegree $(1,1,1,1,1)$ polynomial. With i,j denoting the pair in $\{0,1,2,3,4\}\setminus\{k,m,n\}$, the equations (7.13) define a curve $\mathcal{C}$ with

$$\deg_s(\mathcal{C}) = \delta_{s,i}+\delta_{s,j}\,, \qquad \chi(\mathcal{C}) = 2\,.$$

While this is not the most general form of a degree-2 curve on $(\mathbb{P}^1)^5$, it can be shown that the curves which lie in HΛ are of this form. To ensure that a curve defined by (7.13) lies in HΛ, we must have that, specialising to $Y_i=q_i, Y_j=q_j$, and $Y_k=q_k$,

$$Q^1, Q^2 \in \sqrt{\langle p\rangle} = \langle p\rangle\,.$$

The square root indicates the radical of the ideal $\langle p\rangle$, which in this case is the ideal itself. As p is irreducible and all three polynomials Q^1, Q^2, and p are of multidegree (1,1,1,1,1), this requires $p=CQ^1$ or $p=CQ^2$, with C a constant. Further, we must have either $Q^1=Q^2$ or one of the Q's vanishing at $Y_k=q_k, Y_m=q_m$, and $Y_n=q_n$. We cannot have both Q's vanishing after this specialisation. In general there are 24 values of $\{q_k,q_m,q_n\}$ for which these conditions are satisfied. There are 10 ways of choosing the triple k,m,n, and so we find 240 curves of degree 2 on HΛ. In the $\mathbb{Z}_5$, $\mathbb{Z}_5\times\mathbb{Z}_2$, and $\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$ symmetric cases, these curves come in families of 5, 10, and 20, respectively, so taking the quotients gives exactly 48 curves on H$\Lambda/\mathbb{Z}_5$, 24 on H$\Lambda/\mathbb{Z}_5\times\mathbb{Z}_2$, and 12 on H$\Lambda/\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$. This agrees with the results of [3].

Again, an alternative point-of-view can be obtained by viewing these quadrics as fibres of the elliptic fibrations $\widehat{\text{H}\Lambda}_i \to \mathbb{P}^1_m\times\mathbb{P}^1_n$. Consider the case $(i,j,k,m,n)=(4,2,3,0,1)$. The quadrics $\mathcal{C}_{(0,0,1,1,0)}$ are isomorphic to $\mathbb{P}^1$ with the map given by

$$z\mapsto \big(q_0,q_1,z,r_1(z),q_4\big)\,.$$

Upon projection by π_4, this becomes a quadric on $\widehat{\text{H}\Lambda}_4$ with embedding

$$z\mapsto (q_0,q_1,z,r_1(z))\,.$$

The fibre to which this belongs is given by $\widehat{Q}^4(q_0,q_1,Y_2,Y_3)$, which factorises into two degree-(1,1) factors, with the first one corresponding to the quadric $\widehat{\mathcal{C}}_{(0,0,1,1)}$. The second curve $\widehat{\mathcal{C}}_{(0,0,1,1)}$ is also a quadric on $\widehat{\text{H}\Lambda}_4$, but can be lifted to H$\Lambda$. To do this, we again set $Y_4=-\frac{\beta_4}{\alpha_4}$, to ensure that the lifted curve lies in HΛ. Expressing α_4 and β_4 in terms of z, the parameter on the curve, this is generically a degree-2 rational function. Thus the lift $\mathcal{C}_{(0,0,1,1,2)}$ is a degree-4 curve isomorphic to $\mathbb{P}^1$ via

$$z\mapsto (q_0,q_1,z,r_1(z),r_2(z))\,.$$

Similarly, by symmetry we know that there exists a fibre on HΛ which consists of the curves $\mathcal{C}_{(0,0,0,1,1)}$ and $\mathcal{C}_{(0,0,2,1,1)}$. Projecting them down to $\widehat{\text{H}\Lambda}_4$ gives a line and cubic, $\widehat{\mathcal{C}}_{(0,0,0,1)}$ and $\widehat{\mathcal{C}}_{(0,0,2,1)}$, meeting in two points. By symmetry the curves studied above also meet in two points, in accordance with Kodaira's classification. Thus the 72 quadrics of the form $\mathcal{C}_{(0,0,1,1,0)}$, $\mathcal{C}_{(0,0,1,0,1)}$ or $\mathcal{C}_{(0,0,0,1,1)}$ each correspond to a unique node of the discriminant locus $\Delta_{4;\,0,1}$.

7.5 Cubics

Cubic curves whose multidegree is a permutation of $(1,1,1,0,0)$ can also be expressed as complete intersections. The most general cubic curves that can be defined by four multilinear equations are of the form

$$\begin{aligned} Y_m - c_m &= 0\,, \\ Y_n - c_n &= 0\,, \\ p \stackrel{\text{def}}{=} a_0 + a_1 Y_i + a_2 Y_j + a_3 Y_i Y_j &= 0\,, \\ q \stackrel{\text{def}}{=} b_0 + b_1 Y_k + b_2 Y_j + b_3 Y_k Y_j &= 0\,. \end{aligned}$$

This defines a curve $\mathcal{C}_3$ with

$$\deg_i(\mathcal{C}_3) \,=\, \delta_{i,l} + \delta_{i,m} + \delta_{i,n}\,, \qquad \chi(\mathcal{C}_3) \,=\, 2\,.$$

Curves of this form include all cubics lying in HΛ. For the curve defined in this way to lie in HΛ, the ideal generated by these polynomials must contain the polynomials Q^1 and Q^2. This condition is equivalent to requiring that there are coefficients d_a, e_b such that when $Y_m = c_m$, $Y_n = c_n$

$$Q^1 \,=\, d_1\,p + d_2\,q + d_3\,p\,Y_k + d_4\,q\,Y_i\,, \qquad Q^2 \,=\, e_1\,p + e_2\,q + e_3\,p\,Y_k + e_4\,q\,Y_i\,.$$

For a quintuple (i, j, k, m, n) there are in general exactly 112 solutions to these equations. Summing over the 10 distinct choices of (i, j, k, m, n) gives us 1120 curves of degree 3, which once again come in $\mathbb{Z}_5$, $\mathbb{Z}_5{\times}\mathbb{Z}_2$, and $\mathbb{Z}_5{\times}\mathbb{Z}_2{\times}\mathbb{Z}_2$ invariant families in the symmetric cases. Taking the quotients with respect to $\mathbb{Z}_5$, $\mathbb{Z}_5{\times}\mathbb{Z}_2$, and $\mathbb{Z}_5{\times}\mathbb{Z}_2{\times}\mathbb{Z}_2$ leave 224, 112, and 56 curves of degree 3 respectively, in agreement with [3].

As was the case with the lines and quadrics, the cubics also appear as singular fibres of elliptic fibrations, and in fact account for the remaining 56 nodes of the discriminant locus $\Delta_{i;\,m,n} = 0$. Take again $(i, j, k, m, n) = (4, 2, 3, 0, 1)$ to expedite the discussion, and consider the cubic curves $\mathcal{C}_{(0,0,1,1,1)}$. The projection of this curve to $\widehat{\text{H}\Lambda}_4$ is a quadric $\widehat{\mathcal{C}}_{(0,0,1,1)}$. As before, this indicates that the polynomial $\widehat{Q}^4(Y_2, Y_3)$ factorises into two components, both of degree $(1, 1)$. The isomorphisms with $\mathbb{P}^1$ are of the form

$$z \mapsto (c_0, c_1, z, r_1(z))\,.$$

The quantity $\frac{\beta_4}{\alpha_4}$ determining the lift to a curve on HΛ is a priori a ratio of two degree-2 polynomials. However, this is a component of a reducible elliptic fibre inside of which we already have a curve of total degree 3, therefore the two polynomials α_4, β_4 must share a factor so that the lifts are curves $\mathcal{C}_{(0,0,1,1,1)}$. The isomorphisms with $\mathbb{P}^1$ are given by

$$z \mapsto (c_0, c_1, z, r_1(z), \widetilde{r}_1(z))\,.$$

7.6 Summary

This completes the classifications of fibres over the nodes of the discriminant curves on singular varieties $\widehat{\text{H}\Lambda}_i$ (over the base $\mathbb{P}_m \times \mathbb{P}_n$), and their lifts to HΛ. We summarise our findings in Table 10 and Table 11, taking $i = 4$, $m = 0$, $n = 1$ for concreteness.

Table 10: Factorisations of $\widehat{Q}_4$ over the nodes of the discriminant curve $\Delta_{4;0,1} = \Delta_{3;0,1} = \Delta_{2;0,1}$ and the corresponding curves on the non-singular variety $\mathrm{H}\Lambda$.

Type	Degree 1	Degree 2	Curve 1	Curve 2	Number
Line	(0,0,0,0)	(0,0,2,2)	L_4	$\mathcal{C}_{(0,0,2,2,1)}$	24
Line	(0,0,1,0)	(0,0,1,2)	L_2	$\mathcal{C}_{(0,0,1,2,2)}$	24
Line	(0,0,0,1)	(0,0,2,1)	L_3	$\mathcal{C}_{(0,0,2,1,2)}$	24
Quadric	(0,0,1,1)	(0,0,1,1)	$\mathcal{C}_{(0,0,1,1,0)}$	$\mathcal{C}_{(0,0,1,1,2)}$	24
Quadric	(0,0,0,1)	(0,0,2,1)	$\mathcal{C}_{(0,0,1,0,1)}$	$\mathcal{C}_{(0,0,1,2,1)}$	24
Quadric	(0,0,1,0)	(0,0,1,2)	$\mathcal{C}_{(0,0,0,1,1)}$	$\mathcal{C}_{(0,0,2,1,1)}$	24
Cubic	(0,0,1,1)	(0,0,1,1)	$\mathcal{C}_{(0,0,1,1,1)}$	$\mathcal{C}_{(0,0,1,1,1)}$	56

Table 11: The results of this section, giving the curve-counts for some low degrees. The numbers that are related to these by a cyclic permutation are omitted. Note the agreement with the tables in appendix E.

$\mathfrak{p}$	$n_{\mathfrak{p}}$
(0,0,0,0,1)	24
(0,0,0,1,1)	24
(0,0,0,0,2)	0
(0,0,1,1,1)	112
(0,0,0,1,2)	0
(0,0,0,0,3)	0
(0,0,1,1,2)	24
(0,0,0,1,3)	0
(0,0,0,0,4)	0
(0,0,1,2,2)	24
(0,0,0,1,4)	0
(0,0,0,0,5)	0

8 Outlook and Discussion

8.1 Coxeter groups and higher genus invariants

As the Coxeter group we have described can be seen as acting on the divisors of the manifold, we expect that the identities between instanton numbers obtained from these symmetries persist to all genera for the mirror Hulek-Verrill. For other manifolds admitting such flops, with a similar group appearing at genus 0, one also expects the same group to appear at every genus. The set of such manifolds includes a number of complete intersection Calabi-Yau manifolds. For instance, there is a Coxeter symmetry appearing in the tables of [36], appendix B, which gives the instanton numbers at genera 0,1,2 for a two-parameter split of the quintic threefold. The tables at each genus exhibit the same infinite dihedral symmetry.

This raises a tantalising possibility, the focus of ongoing work, to use the symmetry as an aid to computing higher genus instanton numbers. For compact Calabi-Yau threefolds the set of instanton numbers for a given genus is determined by BCOV recursion [15, 37] *up to* a set of functions $f^{(h)}$, $2 \leqslant h \leqslant g$, the holomorphic ambiguities. These are rational functions of the moduli, with bounded degrees in the numerator and denominator. In order to fix the ambiguities, one must fix these ambiguities, which can be done, for example, if one can independently compute some instanton numbers, or find relations between instanton numbers.

Coxeter symmetry could conceivably provide a way of establishing such independent relations, making it possible to compute instanton numbers to higher genera for the compact multiparameter geometries possessing these symmetries. However, even if this were true, there would remain the practical matter of implementing the recursion. Due to how the Coxeter symmetries affect the degrees of instanton numbers, it may be required to work to a very high degree in order to fix the ambiguities, and in our present work exceeding degree-29 proved to be challenging.

Moreover, there exists a simple recipe, obvious to readers of certain schools, for obtaining the instanton numbers on a manifold from those on a split of the manifold. This was not crucial to our main line of discussion, but it is satisfying to see that the instanton numbers $\nu_{i,j,k,l}$ for the family of tetraquadrics, which can be computed by methods similar to those described in this paper, match perfectly with the combinations $\sum_m n_{i,j,k,l,m}$. This gives a hope for making progress on geometries with no Coxeter symmetry, such as the quintic, by instead working on a split and then projecting out the last index as above. In fact, the tetraquadric actually possesses its own Coxeter group which is compatible with that of the mirror Hulek-Verrill manifold through the above projection.

The instanton numbers play a role in the microstate counting for 4d $\mathcal{N}=2$ black holes. One perspective on this problem considers the elliptic genus of the Maldacena-Strominger-Witten CFT [38], which describes the microscopic degrees of freedom in an M-theory construction. The Fourier coefficients of this quantity are specific combinations of the instanton numbers at different genera, and it is interesting to consider what Coxeter symmetry tells you about these combinations.

8.2 Separation of variables, periods, and amplitudes

The section §3 was written with an explicit focus on the Hulek-Verrill manifold, but most of the discussion generalises directly to other multiparameter manifolds. In particular, in spite of lacking a full Picard-Fuchs system, we can use the knowledge of the triple intersection numbers of the mirror, as in [26], to obtain the periods of the Hulek-Verrill manifolds. We formulate this method of finding periods as expanding the periods in the mirror cohomology algebra elements. Additionally, by studying lines in the moduli space, we are able to obtain the full monodromy data and find the periods everywhere in the moduli space. This process is also greatly simplified by symmetry.

The periods can be expressed in terms of Bessel function moments. These expressions are particularly interesting where banana Feynman diagrams are concerned, as they facilitate analytic study of the cut diagram in regimes of parameter space where the series expression for the fundamental period does not converge.

8.3 The Hulek-Verrill manifolds and modularity

As the methods of computing local zeta functions for threefolds improve, it becomes possible to consider period evaluations and expressions for these in terms of automorphic L-values. In this paper we have seen that the periods can be expressed as Bessel moments, so modularity considerations make it possible to express these moments in terms of L-values. More generally, the periods of any favourable complete intersection can be written as integrals of products of Meijer-G functions. There is much work to be done in exploiting modularity, or more generally automorphy, in order to obtain interesting identities for such integrals, which may prompt new questions.

We will return to the Hulek-Verrill manifold as a primary example in future work. One project considers the zeta function for the five-parameter family and how modular behaviour of this relates to supersymmetric vacua of IIB supergravity flux compactifications. Our work in this paper serves as crucial input for this future project [39].

There are many interesting relations between the mirror Hulek-Verrill and the tetraquadric that follow from the splitting. The latter does not seem to possess a rank-two attractor on the maximally symmetric locus, i.e. the one-parameter family AESZ16 [40]. The former does have rank-two attractors, and we will return elsewhere to studying such pairs of manifolds [41].

Although the zeta function for the one-parameter quotient of the Hulek-Verrill manifold has been extensively tabulated, the same is not true for the full multiparameter family. Progress with this problem is contingent upon possessing a good description of the periods, which we provide in this paper. The results of this paper should make possible a search for rank-two attractors 'off the diagonal', where not all moduli are equal.

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A Toric geometry data

Here we gather some data related to the polytopes and toric varieties discussed in §2.

The polytope $\widehat{\Delta}$ and the ambient variety $\mathbb{P}_{\widehat{\Delta}}$

Vertices of $\widehat{\Delta}$

u_1	(-1, 0, 0, 0)	u_{11}	(0, 0, 0, 1)
u_2	(-1, 0, 0, 1)	u_{12}	(0, 0, 1,-1)
u_3	(-1, 0, 1, 0)	u_{13}	(0, 0, 1, 0)
u_4	(-1, 1, 0, 0)	u_{14}	(0, 1,-1, 0)
u_5	(0,-1, 0, 0)	u_{15}	(0, 1, 0,-1)
u_6	(0,-1, 0, 1)	u_{16}	(0, 1, 0, 0)
u_7	(0,-1, 1, 0)	u_{17}	(1,-1, 0, 0)
u_8	(0, 0,-1, 0)	u_{18}	(1, 0,-1, 0)
u_9	(0, 0,-1, 1)	u_{19}	(1, 0, 0,-1)
u_{10}	(0, 0, 0,-1)	u_{20}	(1, 0, 0, 0)

Faces of $\widehat{\Delta}$

ρ_1	$y_1 = 1$	ρ_{16}	$-y_4 = 1$
ρ_2	$-y_1 = 1$	ρ_{17}	$y_1 + y_4 = 1$
ρ_3	$y_2 = 1$	ρ_{18}	$-y_1 - y_4 = 1$
ρ_4	$-y_2 = 1$	ρ_{19}	$y_2 + y_4 = 1$
ρ_5	$y_1 + y_2 = 1$	ρ_{20}	$-y_2 - y_4 = 1$
ρ_6	$-y_1 - y_2 = 1$	ρ_{21}	$y_1 + y_2 + y_4 = 1$
ρ_7	$y_3 = 1$	ρ_{22}	$-y_1 - y_2 - y_4 = 1$
ρ_8	$-y_3 = 1$	ρ_{23}	$y_3 + y_4 = 1$
ρ_9	$y_1 + y_3 = 1$	ρ_{24}	$-y_3 - y_4 = 1$
ρ_{10}	$-y_1 - y_3 = 1$	ρ_{25}	$y_1 + y_3 + y_4 = 1$
ρ_{11}	$y_2 + y_3 = 1$	ρ_{26}	$-y_1 - y_3 - y_4 = 1$
ρ_{12}	$-y_2 - y_3 = 1$	ρ_{27}	$y_2 + y_3 + y_4 = 1$
ρ_{13}	$y_1 + y_2 + y_3 = 1$	ρ_{28}	$-y_2 - y_3 - y_4 = 1$
ρ_{14}	$-y_1 - y_2 - y_3 = 1$	ρ_{29}	$y_1 + y_2 + y_3 + y_4 = 1$
ρ_{15}	$y_4 = 1$	ρ_{30}	$-y_1 - y_2 - y_3 - y_4 = 1$

We form a matrix $\widehat{\text{M}}$ out of these vectors,

$$\widehat{\text{M}} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_{20} \end{pmatrix} = \big(o_1, o_2, o_3, o_4\big)\,.$$

The nullspace of $\widehat{\text{M}}^T$, expressed in a convenient basis, gives 16 relations between these vectors:

$$\begin{aligned} u_i + u_{21-i} &= 0\,, & 1 \le i \le 10\,, & \\ u_1 - u_5 + u_{17} &= 0\,, & u_1 - u_8 + u_{18} = 0\,, & \quad u_1 - u_{10} + u_{19} = 0\,, \\ u_5 - u_8 + u_{14} &= 0\,, & u_5 - u_{10} + u_{15} = 0\,, & \quad u_8 - u_{10} + u_{12} = 0\,. \end{aligned}$$

Each of these relations corresponds to a scaling symmetry $\mathbb{C}^* \subset (\mathbb{C}^*)^{16}$. For example, the relations $u_1 + u_{20} = 0$ and $u_8 - u_{10} + u_{12} = 0$ correspond to scalings

$$\begin{aligned} \mathbb{C}^*_1 &: (\eta_1, \ldots, \eta_{20}) \mapsto (\lambda_1 \eta_1, \eta_2, \ldots, \eta_{19}, \lambda_1 \eta_{20})\,, \\ \mathbb{C}^*_{16} &: (\eta_1, \ldots, \eta_{20}) \mapsto (\eta_1, \eta_2, \ldots, \eta_7, \lambda_{16}\eta_8, \eta_9, \lambda_{16}^{-1}\eta_{10}, \eta_{11}, \lambda_{16}\eta_{12}, \eta_{13} \ldots \eta_{19}, \eta_{20})\,. \end{aligned}$$

There are four invariant combinations of coordinates that we can identify with the coordinates on the torus $\mathbb{T}^4 \subset \mathbb{P}_{\widehat{\Delta}}$. These can be taken to be

$$\begin{aligned} H_1 &= \eta^{o_1} = \frac{\eta_{17}\eta_{18}\eta_{19}\eta_{20}}{\eta_1\eta_2\eta_3\eta_4}\,, & H_2 &= \eta^{o_2} = \frac{\eta_4\eta_{14}\eta_{15}\eta_{16}}{\eta_5\eta_6\eta_7\eta_{17}}\,, \\ H_3 &= \eta^{o_3} = \frac{\eta_3\eta_7\eta_{12}\eta_{13}}{\eta_8\eta_9\eta_{14}\eta_{18}}\,, & H_4 &= \eta^{o_4} = \frac{\eta_2\eta_6\eta_9\eta_{11}}{\eta_{10}\eta_{12}\eta_{15}\eta_{19}}\,. \end{aligned} \tag{A.1}$$

The dual polytope $\widehat{\Delta}^*$ and the ambient variety $\mathbb{P}_{\widehat{\Delta}^*}$

Vertices of $\widehat{\Delta}^*$			
v_1	(1, 0, 0, 0)	v_{16}	(0, 0, 0,-1)
v_2	(-1, 0, 0, 0)	v_{17}	(1, 0, 0, 1)
v_3	(0, 1, 0, 0)	v_{18}	(-1, 0, 0,-1)
v_4	(0,-1, 0, 0)	v_{19}	(0, 1, 0, 1)
v_5	(1, 1, 0, 0)	v_{20}	(0,-1, 0,-1)
v_6	(-1,-1, 0, 0)	v_{21}	(1, 1, 0, 1)
v_7	(0, 0, 1, 0)	v_{22}	(-1,-1, 0,-1)
v_8	(0, 0,-1, 0)	v_{23}	(0, 0, 1, 1)
v_9	(1, 0, 1, 0)	v_{24}	(0, 0,-1,-1)
v_{10}	(-1, 0,-1, 0)	v_{25}	(1, 0, 1, 1)
v_{11}	(0, 1, 1, 0)	v_{26}	(-1, 0,-1,-1)
v_{12}	(0,-1,-1, 0)	v_{27}	(0, 1, 1, 1)
v_{13}	(1, 1, 1, 0)	v_{28}	(0,-1,-1,-1)
v_{14}	(-1,-1,-1, 0)	v_{29}	(1, 1, 1, 1)
v_{15}	(0, 0, 0, 1)	v_{30}	(-1,-1,-1,-1)

Faces of $\widehat{\Delta}^*$			
τ_1	$-x_1=1$	τ_{11}	$x_4=1$
τ_2	$-x_1+x_4=1$	τ_{12}	$x_3-x_4=1$
τ_3	$-x_1+x_3=1$	τ_{13}	$x_3=1$
τ_4	$-x_1+x_2=1$	τ_{14}	$x_2-x_3=1$
τ_5	$-x_2=1$	τ_{15}	$x_2-x_4=1$
τ_6	$-x_2+x_4=1$	τ_{16}	$x_2=1$
τ_7	$-x_2+x_3=1$	τ_{17}	$x_1-x_2=1$
τ_8	$-x_3=1$	τ_{18}	$x_1-x_3=1$
τ_9	$-x_3+x_4=1$	τ_{19}	$x_1-x_4=1$
τ_{10}	$-x_4=1$	τ_{20}	$x_1=1$

We form a matrix $\widehat{\text{W}}$ out of these vectors,

$$\widehat{\text{W}} = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_{30} \end{pmatrix} = \left(w_1, w_2, w_3, w_4\right).$$

By finding the nullspace of $\widehat{\text{W}}^T$, we find 26 independent relations between the 30 vectors.

$$v_{2i} + v_{2i-1} = 0, \quad 1 \leq i \leq 15,$$

$$\begin{aligned} v_7 + v_{15} + v_{24} &= 0, & v_3 + v_7 + v_{15} + v_{28} &= 0, \\ v_3 + v_{15} + v_{20} &= 0, & v_1 + v_7 + v_{15} + v_{26} &= 0, \\ v_1 + v_{15} + v_{18} &= 0, & v_1 + v_3 + v_{15} + v_{22} &= 0, \\ v_3 + v_7 + v_{12} &= 0, & v_1 + v_3 + v_7 + v_{14} &= 0, \\ v_1 + v_7 + v_{10} &= 0, & v_3 + v_7 + v_{15} + v_{28} &= 0, \\ v_1 + v_3 + v_6 &= 0. \end{aligned}$$

Again, each of these relations corresponds to a scaling symmetry $\mathbb{C}^* \subset (\mathbb{C}^*)^{26}$. There are four invariant combinations of coordinates that we can identify with the coordinates on the torus $\mathbb{T}^4 \subset X_{\widehat{\Delta}^*}$. These can be taken to be

$$\begin{aligned} \Xi_1 &= \xi^{w_1} = \frac{\xi_1\xi_5\xi_9\xi_{13}\xi_{17}\xi_{21}\xi_{25}\xi_{29}}{\xi_2\xi_6\xi_{10}\xi_{14}\xi_{18}\xi_{22}\xi_{26}\xi_{30}}, & \Xi_2 &= \xi^{w_2} = \frac{\xi_3\xi_5\xi_{11}\xi_{13}\xi_{19}\xi_{21}\xi_{27}\xi_{29}}{\xi_4\xi_6\xi_{12}\xi_{14}\xi_{20}\xi_{22}\xi_{28}\xi_{30}}, \\ \Xi_3 &= \xi^{w_3} = \frac{\xi_7\xi_9\xi_{11}\xi_{13}\xi_{23}\xi_{25}\xi_{27}\xi_{29}}{\xi_8\xi_{10}\xi_{12}\xi_{14}\xi_{24}\xi_{26}\xi_{28}\xi_{30}}, & \Xi_4 &= \xi^{w_4} = \frac{\xi_{15}\xi_{17}\xi_{19}\xi_{21}\xi_{23}\xi_{25}\xi_{27}\xi_{29}}{\xi_{16}\xi_{18}\xi_{20}\xi_{22}\xi_{24}\xi_{26}\xi_{28}\xi_{30}}. \end{aligned} \tag{A.2}$$

The polytope ∇^* and the ambient variety $\mathbb{P}_{\nabla^*}$

Vertices of ∇^*	
u_1	(1, 0, 0, 0, 0)
u_2	(-1, 0, 0, 0, 0)
u_3	(0, 1, 0, 0, 0)
u_4	(0,-1, 0, 0, 0)
u_5	(0, 0, 1, 0, 0)
u_6	(0, 0,-1, 0, 0)
u_7	(0, 0, 0, 1, 0)
u_8	(0, 0, 0,-1, 0)
u_9	(0, 0, 0, 0, 1)
u_{10}	(0, 0, 0, 0,-1)

Faces of ∇^*			
τ_1	$-x_1-x_2-x_3-x_4-x_5=1$	τ_{17}	$x_1-x_2-x_3-x_4-x_5=1$
τ_2	$-x_1-x_2-x_3-x_4+x_5=1$	τ_{18}	$x_1-x_2-x_3-x_4+x_5=1$
τ_3	$-x_1-x_2-x_3+x_4-x_5=1$	τ_{19}	$x_1-x_2-x_3+x_4-x_5=1$
τ_4	$-x_1-x_2-x_3+x_4+x_5=1$	τ_{20}	$x_1-x_2-x_3+x_4+x_5=1$
τ_5	$-x_1-x_2+x_3-x_4-x_5=1$	τ_{21}	$x_1-x_2+x_3-x_4-x_5=1$
τ_6	$-x_1-x_2+x_3-x_4+x_5=1$	τ_{22}	$x_1-x_2+x_3-x_4+x_5=1$
τ_7	$-x_1-x_2+x_3+x_4-x_5=1$	τ_{23}	$x_1-x_2+x_3+x_4-x_5=1$
τ_8	$-x_1-x_2+x_3+x_4+x_5=1$	τ_{24}	$x_1-x_2+x_3+x_4+x_5=1$
τ_9	$-x_1+x_2-x_3-x_4-x_5=1$	τ_{25}	$x_1+x_2-x_3-x_4-x_5=1$
τ_{10}	$-x_1+x_2-x_3-x_4+x_5=1$	τ_{26}	$x_1+x_2-x_3-x_4+x_5=1$
τ_{11}	$-x_1+x_2-x_3+x_4-x_5=1$	τ_{27}	$x_1+x_2-x_3+x_4-x_5=1$
τ_{12}	$-x_1+x_2-x_3+x_4+x_5=1$	τ_{28}	$x_1+x_2-x_3+x_4+x_5=1$
τ_{13}	$-x_1+x_2+x_3-x_4-x_5=1$	τ_{29}	$x_1+x_2+x_3-x_4-x_5=1$
τ_{14}	$-x_1+x_2+x_3-x_4+x_5=1$	τ_{30}	$x_1+x_2+x_3-x_4+x_5=1$
τ_{15}	$-x_1+x_2+x_3+x_4-x_5=1$	τ_{31}	$x_1+x_2+x_3+x_4-x_5=1$
τ_{16}	$-x_1+x_2+x_3+x_4+x_5=1$	τ_{32}	$x_1+x_2+x_3+x_4+x_5=1$

A brief inspection reveals that the ten vertices of this polytope share precisely five relations, $u_{2i}+u_{2i-1}=0$. Each pair of vertices entering into these relations form a set of homogeneous coordinates for a $\mathbb{P}^1$. This demonstrates that $\mathbb{P}_{\nabla^*}\cong\left(\mathbb{P}^1\right)^5$.

The polytope Δ^* and the ambient variety $\mathbb{P}_{\Delta^*}$

Vertices of Δ^*			
u_1	(1, 0, 0, 0, 0)	u_{32}	(0, 0, 0, 0,-1)
u_2	(-1, 0, 0, 0, 0)	u_{33}	(1, 0, 0, 0, 1)
u_3	(0, 1, 0, 0, 0)	u_{34}	(-1, 0, 0, 0,-1)
u_4	(0,-1, 0, 0, 0)	u_{35}	(0, 1, 0, 0, 1)
u_5	(1, 1, 0, 0, 0)	u_{36}	(0,-1, 0, 0,-1)
u_6	(-1,-1, 0, 0, 0)	u_{37}	(1, 1, 0, 0, 1)
u_7	(0, 0, 1, 0, 0)	u_{38}	(-1,-1, 0, 0,-1)
u_8	(0, 0,-1, 0, 0)	u_{39}	(0, 0, 1, 0, 1)
u_9	(1, 0, 1, 0, 0)	u_{40}	(0, 0,-1, 0,-1)
u_{10}	(-1, 0,-1, 0, 0)	u_{41}	(1, 0, 1, 0, 1)
u_{11}	(0, 1, 1, 0, 0)	u_{42}	(-1, 0,-1, 0,-1)
u_{12}	(0,-1,-1, 0, 0)	u_{43}	(0, 1, 1, 0, 1)
u_{13}	(1, 1, 1, 0, 0)	u_{44}	(0,-1,-1, 0,-1)
u_{14}	(-1,-1,-1, 0, 0)	u_{45}	(1, 1, 1, 0, 1)
u_{15}	(0, 0, 0, 1, 0)	u_{46}	(-1,-1,-1, 0,-1)
u_{16}	(0, 0, 0,-1, 0)	u_{47}	(0, 0, 0, 1, 1)
u_{17}	(1, 0, 0, 1, 0)	u_{48}	(0, 0, 0,-1,-1)
u_{18}	(-1, 0, 0,-1, 0)	u_{49}	(1, 0, 0, 1, 1)
u_{19}	(0, 1, 0, 1, 0)	u_{50}	(-1, 0, 0,-1,-1)
u_{20}	(0,-1, 0,-1, 0)	u_{51}	(0, 1, 0, 1, 1)
u_{21}	(1, 1, 0, 1, 0)	u_{52}	(0,-1, 0,-1,-1)
u_{22}	(-1,-1, 0,-1, 0)	u_{53}	(1, 1, 0, 1, 1)
u_{23}	(0, 0, 1, 1, 0)	u_{54}	(-1,-1, 0,-1,-1)
u_{24}	(0, 0,-1,-1, 0)	u_{55}	(0, 0, 1, 1, 1)
u_{25}	(1, 0, 1, 1, 0)	u_{56}	(0, 0,-1,-1,-1)
u_{26}	(-1, 0,-1,-1, 0)	u_{57}	(1, 0, 1, 1, 1)
u_{27}	(0, 1, 1, 1, 0)	u_{58}	(-1, 0,-1,-1,-1)
u_{28}	(0,-1,-1,-1, 0)	u_{59}	(0, 1, 1, 1, 1)
u_{29}	(1, 1, 1, 1, 0)	u_{60}	(0,-1,-1,-1,-1)
u_{30}	(-1,-1,-1,-1, 0)	u_{61}	(1, 1, 1, 1, 1)
u_{31}	(0, 0, 0, 0, 1)	u_{62}	(-1,-1,-1,-1,-1)

Faces of Δ^*	
ρ_1	$y_1 = 1$
ρ_2	$y_2 = 1$
ρ_3	$y_3 = 1$
ρ_4	$y_4 = 1$
ρ_5	$y_5 = 1$
ρ_6	$-y_1 = 1$
ρ_7	$-y_2 = 1$
ρ_8	$-y_3 = 1$
ρ_9	$-y_4 = 1$
ρ_{10}	$-y_5 = 1$
ρ_{11}	$y_1 - y_2 = 1$
ρ_{12}	$y_1 - y_3 = 1$
ρ_{13}	$y_1 - y_4 = 1$
ρ_{14}	$y_1 - y_5 = 1$
ρ_{15}	$-y_1 + y_2 = 1$
ρ_{16}	$y_2 - y_3 = 1$
ρ_{17}	$y_2 - y_4 = 1$
ρ_{18}	$y_2 - y_5 = 1$
ρ_{19}	$-y_1 + y_3 = 1$
ρ_{20}	$-y_2 + y_3 = 1$
ρ_{21}	$y_3 - y_4 = 1$
ρ_{22}	$y_3 - y_5 = 1$
ρ_{23}	$-y_1 + y_4 = 1$
ρ_{24}	$-y_2 + y_4 = 1$
ρ_{25}	$-y_3 + y_4 = 1$
ρ_{26}	$y_4 - y_5 = 1$
ρ_{27}	$-y_1 + y_5 = 1$
ρ_{28}	$-y_2 + y_5 = 1$
ρ_{29}	$-y_3 + y_5 = 1$
ρ_{30}	$-y_4 + y_5 = 1$

As is now familiar, we form a matrix W out of these vectors,

$$\mathrm{W} = \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_{30} \end{pmatrix} = \left(w_1, w_2, w_3, w_4, w_5\right) .$$

By examining the nullspace of W^T, we find 57 independent relations between these 62 vectors.

$$u_{2i} + u_{2i-1} = 0\,, \qquad 1 \leq i \leq 31\,,$$

$$\begin{aligned}
&u_1+u_3+u_6=0\,, && u_1+u_7+u_{10}=0\,, && u_1+u_{15}+u_{18}=0\,,\\
&u_1+u_{31}+u_{34}=0\,, && u_3+u_7+u_{12}=0\,, && u_3+u_{15}+u_{20}=0\,,\\
&u_3+u_{31}+u_{36}=0\,, && u_7+u_{15}+u_{24}=0\,, && u_7+u_{31}+u_{40}=0\,,\\
&u_{15}+u_{31}+u_{48}=0\,, && u_1+u_3+u_7+u_{14}=0\,, && u_1+u_3+u_{15}+u_{22}=0\,,\\
&u_1+u_3+u_{31}+u_{38}=0\,, && u_1+u_7+u_{15}+u_{26}=0\,, && u_1+u_7+u_{31}+u_{42}=0\,,\\
&u_1+u_{15}+u_{31}+u_{50}=0\,, && u_3+u_7+u_{15}+u_{28}=0\,, && u_3+u_7+u_{31}+u_{44}=0\,,\\
&u_3+u_{15}+u_{31}+u_{52}=0\,, && u_7+u_{15}+u_{31}+u_{56}=0\,, && u_1+u_3+u_7+u_{15}+u_{30}=0\,,\\
&u_1+u_3+u_7+u_{31}+u_{46}=0\,, && u_1+u_3+u_{15}+u_{31}+u_{54}=0\,, && u_1+u_7+u_{15}+u_{31}+u_{58}=0\,,\\
&u_3+u_7+u_{15}+u_{31}+u_{60}=0\,, && u_1+u_3+u_7+u_{15}+u_{31}+u_{62}=0\,.
\end{aligned}$$

Each of these relations corresponds to a scaling symmetry $\mathbb{C}^* \subset (\mathbb{C}^*)^{57}$. There are five independent invariant combinations of coordinates that we can identify as coordinates on the torus $\mathbb{T}^5 \subset \mathbb{P}_{\Delta^*}$.

$$\begin{aligned}
\Xi_1 &= \xi^{w_1} = \frac{\xi_1\xi_5\xi_9\xi_{13}\xi_{17}\xi_{21}\xi_{25}\xi_{29}\xi_{33}\xi_{37}\xi_{41}\xi_{45}\xi_{49}\xi_{53}\xi_{57}\xi_{61}}{\xi_2\xi_6\xi_{10}\xi_{14}\xi_{18}\xi_{22}\xi_{26}\xi_{30}\xi_{34}\xi_{38}\xi_{42}\xi_{46}\xi_{50}\xi_{54}\xi_{58}\xi_{62}}\,,\\
\Xi_2 &= \xi^{w_2} = \frac{\xi_3\xi_5\xi_{11}\xi_{13}\xi_{19}\xi_{21}\xi_{27}\xi_{29}\xi_{35}\xi_{37}\xi_{43}\xi_{45}\xi_{51}\xi_{53}\xi_{59}\xi_{61}}{\xi_4\xi_6\xi_{12}\xi_{14}\xi_{20}\xi_{22}\xi_{28}\xi_{30}\xi_{36}\xi_{38}\xi_{44}\xi_{46}\xi_{52}\xi_{54}\xi_{60}\xi_{62}}\,,\\
\Xi_3 &= \xi^{w_3} = \frac{\xi_7\xi_9\xi_{11}\xi_{13}\xi_{23}\xi_{25}\xi_{27}\xi_{29}\xi_{39}\xi_{41}\xi_{43}\xi_{45}\xi_{55}\xi_{57}\xi_{59}\xi_{61}}{\xi_8\xi_{10}\xi_{12}\xi_{14}\xi_{24}\xi_{26}\xi_{28}\xi_{30}\xi_{40}\xi_{42}\xi_{44}\xi_{46}\xi_{56}\xi_{58}\xi_{60}\xi_{62}}\,,\\
\Xi_4 &= \xi^{w_4} = \frac{\xi_{15}\xi_{17}\xi_{19}\xi_{21}\xi_{23}\xi_{25}\xi_{27}\xi_{29}\xi_{47}\xi_{49}\xi_{51}\xi_{53}\xi_{55}\xi_{57}\xi_{59}\xi_{61}}{\xi_{16}\xi_{18}\xi_{20}\xi_{22}\xi_{24}\xi_{26}\xi_{28}\xi_{30}\xi_{48}\xi_{50}\xi_{52}\xi_{54}\xi_{56}\xi_{58}\xi_{60}\xi_{62}}\,,\\
\Xi_5 &= \xi^{w_5} = \frac{\xi_{31}\xi_{33}\xi_{35}\xi_{37}\xi_{39}\xi_{41}\xi_{43}\xi_{45}\xi_{47}\xi_{49}\xi_{51}\xi_{53}\xi_{55}\xi_{57}\xi_{59}\xi_{61}}{\xi_{32}\xi_{34}\xi_{36}\xi_{38}\xi_{40}\xi_{42}\xi_{44}\xi_{46}\xi_{48}\xi_{50}\xi_{52}\xi_{54}\xi_{56}\xi_{58}\xi_{60}\xi_{62}}\,.
\end{aligned} \tag{A.3}$$

The polytope ∇

Vertices of ∇					
u_1	(-1,-1,-1,-1,-1)	u_{13}	(-1, 1, 1,-1,-1)	u_{25}	(1, 1,-1,-1,-1)
u_2	(-1,-1,-1,-1, 1)	u_{14}	(-1, 1, 1,-1, 1)	u_{26}	(1, 1,-1,-1, 1)
u_3	(-1,-1,-1, 1,-1)	u_{15}	(-1, 1, 1, 1,-1)	u_{27}	(1, 1,-1, 1,-1)
u_4	(-1,-1,-1, 1, 1)	u_{16}	(-1, 1, 1, 1, 1)	u_{28}	(1, 1,-1, 1, 1)
u_5	(-1,-1, 1,-1,-1)	u_{17}	(1,-1,-1,-1,-1)	u_{29}	(1, 1, 1,-1,-1)
u_6	(-1,-1, 1,-1, 1)	u_{18}	(1,-1,-1,-1, 1)	u_{30}	(1, 1, 1,-1, 1)
u_7	(-1,-1, 1, 1,-1)	u_{19}	(1,-1,-1, 1,-1)	u_{31}	(1, 1, 1, 1,-1)
u_8	(-1,-1, 1, 1, 1)	u_{20}	(1,-1,-1, 1, 1)	u_{32}	(1, 1, 1, 1, 1)
u_9	(-1, 1,-1,-1,-1)	u_{21}	(1,-1, 1,-1,-1)		
u_{10}	(-1, 1,-1,-1, 1)	u_{22}	(1,-1, 1,-1, 1)		
u_{11}	(-1, 1,-1, 1,-1)	u_{23}	(1,-1, 1, 1,-1)		
u_{12}	(-1, 1,-1, 1, 1)	u_{24}	(1,-1, 1, 1, 1)		

Faces of ∇	
ρ_1	$y_1 = 1$
ρ_2	$-y_1 = 1$
ρ_3	$y_2 = 1$
ρ_4	$-y_2 = 1$
ρ_5	$y_3 = 1$
ρ_6	$-y_3 = 1$
ρ_7	$y_4 = 1$
ρ_8	$-y_4 = 1$
ρ_9	$y_5 = 1$
ρ_{10}	$-y_5 = 1$

The polytope Δ

Vertices of Δ	
v_1	(1, 0, 0, 0, 0)
v_2	(0, 1, 0, 0, 0)
v_3	(0, 0, 1, 0, 0)
v_4	(0, 0, 0, 1, 0)
v_5	(0, 0, 0, 0, 1)
v_6	(-1, 0, 0, 0, 0)
v_7	(0,-1, 0, 0, 0)
v_8	(0, 0,-1, 0, 0)
v_9	(0, 0, 0,-1, 0)
v_{10}	(0, 0, 0, 0,-1)
v_{11}	(1,-1, 0, 0, 0)
v_{12}	(1, 0,-1, 0, 0)
v_{13}	(1, 0, 0,-1, 0)
v_{14}	(1, 0, 0, 0,-1)
v_{15}	(-1, 1, 0, 0, 0)
v_{16}	(0, 1,-1, 0, 0)
v_{17}	(0, 1, 0,-1, 0)
v_{18}	(0, 1, 0, 0,-1)
v_{19}	(-1, 0, 1, 0, 0)
v_{20}	(0,-1, 1, 0, 0)
v_{21}	(0, 0, 1,-1, 0)
v_{22}	(0, 0, 1, 0,-1)
v_{23}	(-1, 0, 0, 1, 0)
v_{24}	(0,-1, 0, 1, 0)
v_{25}	(0, 0,-1, 1, 0)
v_{26}	(0, 0, 0, 1,-1)
v_{27}	(-1, 0, 0, 0, 1)
v_{28}	(0,-1, 0, 0, 1)
v_{29}	(0, 0,-1, 0, 1)
v_{30}	(0, 0, 0,-1, 1)

Faces of Δ			
τ_1	$x_1 = 1$	τ_{32}	$-x_5 = 1$
τ_2	$-x_1 = 1$	τ_{33}	$x_1 + x_5 = 1$
τ_3	$x_2 = 1$	τ_{34}	$-x_1 - x_5 = 1$
τ_4	$-x_2 = 1$	τ_{35}	$x_2 + x_5 = 1$
τ_5	$x_1 + x_2 = 1$	τ_{36}	$-x_2 - x_5 = 1$
τ_6	$-x_1 - x_2 = 1$	τ_{37}	$x_1 + x_2 + x_5 = 1$
τ_7	$x_3 = 1$	τ_{38}	$-x_1 - x_2 - x_5 = 1$
τ_8	$-x_3 = 1$	τ_{39}	$x_3 + x_5 = 1$
τ_9	$x_1 + x_3 = 1$	τ_{40}	$-x_3 - x_5 = 1$
τ_{10}	$-x_1 - x_3 = 1$	τ_{41}	$x_1 + x_3 + x_5 = 1$
τ_{11}	$x_2 + x_3 = 1$	τ_{42}	$-x_1 - x_3 - x_5 = 1$
τ_{12}	$-x_2 - x_3 = 1$	τ_{43}	$x_2 + x_3 + x_5 = 1$
τ_{13}	$x_1 + x_2 + x_3 = 1$	τ_{44}	$-x_2 - x_3 - x_5 = 1$
τ_{14}	$-x_1 - x_2 - x_3 = 1$	τ_{45}	$x_1 + x_2 + x_3 + x_5 = 1$
τ_{15}	$x_4 = 1$	τ_{46}	$-x_1 - x_2 - x_3 - x_5 = 1$
τ_{16}	$-x_4 = 1$	τ_{47}	$x_4 + x_5 = 1$
τ_{17}	$x_1 + x_4 = 1$	τ_{48}	$-x_4 - x_5 = 1$
τ_{18}	$-x_1 - x_4 = 1$	τ_{49}	$x_1 + x_4 + x_5 = 1$
τ_{19}	$x_2 + x_4 = 1$	τ_{50}	$-x_1 - x_4 - x_5 = 1$
τ_{20}	$-x_2 - x_4 = 1$	τ_{51}	$x_2 + x_4 + x_5 = 1$
τ_{21}	$x_1 + x_2 + x_4 = 1$	τ_{52}	$-x_2 - x_4 - x_5 = 1$
τ_{22}	$-x_1 - x_2 - x_4 = 1$	τ_{53}	$x_1 + x_2 + x_4 + x_5 = 1$
τ_{23}	$x_3 + x_4 = 1$	τ_{54}	$-x_1 - x_2 - x_4 - x_5 = 1$
τ_{24}	$-x_3 - x_4 = 1$	τ_{55}	$x_3 + x_4 + x_5 = 1$
τ_{25}	$x_1 + x_3 + x_4 = 1$	τ_{56}	$-x_3 - x_4 - x_5 = 1$
τ_{26}	$-x_1 - x_3 - x_4 = 1$	τ_{57}	$x_1 + x_3 + x_4 + x_5 = 1$
τ_{27}	$x_2 + x_3 + x_4 = 1$	τ_{58}	$-x_1 - x_3 - x_4 - x_5 = 1$
τ_{28}	$-x_2 - x_3 - x_4 = 1$	τ_{59}	$x_2 + x_3 + x_4 + x_5 = 1$
τ_{29}	$x_1 + x_2 + x_3 + x_4 = 1$	τ_{60}	$-x_2 - x_3 - x_4 - x_5 = 1$
τ_{30}	$-x_1 - x_2 - x_3 - x_4 = 1$	τ_{61}	$x_1 + x_2 + x_3 + x_4 + x_5 = 1$
τ_{31}	$x_5 = 1$	τ_{62}	$-x_1 - x_2 - x_3 - x_4 - x_5 = 1$

B Series expressions for the Bessel integrals

The symbol $\mathbf{p}$ is understood to denote a multi-index $(p_1, p_2, p_3, p_4, p_5)$. We adopt a notation $c_{\mathbf{p}}$ for the multinomial coefficients. Recall also the harmonic numbers H_n and Polygamma functions ψ.

$$c_{\mathbf{p}} \overset{\text{def}}{=} \binom{\deg(\boldsymbol{p})}{\boldsymbol{p}}^2 = \left(\frac{\left(\sum_{i=1}^5 p_i\right)!}{\prod_{i=1}^5 p_i!}\right)^2, \qquad H_n = \sum_{k=1}^n \frac{1}{k}, \qquad \psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log\Gamma(z).$$

For positive integers m one has the following special values for ψ and its derivatives:

$$\psi(m) = H_{m-1} - \gamma, \qquad \psi^{(1)}(m) = \frac{\pi^2}{6} - \sum_{k=1}^{m-1} \frac{1}{k^2}, \qquad \psi^{(2)}(m) = 2\left(-\zeta(3) + \sum_{k=1}^{m-1} \frac{1}{k^3}\right),$$

with γ the Euler-Mascheroni constant.

With n understood to be a positive integer, we will make frequent use of the following integrals, valid for $\text{Re}[\varphi] > 0$.

$$\begin{aligned}
\int_0^\infty \mathrm{d}z\, K_0(\sqrt{\varphi}z) z^{2n+1} &= 4^n (n!)^2 \varphi^{-1-n}\,, \\
\int_0^\infty \mathrm{d}z\, K_0(\sqrt{\varphi}z) \log\left(\frac{z}{2}\right) z^{2n+1} &= 4^n (n!)^2 \varphi^{-1-n} \left(\psi(n+1) - \frac{1}{2}\log(\varphi)\right), \\
\int_0^\infty \mathrm{d}z\, K_0(\sqrt{\varphi}z) \log\left(\frac{z}{2}\right)^2 z^{2n+1} &= 4^{n-1} (n!)^2 \varphi^{-1-n} \left(2\psi^{(1)}(n+1) - 2\psi(n+1) + \log(\varphi)\right), \\
\int_0^\infty \mathrm{d}z\, K_0(\sqrt{\varphi}z) \log\left(\frac{z}{2}\right)^3 z^{2n+1} &= 4^{n-1} (n!)^2 \varphi^{-1-n} \Big(\psi^{(2)}(n+1) \\
&\quad -3\left(\log\varphi - 2\psi(n+1)\right)\psi^{(1)}(n+1) - \frac{1}{2}\left(\log\varphi - 2\psi(n+1)\right)^3\Big)\,.
\end{aligned} \tag{B.1}$$

To derive the formulae (3.17) and (3.32), recall the following series expressions for the Bessel functions $I_0(x)$ and $K_0(x)$, which can be substituted in the integrals (3.30)-(3.31). One should substitute all Bessel functions for their series below, barring one K_0. Then integrating termwise and applying the above identities allows one obtain the formulae (3.17) and (3.32).

$$I_0(x) = \sum_{n=0}^\infty \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \qquad K_0(x) = -\log\left(\frac{x}{2}\right) I_0(x) + \sum_{n=0}^\infty \frac{\psi(n+1)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}. \tag{B.2}$$

C Parameter counting

The polynomials (2.19)-(2.20) defining the manifolds HΛ and their various quotients contain a number of parameters, which can be viewed as the complex structure parameters of the family HΛ. Naïvely it would seem that there are more free parameters in the defining polynomials than there are complex structure parameters. However, a more careful consideration will show that upon correctly accounting for redundancies, the parameter counts indeed agree.

Consider, for concreteness, the varieties in the family HΛ which are symmetric under $\mathbb{Z}_5 \times \mathbb{Z}_2$, which we take to be those generated by S and V as in (2.18).

We wish to determine the independent parameters in the polynomials Q^1 and Q^2 defining this symmetric variety. There are at least two sources of redundancy. The first is that different polynomials can generate the same ideal. The second arises from automorphisms of the ambient variety $\left(\mathbb{P}^1\right)^5$.

We begin by considering the most general $\mathbb{Z}_5$-invariant polynomials:

$$\begin{aligned}
Q^1 &= A_0 m_{11111} + A_1 m_{10000} + A_2 m_{11000} + A_3 m_{10100} + A_4 m_{11100} + A_5 m_{11010} + A_6 m_{11110} + A_7 m_{00000}\,, \\
Q^2 &= B_0 m_{11111} + B_1 m_{10000} + B_2 m_{11000} + B_3 m_{10100} + B_4 m_{11100} + B_5 m_{11010} + B_6 m_{11110} + B_7 m_{00000}\,.
\end{aligned}$$

To have a variety that is invariant under the $\mathbb{Z}_2$ transformation

$$V \;:\; Y_{i,0} \longleftrightarrow Y_{i,1}\,, \qquad \text{for all } i\,.$$

We demand that the ideal $\langle Q^1, Q^2\rangle$ is invariant under the action of V. In this case this reduces to demanding that VQ^1 and VQ^2 are linear combinations of Q^1 and Q^2:

$$\begin{pmatrix} VQ^1 \\ VQ^2 \end{pmatrix} = \mathrm{M}\begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}, \qquad \text{for some} \qquad \mathrm{M} \in \mathrm{GL}(2,\mathbb{C})\,. \tag{C.1}$$

Clearly $V^2 = \mathrm{Id}$ from which it follows that $\mathrm{M}^2 = 1$. In the generic case, the matrix M takes the form

$$\mathrm{M} = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}.$$

This has the Jordan normal form

$$\mathrm{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, by redefining Q^1 and Q^2 suitably, the condition (C.1) becomes

$$VQ^1 = -Q^1\,, \qquad \text{and} \qquad VQ^2 = Q^2\,. \tag{C.2}$$

The only residual redefinitions of Q^1 and Q^2 are those that keep the diagonalised M fixed, that is rescalings of Q^1 and Q^2. Leaving these scalings unfixed for the time being, the condition (C.2) can be solved to give

$$A_{7-i} = A_i\,, \qquad B_{7-i} = B_i\,. \tag{C.3}$$

Demanding the condition (C.3) fixes most of the automorphisms of $(\mathbb{P}^1)^5/\mathbb{Z}_5$, but there is one remaining family of Möbius automorphisms of the form

$$T: \qquad \frac{Y_{i,0}}{Y_{i,1}} \mapsto \frac{Y_{i,0} + k\,Y_{i,1}}{k\,Y_{i,0} + Y_{i,1}}\,, \qquad \text{with} \qquad k \in \mathbb{C}\setminus\{1,-1\}\,, \qquad \text{for all } i\,.$$

Transformations of this form preserve the condition (C.3). The images of Q^1 and Q^2 can be written down, but the generic form is slightly complicated. We note that

$$T(Q^1) = \frac{(k-1)\left(-A_1k^3 - A_1k^2 - A_2k^2 - A_3k^2 + A_0\left(k^4+k^3+k^2+k+1\right) - A_1k\right)}{k^5}m_{00000} + \ldots$$

By choosing k suitably, we can force the coefficient of m_{00000} to vanish. Upon redefining the remaining parameters the polynomials Q^1 and Q^2 become

$$\begin{aligned} Q^1 &= A_1m_{10000}+A_2m_{11000}+A_3m_{10100}-A_2m_{11100}-A_3m_{11010}-A_1m_{11110}\,,\\ Q^2 &= B_0m_{11111}+B_1m_{10000}+B_2m_{11000}+B_3m_{10100}+B_2m_{11100}+B_3m_{11010}+B_1m_{11110}+B_0m_{00000}\,. \end{aligned}$$

Finally, we can eliminate two parameters by rescaling. This leaves two polynomials with five independent parameters.

$$\begin{aligned} Q^1 &= m_{10000} + a_1m_{11000} + a_2m_{10100} - a_2m_{11100} - a_1m_{11010} - m_{11110}\,,\\ Q^2 &= m_{11111} + a_3m_{10000} + a_4m_{11000} + a_5m_{10100} + a_4m_{11100} + a_5m_{11010} + a_3m_{11110} + m_{00000}\,. \end{aligned}$$

D Recurrences for elements of webs

Consider the map $\psi_{\boldsymbol{I}} : \boldsymbol{J} \mapsto -2H(\boldsymbol{J},\boldsymbol{I})$, which gives a map $\mathbb{Z}^5 \to \mathbb{Z}$. Let us denote by ℓ the greatest common divisor of the nonzero $\psi_{\boldsymbol{I}}(\boldsymbol{J})$ over all $\boldsymbol{J} \in \mathbb{Z}^5$.

Claim 1 *Let $\boldsymbol{J} = w\boldsymbol{I}$, then $J_i - I_i = 0 \mod \ell$ for all i. Specifically,*

$$J_i - I_i \;=\; -2H(\boldsymbol{n}_i(\boldsymbol{J}),\boldsymbol{I})\,, \tag{D.1}$$

where $\boldsymbol{n}_i$ satisfies the recurrence relations

$$\begin{aligned} \boldsymbol{n}_i(g_i\boldsymbol{J}) &= \boldsymbol{n}_i(\boldsymbol{J}) + w^{-1}\boldsymbol{e}^i\,,\\ \boldsymbol{n}_i(g_j\boldsymbol{J}) &= \boldsymbol{n}_i(\boldsymbol{J})\,. \end{aligned} \tag{D.2}$$

Let us show this by induction. The initial step is straightforward, as we need to only check the component on which the duality acts.

$$(g_j\boldsymbol{I})_j - I_j \;=\; \deg\boldsymbol{I} - 3I_j \;=\; -2H(\mathbf{e}^j,\boldsymbol{I}) \;=\; 0 \mod \ell\,.$$

For the induction step, assume $\boldsymbol{J} = w\boldsymbol{I}$ and that for some vectors $\boldsymbol{n}_i$

$$J_i - I_i = -2H(\boldsymbol{n}_i(\boldsymbol{J}),\boldsymbol{I})\,.$$

Then it is enough to check that $(g_i\boldsymbol{J})_i - I_i = (g_i\boldsymbol{J})_i - J_i \mod \ell$ vanishes:

$$(g_i\boldsymbol{J})_i - I_i \;=\; (g_i\boldsymbol{J})_i - J_i \;=\; \deg\boldsymbol{J} - 3J_i \;=\; -2H(\mathbf{e}^i,\boldsymbol{J}) \mod \ell\,.$$

Now recalling that the bilinear H is invariant under the action of the Coxeter group, we have that

$$-2H(\mathbf{e}^i,\boldsymbol{J}) \;=\; -2H(\mathbf{e}^i,w\boldsymbol{I}) \;=\; -2H(w^{-1}\mathbf{e}^i,\boldsymbol{I}) \;=\; 0 \mod \ell\,.$$

Keeping track of the terms that vanish modulo ℓ gives the recurrence formulae (D.2).

An immediate consequence of this is the following claim:

Claim 2 *If $w = g_1 \dots g_l$ is a word in the duality operations and $\boldsymbol{J} = w\boldsymbol{I}$, $\deg\boldsymbol{J} - \deg\boldsymbol{I} = 0 \mod \ell$. Specifically,*

$$\deg\boldsymbol{J} - \deg\boldsymbol{I} \;=\; -2H(\boldsymbol{N}(\boldsymbol{J}),\boldsymbol{I})\,, \qquad \textit{with} \qquad \boldsymbol{N}(\boldsymbol{J}) = \sum_i \boldsymbol{n}_i(\boldsymbol{J})\,. \tag{D.3}$$

E Instanton numbers

In this appendix we tabulate nonzero instanton numbers up to degree 29 for genera 0 and 1. Our tables only give one multidegree in each S_5 orbit. For example the number $n_{(0,1,0,0,0)}$ is not explicitly given, but this number equals $n_{(1,0,0,0,0)} = 24$ which does appear. Furthermore, instanton numbers that vanish are omitted.

E.1 Genus-0 instantons

Table 12: The genus-zero instanton numbers n_I for $\deg(I) \leqslant 29$.

$\deg(I) = 1$

I	n_I	I	n_I
$(1,0,0,0,0)$	24		

$\deg(I) = 2$

I	n_I	I	n_I
$(1,1,0,0,0)$	24		

$\deg(I) = 3$

I	n_I	I	n_I
$(1,1,1,0,0)$	112		

$\deg(I) = 4$

I	n_I	I	n_I
$(1,1,1,1,0)$	1104	$(2,1,1,0,0)$	24

$\deg(I) = 5$

I	n_I	I	n_I
$(1,1,1,1,1)$	19200	$(2,1,1,1,0)$	1104
$(2,2,1,0,0)$	24		

$\deg(I) = 6$

I	n_I	I	n_I
$(2,1,1,1,1)$	45408	$(2,2,1,1,0)$	2800
$(2,2,2,0,0)$	80	$(3,1,1,1,0)$	112

$\deg(I)=7$			
I	n_I	I	n_I
$(2,2,1,1,1)$	212880	$(2,2,2,1,0)$	14496
$(3,1,1,1,1)$	19200	$(3,2,1,1,0)$	1104
$(3,2,2,0,0)$	24		

$\deg(I)=8$			
I	n_I	I	n_I
$(2,2,2,1,1)$	1691856	$(2,2,2,2,0)$	122352
$(3,2,1,1,1)$	212880	$(3,2,2,1,0)$	14496
$(3,3,1,1,0)$	1104	$(3,3,2,0,0)$	24
$(4,1,1,1,1)$	1104	$(4,2,1,1,0)$	24

$\deg(I)=9$			
I	n_I	I	n_I
$(2,2,2,2,1)$	20299992	$(3,2,2,1,1)$	3222112
$(3,2,2,2,0)$	234048	$(3,3,1,1,1)$	434688
$(3,3,2,1,0)$	30624	$(3,3,3,0,0)$	112
$(4,2,1,1,1)$	45408	$(4,2,2,1,0)$	2800
$(4,3,1,1,0)$	112		

$\deg(I)=10$			
I	n_I	I	n_I
$(2,2,2,2,2)$	341681280	$(3,2,2,2,1)$	63576576
$(3,3,2,1,1)$	10883712	$(3,3,2,2,0)$	795936
$(3,3,3,1,0)$	122448	$(4,2,2,1,1)$	1691856
$(4,2,2,2,0)$	122352	$(4,3,1,1,1)$	212880
$(4,3,2,1,0)$	14496	$(4,3,3,0,0)$	24
$(4,4,1,1,0)$	24	$(5,2,1,1,1)$	1104
$(5,2,2,1,0)$	24		

$\deg(I)=11$			
I	n_I	I	n_I
$(3,2,2,2,2)$	1599622824	$(3,3,2,2,1)$	316997280
$(3,3,3,1,1)$	59097600	$(3,3,3,2,0)$	4326048
$(4,2,2,2,1)$	63576576	$(4,3,2,1,1)$	10883712
$(4,3,2,2,0)$	795936	$(4,3,3,1,0)$	122448
$(4,4,1,1,1)$	212880	$(4,4,2,1,0)$	14496
$(4,4,3,0,0)$	24	$(5,2,2,1,1)$	212880
$(5,2,2,2,0)$	14496	$(5,3,1,1,1)$	19200
$(5,3,2,1,0)$	1104		

$\deg(I)=12$			
I	n_I	I	n_I
$(3,3,2,2,2)$	11032046624	$(3,3,3,2,1)$	2322325968

Continued on the following page

deg(I) = 12, *continued*			
I	n_I	I	n_I
(3,3,3,3,0)	33777312	(4,2,2,2,2)	2624447520
(4,3,2,2,1)	529392832	(4,3,3,1,1)	100919904
(4,3,3,2,0)	7371792	(4,4,2,1,1)	19420400
(4,4,2,2,0)	1423104	(4,4,3,1,0)	234048
(4,4,4,0,0)	80	(5,2,2,2,1)	20299992
(5,3,2,1,1)	3222112	(5,3,2,2,0)	234048
(5,3,3,1,0)	30624	(5,4,1,1,1)	45408
(5,4,2,1,0)	2800	(6,2,2,1,1)	2800
(6,2,2,2,0)	80	(6,3,1,1,1)	112

deg(I) = 13			
I	n_I	I	n_I
(3,3,3,2,2)	105371446464	(3,3,3,3,1)	23351460864
(4,3,2,2,2)	27607031136	(4,3,3,2,1)	5950086192
(4,3,3,3,0)	88179456	(4,4,2,2,1)	1426637712
(4,4,3,1,1)	282674592	(4,4,3,2,0)	20578560
(4,4,4,1,0)	795936	(5,2,2,2,2)	1599622824
(5,3,2,2,1)	316997280	(5,3,3,1,1)	59097600
(5,3,3,2,0)	4326048	(5,4,2,1,1)	10883712
(5,4,2,2,0)	795936	(5,4,3,1,0)	122448
(5,4,4,0,0)	24	(5,5,1,1,1)	19200
(5,5,2,1,0)	1104	(6,2,2,2,1)	1691856
(6,3,2,1,1)	212880	(6,3,2,2,0)	14496
(6,3,3,1,0)	1104	(6,4,1,1,1)	1104
(6,4,2,1,0)	24		

deg(I) = 14			
I	n_I	I	n_I
(3,3,3,3,2)	1326841710624	(4,3,3,2,2)	377080188864
(4,3,3,3,1)	85495746528	(4,4,2,2,2)	103492041648
(4,4,3,2,1)	22951602432	(4,4,3,3,0)	347078520
(4,4,4,1,1)	1218252960	(4,4,4,2,0)	88177920
(5,3,2,2,2)	27607031136	(5,3,3,2,1)	5950086192
(5,3,3,3,0)	88179456	(5,4,2,2,1)	1426637712
(5,4,3,1,1)	282674592	(5,4,3,2,0)	20578560
(5,4,4,1,0)	795936	(5,5,2,1,1)	10883712
(5,5,2,2,0)	795936	(5,5,3,1,0)	122448
(5,5,4,0,0)	24	(6,2,2,2,2)	341681280
(6,3,2,2,1)	63576576	(6,3,3,1,1)	10883712
(6,3,3,2,0)	795936	(6,4,2,1,1)	1691856
(6,4,2,2,0)	122352	(6,4,3,1,0)	14496
(6,5,1,1,1)	1104	(6,5,2,1,0)	24
(7,2,2,2,1)	14496	(7,3,2,1,1)	1104
(7,3,2,2,0)	24		

deg(I) = 15			
I	n_I	I	n_I
(3,3,3,3,3)	21228933784320	(4,3,3,3,2)	6446376071472
(4,4,3,2,2)	1912895782008	(4,4,3,3,1)	443961562528

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deg(*I*) = 15, *continued*			
I	n_I	I	n_I
(4,4,4,2,1)	126121309632	(4,4,4,3,0)	1935300720
(5,3,3,2,2)	570360079168	(5,3,3,3,1)	130194945024
(5,4,2,2,2)	158730945984	(5,4,3,2,1)	35487082592
(5,4,3,3,0)	539120544	(5,4,4,1,1)	1944767152
(5,4,4,2,0)	140436672	(5,5,2,2,1)	2306418848
(5,5,3,1,1)	464696832	(5,5,3,2,0)	33777312
(5,5,4,1,0)	1423616	(5,5,5,0,0)	112
(6,3,2,2,2)	11032046624	(6,3,3,2,1)	2322325968
(6,3,3,3,0)	33777312	(6,4,2,2,1)	529392832
(6,4,3,1,1)	100919904	(6,4,3,2,0)	7371792
(6,4,4,1,0)	234048	(6,5,2,1,1)	3222112
(6,5,2,2,0)	234048	(6,5,3,1,0)	30624
(6,6,1,1,1)	112	(7,2,2,2,2)	20299992
(7,3,2,2,1)	3222112	(7,3,3,1,1)	434688
(7,3,3,2,0)	30624	(7,4,2,1,1)	45408
(7,4,2,2,0)	2800	(7,4,3,1,0)	112

deg(*I*) = 16			
I	n_I	I	n_I
(4,3,3,3,3)	134508124418928	(4,4,3,3,2)	42411173392368
(4,4,4,2,2)	13138629854976	(4,4,4,3,1)	3114669545280
(4,4,4,4,0)	14386855920	(5,3,3,3,2)	13834674726336
(5,4,3,2,2)	4183230238656	(5,4,3,3,1)	980247769056
(5,4,4,2,1)	285207114048	(5,4,4,3,0)	4392333792
(5,5,2,2,2)	366406656528	(5,5,3,2,1)	83099778720
(5,5,3,3,0)	1272585120	(5,5,4,1,1)	4826161680
(5,5,4,2,0)	347078520	(5,5,5,1,0)	4326048
(6,3,3,2,2)	377080188864	(6,3,3,3,1)	85495746528
(6,4,2,2,2)	103492041648	(6,4,3,2,1)	22951602432
(6,4,3,3,0)	347078520	(6,4,4,1,1)	1218252960
(6,4,4,2,0)	88177920	(6,5,2,2,1)	1426637712
(6,5,3,1,1)	282674592	(6,5,3,2,0)	20578560
(6,5,4,1,0)	795936	(6,5,5,0,0)	24
(6,6,2,1,1)	1691856	(6,6,2,2,0)	122352
(6,6,3,1,0)	14496	(7,3,2,2,2)	1599622824
(7,3,3,2,1)	316997280	(7,3,3,3,0)	4326048
(7,4,2,2,1)	63576576	(7,4,3,1,1)	10883712
(7,4,3,2,0)	795936	(7,4,4,1,0)	14496
(7,5,2,1,1)	212880	(7,5,2,2,0)	14496
(7,5,3,1,0)	1104	(8,2,2,2,2)	122352
(8,3,2,2,1)	14496	(8,3,3,1,1)	1104
(8,3,3,2,0)	24	(8,4,2,1,1)	24

deg(*I*) = 17			
I	n_I	I	n_I
(4,4,3,3,3)	1112487680575968	(4,4,4,3,2)	363393804317664
(4,4,4,4,1)	28258960027296	(5,3,3,3,3)	389973010495488
(5,4,3,3,2)	125365423769760	(5,4,4,2,2)	39692266181304
(5,4,4,3,1)	9502910875584	(5,4,4,4,0)	45007048752
(5,5,3,2,2)	13073262151968	(5,5,3,3,1)	3100342138368
(5,5,4,2,1)	931163905728	(5,5,4,3,0)	14386869840
(5,5,5,1,1)	17798444544	(5,5,5,2,0)	1272585120

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deg(I) = 17, *continued*			
I	n_I	I	n_I
(6,3,3,3,2)	13834674726336	(6,4,3,2,2)	4183230238656
(6,4,3,3,1)	980247769056	(6,4,4,2,1)	285207114048
(6,4,4,3,0)	4392333792	(6,5,2,2,2)	366406656528
(6,5,3,2,1)	83099778720	(6,5,3,3,0)	1272585120
(6,5,4,1,1)	4826161680	(6,5,4,2,0)	347078520
(6,5,5,1,0)	4326048	(6,6,2,2,1)	1426637712
(6,6,3,1,1)	282674592	(6,6,3,2,0)	20578560
(6,6,4,1,0)	795936	(6,6,5,0,0)	24
(7,3,3,2,2)	105371446464	(7,3,3,3,1)	23351460864
(7,4,2,2,2)	27607031136	(7,4,3,2,1)	5950086192
(7,4,3,3,0)	88179456	(7,4,4,1,1)	282674592
(7,4,4,2,0)	20578560	(7,5,2,2,1)	316997280
(7,5,3,1,1)	59097600	(7,5,3,2,0)	4326048
(7,5,4,1,0)	122448	(7,6,2,1,1)	212880
(7,6,2,2,0)	14496	(7,6,3,1,0)	1104
(8,3,2,2,2)	63576576	(8,3,3,2,1)	10883712
(8,3,3,3,0)	122448	(8,4,2,2,1)	1691856
(8,4,3,1,1)	212880	(8,4,3,2,0)	14496
(8,4,4,1,0)	24	(8,5,2,1,1)	1104
(8,5,2,2,0)	24		

deg(I) = 18			
I	n_I	I	n_I
(4,4,4,3,3)	11630106886504344	(4,4,4,4,2)	3920585033699328
(5,4,3,3,3)	4272828104425920	(5,4,4,3,2)	1423524718242752
(5,4,4,4,1)	114110495895360	(5,5,3,3,2)	507096396665312
(5,5,4,2,2)	164605655104880	(5,5,4,3,1)	39821013536096
(5,5,4,4,0)	193411225936	(5,5,5,2,1)	4217701870608
(5,5,5,3,0)	65215603200	(6,3,3,3,3)	552486590320032
(6,4,3,3,2)	178677828494464	(6,4,4,2,2)	56949598227232
(6,4,4,3,1)	13674852866304	(6,4,4,4,0)	65215569408
(6,5,3,2,2)	18954386538304	(6,5,3,3,1)	4510722900128
(6,5,4,2,1)	1367836823744	(6,5,4,3,0)	21143067840
(6,5,5,1,1)	27120466144	(6,5,5,2,0)	1935300720
(6,6,2,2,2)	551803842816	(6,6,3,2,1)	125948336640
(6,6,3,3,0)	1935300720	(6,6,4,1,1)	7510615200
(6,6,4,2,0)	539115744	(6,6,5,1,0)	7371792
(6,6,6,0,0)	80	(7,3,3,3,2)	6446376071472
(7,4,3,2,2)	1912895782008	(7,4,3,3,1)	443961562528
(7,4,4,2,1)	126121309632	(7,4,4,3,0)	1935300720
(7,5,2,2,2)	158730945984	(7,5,3,2,1)	35487082592
(7,5,3,3,0)	539120544	(7,5,4,1,1)	1944767152
(7,5,4,2,0)	140436672	(7,5,5,1,0)	1423616
(7,6,2,2,1)	529392832	(7,6,3,1,1)	100919904
(7,6,3,2,0)	7371792	(7,6,4,1,0)	234048
(7,7,2,1,1)	45408	(7,7,2,2,0)	2800
(7,7,3,1,0)	112	(8,3,3,2,2)	11032046624
(8,3,3,3,1)	2322325968	(8,4,2,2,2)	2624447520
(8,4,3,2,1)	529392832	(8,4,3,3,0)	7371792
(8,4,4,1,1)	19420400	(8,4,4,2,0)	1423104
(8,5,2,2,1)	20299992	(8,5,3,1,1)	3222112
(8,5,3,2,0)	234048	(8,5,4,1,0)	2800
(8,6,2,1,1)	2800	(8,6,2,2,0)	80
(9,3,2,2,2)	234048	(9,3,3,2,1)	30624
(9,3,3,3,0)	112	(9,4,2,2,1)	2800

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deg(I) = 18, *continued*			
I	n_I	I	n_I
(9,4,3,1,1)	112		

deg(I) = 19			
I	n_I	I	n_I
(4,4,4,4,3)	1495834072023671 76	(5,4,4,3,3)	573091296207111 36
(5,4,4,4,2)	196801577604071 04	(5,5,3,3,3)	216719629053204 48
(5,5,4,3,2)	7371081117191712	(5,5,4,4,1)	609209937409968
(5,5,5,2,2)	905275754212800	(5,5,5,3,1)	221145135246336
(5,5,5,4,0)	1096632180480	(6,4,3,3,3)	8236673292611808
(6,4,4,3,2)	2768640614245200	(6,4,4,4,1)	224917616990784
(6,5,3,3,2)	1000740719949936	(6,5,4,2,2)	328447354833120
(6,5,4,3,1)	79804026346992	(6,5,4,4,0)	391409808576
(6,5,5,2,1)	8748592415904	(6,5,5,3,0)	135171775392
(6,6,3,2,2)	39360165257928	(6,6,3,3,1)	9425697295296
(6,6,4,2,1)	2910089695872	(6,6,4,3,0)	45007048752
(6,6,5,1,1)	61773182400	(6,6,5,2,0)	4392333792
(6,6,6,1,0)	20578560	(7,3,3,3,3)	389973010495488
(7,4,3,3,2)	125365423769760	(7,4,4,2,2)	39692266181304
(7,4,4,3,1)	9502910875584	(7,4,4,4,0)	45007048752
(7,5,3,2,2)	13073262151968	(7,5,3,3,1)	3100342138368
(7,5,4,2,1)	931163905728	(7,5,4,3,0)	14386869840
(7,5,5,1,1)	17798444544	(7,5,5,2,0)	1272585120
(7,6,2,2,2)	366406656528	(7,6,3,2,1)	83099778720
(7,6,3,3,0)	1272585120	(7,6,4,1,1)	4826161680
(7,6,4,2,0)	347078520	(7,6,5,1,0)	4326048
(7,6,6,0,0)	24	(7,7,2,2,1)	316997280
(7,7,3,1,1)	59097600	(7,7,3,2,0)	4326048
(7,7,4,1,0)	122448	(8,3,3,3,2)	1326841710624
(8,4,3,2,2)	377080188864	(8,4,3,3,1)	85495746528
(8,4,4,2,1)	22951602432	(8,4,4,3,0)	347078520
(8,5,2,2,2)	27607031136	(8,5,3,2,1)	5950086192
(8,5,3,3,0)	88179456	(8,5,4,1,1)	282674592
(8,5,4,2,0)	20578560	(8,5,5,1,0)	122448
(8,6,2,2,1)	63576576	(8,6,3,1,1)	10883712
(8,6,3,2,0)	795936	(8,6,4,1,0)	14496
(8,7,2,1,1)	1104	(8,7,2,2,0)	24
(9,3,3,2,2)	316997280	(9,3,3,3,1)	59097600
(9,4,2,2,2)	63576576	(9,4,3,2,1)	10883712
(9,4,3,3,0)	122448	(9,4,4,1,1)	212880
(9,4,4,2,0)	14496	(9,5,2,2,1)	212880
(9,5,3,1,1)	19200	(9,5,3,2,0)	1104

deg(I) = 20			
I	n_I	I	n_I
(4,4,4,4,4)	23157586017060115 20	(5,4,4,4,3)	9202466920526724 48
(5,5,4,3,3)	3621767329918822 56	(5,5,4,4,2)	1266563775077366 16
(5,5,5,3,2)	48949713376347552	(5,5,5,4,1)	4162140562025760
(5,5,5,5,0)	7888589144400	(6,4,4,3,3)	145074948270672288
(6,4,4,4,2)	50310287851264512	(6,5,3,3,3)	55724768553096576
(6,5,4,3,2)	19159936729163904	(6,5,4,4,1)	1608297381675072
(6,5,5,2,2)	2428815576573408	(6,5,5,3,1)	596073535387056
(6,5,5,4,0)	2981800050480	(6,6,3,3,2)	2713101057421728

Continued on the following page

I	n_I	I	n_I
$\deg(I) = 20$, *continued*			
(6,6,4,2,2)	903893653068672	(6,6,4,3,1)	220840621188096
(6,6,4,4,0)	1096632086784	(6,6,5,2,1)	25377635878296
(6,6,5,3,0)	391409808576	(6,6,6,1,1)	203336907216
(6,6,6,2,0)	14386855920	(7,4,3,3,3)	8236673292611808
(7,4,4,3,2)	2768640614245200	(7,4,4,4,1)	224917616990784
(7,5,3,3,2)	1000740719949936	(7,5,4,2,2)	328447354833120
(7,5,4,3,1)	79804026346992	(7,5,4,4,0)	391409808576
(7,5,5,2,1)	8748592415904	(7,5,5,3,0)	135171775392
(7,6,3,2,2)	39360165257928	(7,6,3,3,1)	9425697295296
(7,6,4,2,1)	2910089695872	(7,6,4,3,0)	45007048752
(7,6,5,1,1)	61773182400	(7,6,5,2,0)	4392333792
(7,6,6,1,0)	20578560	(7,7,2,2,2)	366406656528
(7,7,3,2,1)	83099778720	(7,7,3,3,0)	1272585120
(7,7,4,1,1)	4826161680	(7,7,4,2,0)	347078520
(7,7,5,1,0)	4326048	(7,7,6,0,0)	24
(8,3,3,3,3)	134508124418928	(8,4,3,3,2)	42411173392368
(8,4,4,2,2)	13138629854976	(8,4,4,3,1)	3114669545280
(8,4,4,4,0)	14386855920	(8,5,3,2,2)	4183230238656
(8,5,3,3,1)	980247769056	(8,5,4,2,1)	285207114048
(8,5,4,3,0)	4392333792	(8,5,5,1,1)	4826161680
(8,5,5,2,0)	347078520	(8,6,2,2,2)	103492041648
(8,6,3,2,1)	22951602432	(8,6,3,3,0)	347078520
(8,6,4,1,1)	1218252960	(8,6,4,2,0)	88177920
(8,6,5,1,0)	795936	(8,7,2,2,1)	63576576
(8,7,3,1,1)	10883712	(8,7,3,2,0)	795936
(8,7,4,1,0)	14496	(8,8,2,1,1)	24
(9,3,3,3,2)	105371446464	(9,4,3,2,2)	27607031136
(9,4,3,3,1)	5950086192	(9,4,4,2,1)	1426637712
(9,4,4,3,0)	20578560	(9,5,2,2,2)	1599622824
(9,5,3,2,1)	316997280	(9,5,3,3,0)	4326048
(9,5,4,1,1)	10883712	(9,5,4,2,0)	795936
(9,5,5,1,0)	1104	(9,6,2,2,1)	1691856
(9,6,3,1,1)	212880	(9,6,3,2,0)	14496
(9,6,4,1,0)	24	(10,3,3,2,2)	795936
(10,3,3,3,1)	122448	(10,4,2,2,2)	122352
(10,4,3,2,1)	14496	(10,4,3,3,0)	24
(10,4,4,1,1)	24	(10,5,2,2,1)	24

I	n_I	I	n_I
$\deg(I) = 21$			
(5,4,4,4,4)	17389206433621316832	(5,5,4,4,3)	7079567101109436512
(5,5,5,3,3)	2860072289627444736	(5,5,5,4,2)	1017289744237857120
(5,5,5,5,1)	35306571598392576	(6,4,4,4,3)	2968386268852263168
(6,5,4,3,3)	1187054464752608224	(6,5,4,4,2)	419478239436537264
(6,5,5,3,2)	165119843412344816	(6,5,5,4,1)	14258867760974432
(6,5,5,5,0)	27765085214112	(6,6,3,3,3)	190193228131870512
(6,6,4,3,2)	66233922634330080	(6,6,4,4,1)	5658979212554128
(6,6,5,2,2)	8718347106041576	(6,6,5,3,1)	2150266975191936
(6,6,5,4,0)	10848408360480	(6,6,6,2,1)	99894446151552
(6,6,6,3,0)	1535514818112	(7,4,4,3,3)	196866216448867200
(7,4,4,4,2)	68481669752665152	(7,5,3,3,3)	75992812385562624
(7,5,4,3,2)	26217346711258048	(7,5,4,4,1)	2211223893638272
(7,5,5,2,2)	3356453655323136	(7,5,5,3,1)	824874647838720
(7,5,5,4,0)	4136092936448	(7,6,3,3,2)	3761948244770304
(7,6,4,2,2)	1259132047619264	(7,6,4,3,1)	308134225628128

Continued on the following page

$\deg(I) = 21$, *continued*			
I	n_I	I	n_I
(7,6,4,4,0)	1535514818112	(7,6,5,2,1)	35929933424832
(7,6,5,3,0)	553728279360	(7,6,6,1,1)	299302640864
(7,6,6,2,0)	21143067840	(7,7,3,2,2)	56389985840000
(7,7,3,3,1)	13542066341888	(7,7,4,2,1)	4216297529824
(7,7,4,3,0)	65215603200	(7,7,5,1,1)	92396257280
(7,7,5,2,0)	6558863360	(7,7,6,1,0)	33777312
(7,7,7,0,0)	112	(8,4,3,3,3)	4272828104425920
(8,4,4,3,2)	1423524718242752	(8,4,4,4,1)	114110495895360
(8,5,3,3,2)	507096396665312	(8,5,4,2,2)	164605655104880
(8,5,4,3,1)	39821013536096	(8,5,4,4,0)	193411225936
(8,5,5,2,1)	4217701870608	(8,5,5,3,0)	65215603200
(8,6,3,2,2)	18954386538304	(8,6,3,3,1)	4510722900128
(8,6,4,2,1)	1367836823744	(8,6,4,3,0)	21143067840
(8,6,5,1,1)	27120466144	(8,6,5,2,0)	1935300720
(8,6,6,1,0)	7371792	(8,7,2,2,2)	158730945984
(8,7,3,2,1)	35487082592	(8,7,3,3,0)	539120544
(8,7,4,1,1)	1944767152	(8,7,4,2,0)	140436672
(8,7,5,1,0)	1423616	(8,8,2,2,1)	20299992
(8,8,3,1,1)	3222112	(8,8,3,2,0)	234048
(8,8,4,1,0)	2800	(9,3,3,3,3)	21228933784320
(9,4,3,3,2)	6446376071472	(9,4,4,2,2)	1912895782008
(9,4,4,3,1)	443961562528	(9,4,4,4,0)	1935300720
(9,5,3,2,2)	570360079168	(9,5,3,3,1)	130194945024
(9,5,4,2,1)	35487082592	(9,5,4,3,0)	539120544
(9,5,5,1,1)	464696832	(9,5,5,2,0)	33777312
(9,6,2,2,2)	11032046624	(9,6,3,2,1)	2322325968
(9,6,3,3,0)	33777312	(9,6,4,1,1)	100919904
(9,6,4,2,0)	7371792	(9,6,5,1,0)	30624
(9,7,2,2,1)	3222112	(9,7,3,1,1)	434688
(9,7,3,2,0)	30624	(9,7,4,1,0)	112
(10,3,3,3,2)	2322325968	(10,4,3,2,2)	529392832
(10,4,3,3,1)	100919904	(10,4,4,2,1)	19420400
(10,4,4,3,0)	234048	(10,5,2,2,2)	20299992
(10,5,3,2,1)	3222112	(10,5,3,3,0)	30624
(10,5,4,1,1)	45408	(10,5,4,2,0)	2800
(10,6,2,2,1)	2800	(10,6,3,1,1)	112

$\deg(I) = 22$			
I	n_I	I	n_I
(5,5,4,4,4)	159832960277398698312	(5,5,5,4,3)	66573482065327669440
(5,5,5,5,2)	9952370045915290464	(6,4,4,4,4)	69702170473826178048
(6,5,4,4,3)	28814753795787304128	(6,5,5,3,3)	11833136668383611040
(6,5,5,4,2)	4252005327651223776	(6,5,5,5,1)	152435152838866176
(6,6,4,3,3)	5023740750844977792	(6,6,4,4,2)	1795314514514344416
(6,6,5,3,2)	721163569257189312	(6,6,5,4,1)	63276065657309280
(6,6,5,5,0)	126532108859856	(6,6,6,2,2)	40767562975883520
(6,6,6,3,1)	10102374952223232	(6,6,6,4,0)	51294956593632
(7,4,4,4,3)	5273427759409817952	(7,5,4,3,3)	2124595552827372432
(7,5,4,4,2)	754387255278771840	(7,5,5,3,2)	299477728365291600
(7,5,5,4,1)	26040136828870752	(7,5,5,5,0)	51294957112992
(7,6,3,3,3)	346896207708697296	(7,6,4,3,2)	121505012171479176
(7,6,4,4,1)	10462960782869952	(7,6,5,2,2)	16270300857476160
(7,6,5,3,1)	4021264698687264	(7,6,5,4,0)	20350993239840
(7,6,6,2,1)	194378107421760	(7,6,6,3,0)	2981800050480
(7,7,3,3,2)	7173870919736064	(7,7,4,2,2)	2422178398686816

Continued on the following page

deg(I) = 22, *continued*			
I	n_I	I	n_I
(7,7,4,3,1)	594508678788528	(7,7,4,4,0)	2981800050480
(7,7,5,2,1)	71274491245200	(7,7,5,3,0)	1096632180480
(7,7,6,1,1)	639016897824	(7,7,6,2,0)	45007048752
(7,7,7,1,0)	88179456	(8,4,4,3,3)	145074948270672288
(8,4,4,4,2)	50310287851264512	(8,5,3,3,3)	55724768553096576
(8,5,4,3,2)	19159936729163904	(8,5,4,4,1)	1608297381675072
(8,5,5,2,2)	2428815576573408	(8,5,5,3,1)	596073535387056
(8,5,5,4,0)	2981800050480	(8,6,3,3,2)	2713101057421728
(8,6,4,2,2)	903893653068672	(8,6,4,3,1)	220840621188096
(8,6,4,4,0)	1096632086784	(8,6,5,2,1)	25377635878296
(8,6,5,3,0)	391409808576	(8,6,6,1,1)	203336907216
(8,6,6,2,0)	14386855920	(8,7,3,2,2)	39360165257928
(8,7,3,3,1)	9425697295296	(8,7,4,2,1)	2910089695872
(8,7,4,3,0)	45007048752	(8,7,5,1,1)	61773182400
(8,7,5,2,0)	4392333792	(8,7,6,1,0)	20578560
(8,7,7,0,0)	24	(8,8,2,2,2)	103492041648
(8,8,3,2,1)	22951602432	(8,8,3,3,0)	347078520
(8,8,4,1,1)	1218252960	(8,8,4,2,0)	88177920
(8,8,5,1,0)	795936	(9,4,3,3,3)	1112487680575968
(9,4,4,3,2)	363393804317664	(9,4,4,4,1)	28258960027296
(9,5,3,3,2)	125365423769760	(9,5,4,2,2)	39692266181304
(9,5,4,3,1)	9502910875584	(9,5,4,4,0)	45007048752
(9,5,5,2,1)	931163905728	(9,5,5,3,0)	14386869840
(9,6,3,2,2)	4183230238656	(9,6,3,3,1)	980247769056
(9,6,4,2,1)	285207114048	(9,6,4,3,0)	4392333792
(9,6,5,1,1)	4826161680	(9,6,5,2,0)	347078520
(9,6,6,1,0)	795936	(9,7,2,2,2)	27607031136
(9,7,3,2,1)	5950086192	(9,7,3,3,0)	88179456
(9,7,4,1,1)	282674592	(9,7,4,2,0)	20578560
(9,7,5,1,0)	122448	(9,8,2,2,1)	1691856
(9,8,3,1,1)	212880	(9,8,3,2,0)	14496
(9,8,4,1,0)	24	(10,3,3,3,3)	1326841710624
(10,4,3,3,2)	377080188864	(10,4,4,2,2)	103492041648
(10,4,4,3,1)	22951602432	(10,4,4,4,0)	88177920
(10,5,3,2,2)	27607031136	(10,5,3,3,1)	5950086192
(10,5,4,2,1)	1426637712	(10,5,4,3,0)	20578560
(10,5,5,1,1)	10883712	(10,5,5,2,0)	795936
(10,6,2,2,2)	341681280	(10,6,3,2,1)	63576576
(10,6,3,3,0)	795936	(10,6,4,1,1)	1691856
(10,6,4,2,0)	122352	(10,6,5,1,0)	24
(10,7,2,2,1)	14496	(10,7,3,1,1)	1104
(10,7,3,2,0)	24	(11,3,3,3,2)	4326048
(11,4,3,2,2)	795936	(11,4,3,3,1)	122448
(11,4,4,2,1)	14496	(11,4,4,3,0)	24
(11,5,2,2,2)	14496	(11,5,3,2,1)	1104

deg(I) = 23			
I	n_I	I	n_I
(5,5,5,4,4)	1763388068567027864736	(5,5,5,5,3)	749974117128947225088
(6,5,4,4,4)	790863904443723569376	(6,5,5,4,3)	334370838391810248432
(6,5,5,5,2)	51265779665018065536	(6,6,4,4,3)	147828049737997120632
(6,6,5,3,3)	61757539943858380704	(6,6,5,4,2)	22420496977021999680
(6,6,5,5,1)	830212985215356432	(6,6,6,3,2)	3982038442779651984
(6,6,6,4,1)	354725791310991552	(6,6,6,5,0)	725912434085952
(7,4,4,4,4)	157391952685989052728	(7,5,4,4,3)	65617907601886711296

Continued on the following page

deg(I) = 23, *continued*			
I	n_I	I	n_I
(7,5,5,3,3)	27190620887571766272	(7,5,5,4,2)	9824161857371476896
(7,5,5,5,1)	358330187751266304	(7,6,4,3,3)	11689593863624674656
(7,6,4,4,2)	4202606608677077184	(7,6,5,3,2)	1706830027589928192
(7,6,5,4,1)	150993571342096992	(7,6,5,5,0)	305922925426848
(7,6,6,2,2)	100108346194477248	(7,6,6,3,1)	24863416450991904
(7,6,6,4,0)	126532108859856	(7,7,3,3,3)	841539378868429824
(7,7,4,3,2)	297115911452589936	(7,7,4,4,1)	25857038420140320
(7,7,5,2,2)	40746789567213888	(7,7,5,3,1)	10097809547695104
(7,7,5,4,0)	51294957112992	(7,7,6,2,1)	515881389602064
(7,7,6,3,0)	7888589144400	(7,7,7,1,1)	1927069671936
(7,7,7,2,0)	135171775392	(8,4,4,4,3)	5273427759409817952
(8,5,4,3,3)	2124595552827372432	(8,5,4,4,2)	754387255278771840
(8,5,5,3,2)	299477728365291600	(8,5,5,4,1)	26040136828870752
(8,5,5,5,0)	51294957112992	(8,6,3,3,3)	346896207708697296
(8,6,4,3,2)	121505012171479176	(8,6,4,4,1)	10462960782869952
(8,6,5,2,2)	16270300857476160	(8,6,5,3,1)	4021264698687264
(8,6,5,4,0)	20350993239840	(8,6,6,2,1)	194378107421760
(8,6,6,3,0)	2981800050480	(8,7,3,3,2)	7173870919736064
(8,7,4,2,2)	2422178398686816	(8,7,4,3,1)	594508678788528
(8,7,4,4,0)	2981800050480	(8,7,5,2,1)	71274491245200
(8,7,5,3,0)	1096632180480	(8,7,6,1,1)	639016897824
(8,7,6,2,0)	45007048752	(8,7,7,1,0)	88179456
(8,8,3,2,2)	39360165257928	(8,8,3,3,1)	9425697295296
(8,8,4,2,1)	2910089695872	(8,8,4,3,0)	45007048752
(8,8,5,1,1)	61773182400	(8,8,5,2,0)	4392333792
(8,8,6,1,0)	20578560	(8,8,7,0,0)	24
(9,4,4,3,3)	57309129620711136	(9,4,4,4,2)	19680157760407104
(9,5,3,3,3)	21671962905320448	(9,5,4,3,2)	7371081117191712
(9,5,4,4,1)	609209937409968	(9,5,5,2,2)	905275754212800
(9,5,5,3,1)	221145135246336	(9,5,5,4,0)	1096632180480
(9,6,3,3,2)	1000740719949936	(9,6,4,2,2)	328447354833120
(9,6,4,3,1)	79804026346992	(9,6,4,4,0)	391409808576
(9,6,5,2,1)	8748592415904	(9,6,5,3,0)	135171775392
(9,6,6,1,1)	61773182400	(9,6,6,2,0)	4392333792
(9,7,3,2,2)	13073262151968	(9,7,3,3,1)	3100342138368
(9,7,4,2,1)	931163905728	(9,7,4,3,0)	14386869840
(9,7,5,1,1)	17798444544	(9,7,5,2,0)	1272585120
(9,7,6,1,0)	4326048	(9,8,2,2,2)	27607031136
(9,8,3,2,1)	5950086192	(9,8,3,3,0)	88179456
(9,8,4,1,1)	282674592	(9,8,4,2,0)	20578560
(9,8,5,1,0)	122448	(9,9,2,2,1)	212880
(9,9,3,1,1)	19200	(9,9,3,2,0)	1104
(10,4,3,3,3)	134508124418928	(10,4,4,3,2)	42411173392368
(10,4,4,4,1)	3114669545280	(10,5,3,3,2)	13834674726336
(10,5,4,2,2)	4183230238656	(10,5,4,3,1)	980247769056
(10,5,4,4,0)	4392333792	(10,5,5,2,1)	83099778720
(10,5,5,3,0)	1272585120	(10,6,3,2,2)	377080188864
(10,6,3,3,1)	85495746528	(10,6,4,2,1)	22951602432
(10,6,4,3,0)	347078520	(10,6,5,1,1)	282674592
(10,6,5,2,0)	20578560	(10,6,6,1,0)	14496
(10,7,2,2,2)	1599622824	(10,7,3,2,1)	316997280
(10,7,3,3,0)	4326048	(10,7,4,1,1)	10883712
(10,7,4,2,0)	795936	(10,7,5,1,0)	1104
(10,8,2,2,1)	14496	(10,8,3,1,1)	1104
(10,8,3,2,0)	24	(11,3,3,3,3)	23351460864
(11,4,3,3,2)	5950086192	(11,4,4,2,2)	1426637712
(11,4,4,3,1)	282674592	(11,4,4,4,0)	795936

Continued on the following page

deg(I) = 23, *continued*			
I	n_I	I	n_I
(11,5,3,2,2)	316997280	(11,5,3,3,1)	59097600
(11,5,4,2,1)	10883712	(11,5,4,3,0)	122448
(11,5,5,1,1)	19200	(11,5,5,2,0)	1104
(11,6,2,2,2)	1691856	(11,6,3,2,1)	212880
(11,6,3,3,0)	1104	(11,6,4,1,1)	1104
(11,6,4,2,0)	24		

deg(I) = 24			
I	n_I	I	n_I
(5,5,5,5,4)	22958958469178899286112	(6,5,5,4,4)	10565118218002014469248
(6,5,5,5,3)	4556958521329222612288	(6,6,4,4,4)	4831190355131709036288
(6,6,5,4,3)	2073506802039240412736	(6,6,5,5,2)	325919053252224299168
(6,6,6,3,3)	398117530652334602832	(6,6,6,4,2)	145958872012447992432
(6,6,6,5,1)	5571858588504821712	(6,6,6,6,0)	5101035241706976
(7,5,4,4,4)	2234583126440197477248	(7,5,5,4,3)	953357306203227960000
(7,5,5,5,2)	148388879628408287328	(7,6,4,4,3)	426992405813739053760
(7,6,5,3,3)	180227194372605425904	(7,6,5,4,2)	65822715717348500960
(7,6,5,5,1)	2483294974158049312	(7,6,6,3,2)	12017787465197578008
(7,6,6,4,1)	1079565550915867008	(7,6,6,5,0)	2235977596096128
(7,7,4,3,3)	35230204567671156080	(7,7,4,4,2)	12756877670900976952
(7,7,5,3,2)	5251183397613765152	(7,7,5,4,1)	468967501173905952
(7,7,5,5,0)	964399018545152	(7,7,6,2,2)	322136481160659232
(7,7,6,3,1)	80203179581197904	(7,7,6,4,0)	408865565088240
(7,7,7,2,1)	1823829689450016	(7,7,7,3,0)	27765085214112
(8,4,4,4,4)	205889395932163617312	(8,5,4,4,3)	86069413996832124352
(8,5,5,3,3)	35767925041629127584	(8,5,5,4,2)	12945658352928829152
(8,5,5,5,1)	474753531842519136	(8,6,4,3,3)	15439042985968145360
(8,6,4,4,2)	5561099164846002688	(8,6,5,3,2)	2266457718090172160
(8,6,5,4,1)	201010148340707968	(8,6,5,5,0)	408865565088240
(8,6,6,2,2)	134490191192202528	(8,6,6,3,1)	33425310221788864
(8,6,6,4,0)	170193514498560	(8,7,3,3,3)	1126346577851592960
(8,7,4,3,2)	398660145480045856	(8,7,4,4,1)	34807080531177792
(8,7,5,2,2)	55084399995750144	(8,7,5,3,1)	13661798641742976
(8,7,5,4,0)	69468841810240	(8,7,6,2,1)	710229601026304
(8,7,6,3,0)	10848408360480	(8,7,7,1,1)	2760956499680
(8,7,7,2,0)	193411225936	(8,8,3,3,2)	9867511282052976
(8,8,4,2,2)	3345447617629184	(8,8,4,3,1)	822222499846400
(8,8,4,4,0)	4136092740352	(8,8,5,2,1)	99885127862240
(8,8,5,3,0)	1535514818112	(8,8,6,1,1)	927231904320
(8,8,6,2,0)	65215569408	(8,8,7,1,0)	140436672
(8,8,8,0,0)	80	(9,4,4,4,3)	2968386268852263168
(9,5,4,3,3)	1187054464752608224	(9,5,4,4,2)	419478239436537264
(9,5,5,3,2)	165119843412344816	(9,5,5,4,1)	14258867760974432
(9,5,5,5,0)	27765085214112	(9,6,3,3,3)	190193228131870512
(9,6,4,3,2)	66233922634330080	(9,6,4,4,1)	5658979212554128
(9,6,5,2,2)	8718347106041576	(9,6,5,3,1)	2150266975191936
(9,6,5,4,0)	10848408360480	(9,6,6,2,1)	99894446151552
(9,6,6,3,0)	1535514818112	(9,7,3,3,2)	3761948244770304
(9,7,4,2,2)	1259132047619264	(9,7,4,3,1)	308134225628128
(9,7,4,4,0)	1535514818112	(9,7,5,2,1)	35929933424832
(9,7,5,3,0)	553728279360	(9,7,6,1,1)	299302640864
(9,7,6,2,0)	21143067840	(9,7,7,1,0)	33777312
(9,8,3,2,2)	18954386538304	(9,8,3,3,1)	4510722900128
(9,8,4,2,1)	1367836823744	(9,8,4,3,0)	21143067840
(9,8,5,1,1)	27120466144	(9,8,5,2,0)	1935300720

Continued on the following page

deg(I) = 24, *continued*

I	n_I	I	n_I
(9, 8, 6, 1, 0)	7371792	(9, 9, 2, 2, 2)	11032046624
(9, 9, 3, 2, 1)	2322325968	(9, 9, 3, 3, 0)	33777312
(9, 9, 4, 1, 1)	100919904	(9, 9, 4, 2, 0)	7371792
(9, 9, 5, 1, 0)	30624	(10, 4, 4, 3, 3)	11630106886504344
(10, 4, 4, 4, 2)	3920585033699328	(10, 5, 3, 3, 3)	4272828104425920
(10, 5, 4, 3, 2)	1423524718242752	(10, 5, 4, 4, 1)	114110495895360
(10, 5, 5, 2, 2)	164605655104880	(10, 5, 5, 3, 1)	39821013536096
(10, 5, 5, 4, 0)	193411225936	(10, 6, 3, 3, 2)	178677828494464
(10, 6, 4, 2, 2)	56949598227232	(10, 6, 4, 3, 1)	13674852866304
(10, 6, 4, 4, 0)	65215569408	(10, 6, 5, 2, 1)	1367836823744
(10, 6, 5, 3, 0)	21143067840	(10, 6, 6, 1, 1)	7510615200
(10, 6, 6, 2, 0)	539115744	(10, 7, 3, 2, 2)	1912895782008
(10, 7, 3, 3, 1)	443961562528	(10, 7, 4, 2, 1)	126121309632
(10, 7, 4, 3, 0)	1935300720	(10, 7, 5, 1, 1)	1944767152
(10, 7, 5, 2, 0)	140436672	(10, 7, 6, 1, 0)	234048
(10, 8, 2, 2, 2)	2624447520	(10, 8, 3, 2, 1)	529392832
(10, 8, 3, 3, 0)	7371792	(10, 8, 4, 1, 1)	19420400
(10, 8, 4, 2, 0)	1423104	(10, 8, 5, 1, 0)	2800
(10, 9, 2, 2, 1)	2800	(10, 9, 3, 1, 1)	112
(11, 4, 3, 3, 3)	6446376071472	(11, 4, 4, 3, 2)	1912895782008
(11, 4, 4, 4, 1)	126121309632	(11, 5, 3, 3, 2)	570360079168
(11, 5, 4, 2, 2)	158730945984	(11, 5, 4, 3, 1)	35487082592
(11, 5, 4, 4, 0)	140436672	(11, 5, 5, 2, 1)	2306418848
(11, 5, 5, 3, 0)	33777312	(11, 6, 3, 2, 2)	11032046624
(11, 6, 3, 3, 1)	2322325968	(11, 6, 4, 2, 1)	529392832
(11, 6, 4, 3, 0)	7371792	(11, 6, 5, 1, 1)	3222112
(11, 6, 5, 2, 0)	234048	(11, 7, 2, 2, 2)	20299992
(11, 7, 3, 2, 1)	3222112	(11, 7, 3, 3, 0)	30624
(11, 7, 4, 1, 1)	45408	(11, 7, 4, 2, 0)	2800
(12, 3, 3, 3, 3)	33777312	(12, 4, 3, 3, 2)	7371792
(12, 4, 4, 2, 2)	1423104	(12, 4, 4, 3, 1)	234048
(12, 4, 4, 4, 0)	80	(12, 5, 3, 2, 2)	234048
(12, 5, 3, 3, 1)	30624	(12, 5, 4, 2, 1)	2800
(12, 6, 2, 2, 2)	80		

deg(I) = 25

I	n_I	I	n_I
(5, 5, 5, 5, 5)	34771859808804178932864O	(6, 5, 5, 5, 4)	163766423699355653551056
(6, 6, 5, 4, 4)	76746430278444036385392	(6, 6, 5, 5, 3)	33557221088952835248384
(6, 6, 6, 4, 3)	15579801166584314831616	(6, 6, 6, 5, 2)	2507158978553441682912
(6, 6, 6, 6, 1)	45075021198059982144	(7, 5, 5, 4, 4)	36602428260502812573792
(7, 5, 5, 5, 3)	15930480413967177684480	(7, 6, 4, 4, 4)	16950676810888359150336
(7, 6, 5, 4, 3)	7345251761305389562560	(7, 6, 5, 5, 2)	1172715223879828113648
(7, 6, 6, 3, 3)	1445782834458789325920	(7, 6, 6, 4, 2)	533243466879375407808
(7, 6, 6, 5, 1)	20734174826253969312	(7, 6, 6, 6, 0)	19503820669876800
(7, 7, 4, 4, 3)	1560763765722117846528	(7, 7, 5, 3, 3)	666467844013257615360
(7, 7, 5, 4, 2)	245004909605415502560	(7, 7, 5, 5, 1)	9433084896265973760
(7, 7, 6, 3, 2)	46154206945493038080	(7, 7, 6, 4, 1)	4182469007721935136
(7, 7, 6, 5, 0)	8765016259161504	(7, 7, 7, 2, 2)	1329872815417735680
(7, 7, 7, 3, 1)	331877990439469056	(7, 7, 7, 4, 0)	1692511362069504
(8, 5, 4, 4, 4)	3727698169135125498096	(8, 5, 5, 4, 3)	1597193840680108612560
(8, 5, 5, 5, 2)	250363560088071437904	(8, 6, 4, 4, 3)	719772958474534446912
(8, 6, 5, 3, 3)	305275969074406465728	(8, 6, 5, 4, 2)	111802499676622459032
(8, 6, 5, 5, 1)	4254301857683952288	(8, 6, 6, 3, 2)	20678681485694660736
(8, 6, 6, 4, 1)	1864566688856423904	(8, 6, 6, 5, 0)	3881643757375656

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deg(I) = 25, *continued*			
I	n_I	I	n_I
(8,7,4,3,3)	60590920000179493056	(8,7,4,4,2)	22012186784542835520
(8,7,5,3,2)	9117897040377080832	(8,7,5,4,1)	817746667654917168
(8,7,5,5,0)	1692511362069504	(8,7,6,2,2)	571128202199454336
(8,7,6,3,1)	142342287006477504	(8,7,6,4,0)	725912434085952
(8,7,7,2,1)	3377194221012096	(8,7,7,3,0)	51294957112992
(8,8,3,3,3)	2006276928131711424	(8,8,4,3,2)	713471511849776160
(8,8,4,4,1)	62675569121448240	(8,8,5,2,2)	100010833402440120
(8,8,5,3,1)	24840263013168672	(8,8,5,4,0)	126532108859856
(8,8,6,2,1)	1335301022489328	(8,8,6,3,0)	20350993239840
(8,8,7,1,1)	5601159429504	(8,8,7,2,0)	391409808576
(8,8,8,1,0)	347078520	(9,4,4,4,4)	157391952685989052728
(9,5,4,4,3)	65617907601886711296	(9,5,5,3,3)	27190620887571766272
(9,5,5,4,2)	9824161857371476896	(9,5,5,5,1)	358330187751266304
(9,6,4,3,3)	11689593863624674656	(9,6,4,4,2)	4202606608677077184
(9,6,5,3,2)	1706830027589928192	(9,6,5,4,1)	150993571342096992
(9,6,5,5,0)	305922925426848	(9,6,6,2,2)	100108346194477248
(9,6,6,3,1)	24863416450991904	(9,6,6,4,0)	126532108859856
(9,7,3,3,3)	841539378868429824	(9,7,4,3,2)	297115911452589936
(9,7,4,4,1)	25857038420140320	(9,7,5,2,2)	40746789567213888
(9,7,5,3,1)	10097809547695104	(9,7,5,4,0)	51294957112992
(9,7,6,2,1)	515881389602064	(9,7,6,3,0)	7888589144400
(9,7,7,1,1)	1927069671936	(9,7,7,2,0)	135171775392
(9,8,3,3,2)	7173870919736064	(9,8,4,2,2)	2422178398686816
(9,8,4,3,1)	594508678788528	(9,8,4,4,0)	2981800050480
(9,8,5,2,1)	71274491245200	(9,8,5,3,0)	1096632180480
(9,8,6,1,1)	639016897824	(9,8,6,2,0)	45007048752
(9,8,7,1,0)	88179456	(9,8,8,0,0)	24
(9,9,3,2,2)	13073262151968	(9,9,3,3,1)	3100342138368
(9,9,4,2,1)	931163905728	(9,9,4,3,0)	14386869840
(9,9,5,1,1)	17798444544	(9,9,5,2,0)	1272585120
(9,9,6,1,0)	4326048	(10,4,4,4,3)	920246692052672448
(10,5,4,3,3)	362176732991882256	(10,5,4,4,2)	126656377507736616
(10,5,5,3,2)	48949713376347552	(10,5,5,4,1)	4162140562025760
(10,5,5,5,0)	7888589144400	(10,6,3,3,3)	55724768553096576
(10,6,4,3,2)	19159936729163904	(10,6,4,4,1)	1608297381675072
(10,6,5,2,2)	2428815576573408	(10,6,5,3,1)	596073535387056
(10,6,5,4,0)	2981800050480	(10,6,6,2,1)	25377635878296
(10,6,6,3,0)	391409808576	(10,7,3,3,2)	1000740719949936
(10,7,4,2,2)	328447354833120	(10,7,4,3,1)	79804026346992
(10,7,4,4,0)	391409808576	(10,7,5,2,1)	8748592415904
(10,7,5,3,0)	135171775392	(10,7,6,1,1)	61773182400
(10,7,6,2,0)	4392333792	(10,7,7,1,0)	4326048
(10,8,3,2,2)	4183230238656	(10,8,3,3,1)	980247769056
(10,8,4,2,1)	285207114048	(10,8,4,3,0)	4392333792
(10,8,5,1,1)	4826161680	(10,8,5,2,0)	347078520
(10,8,6,1,0)	795936	(10,9,2,2,2)	1599622824
(10,9,3,2,1)	316997280	(10,9,3,3,0)	4326048
(10,9,4,1,1)	10883712	(10,9,4,2,0)	795936
(10,9,5,1,0)	1104	(10,10,2,2,1)	24
(11,4,4,3,3)	1112487680575968	(11,4,4,4,2)	363393804317664
(11,5,3,3,3)	389973010495488	(11,5,4,3,2)	125365423769760
(11,5,4,4,1)	9502910875584	(11,5,5,2,2)	13073262151968
(11,5,5,3,1)	3100342138368	(11,5,5,4,0)	14386869840
(11,6,3,3,2)	13834674726336	(11,6,4,2,2)	4183230238656
(11,6,4,3,1)	980247769056	(11,6,4,4,0)	4392333792
(11,6,5,2,1)	83099778720	(11,6,5,3,0)	1272585120
(11,6,6,1,1)	282674592	(11,6,6,2,0)	20578560

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deg(I) = 25, *continued*

I	n_I	I	n_I
(11, 7, 3, 2, 2)	105371446464	(11, 7, 3, 3, 1)	23351460864
(11, 7, 4, 2, 1)	5950086192	(11, 7, 4, 3, 0)	88179456
(11, 7, 5, 1, 1)	59097600	(11, 7, 5, 2, 0)	4326048
(11, 7, 6, 1, 0)	1104	(11, 8, 2, 2, 2)	63576576
(11, 8, 3, 2, 1)	10883712	(11, 8, 3, 3, 0)	122448
(11, 8, 4, 1, 1)	212880	(11, 8, 4, 2, 0)	14496
(12, 4, 3, 3, 3)	85495746528	(12, 4, 4, 3, 2)	22951602432
(12, 4, 4, 4, 1)	1218252960	(12, 5, 3, 3, 2)	5950086192
(12, 5, 4, 2, 2)	1426637712	(12, 5, 4, 3, 1)	282674592
(12, 5, 4, 4, 0)	795936	(12, 5, 5, 2, 1)	10883712
(12, 5, 5, 3, 0)	122448	(12, 6, 3, 2, 2)	63576576
(12, 6, 3, 3, 1)	10883712	(12, 6, 4, 2, 1)	1691856
(12, 6, 4, 3, 0)	14496	(12, 6, 5, 1, 1)	1104
(12, 6, 5, 2, 0)	24	(12, 7, 2, 2, 2)	14496
(12, 7, 3, 2, 1)	1104		

deg(I) = 26

I	n_I	I	n_I
(6, 5, 5, 5, 5)	2910174233830401416162688	(6, 6, 5, 5, 4)	1393776642755701910391504
(6, 6, 6, 4, 4)	664776534906643820467776	(6, 6, 6, 5, 3)	294433074567120966718080
(6, 6, 6, 6, 2)	22914149837439123291648	(7, 5, 5, 5, 4)	683358195482651060173200
(7, 6, 5, 4, 4)	324239422338282700223616	(7, 6, 5, 5, 3)	143065987059929651882064
(7, 6, 6, 4, 3)	67330722644161744497600	(7, 6, 6, 5, 2)	11001712604766772877568
(7, 6, 6, 6, 1)	204289830851585811840	(7, 7, 4, 4, 4)	73760592549207990341160
(7, 7, 5, 4, 3)	32292308644789613776992	(7, 7, 5, 5, 2)	5240983985031424336512
(7, 7, 6, 3, 3)	6528373563454253739936	(7, 7, 6, 4, 2)	2422666442072438912352
(7, 7, 6, 5, 1)	95977233617823957552	(7, 7, 6, 6, 0)	92700939550359360
(7, 7, 7, 3, 2)	221273208968851435344	(7, 7, 7, 4, 1)	20221514209776438144
(7, 7, 7, 5, 0)	42801528146793216	(8, 5, 5, 4, 4)	76121903698269498879600
(8, 5, 5, 5, 3)	33298242026156998722144	(8, 6, 4, 4, 4)	35504946591945154063104
(8, 6, 5, 4, 3)	15467950978366663032576	(8, 6, 5, 5, 2)	2490930240945503131824
(8, 6, 6, 3, 3)	3087105410543684178144	(8, 6, 6, 4, 2)	1142319760025546317200
(8, 6, 6, 5, 1)	44858935490060258472	(8, 6, 6, 6, 0)	42801528135993600
(8, 7, 4, 4, 3)	3345260435684206623648	(8, 7, 5, 3, 3)	1437664544143525963632
(8, 7, 5, 4, 2)	530380950844598802240	(8, 7, 5, 5, 1)	20644575230653895136
(8, 7, 6, 3, 2)	101657485900434092424	(8, 7, 6, 4, 1)	9254014284061822464
(8, 7, 6, 5, 0)	19503820669876800	(8, 7, 7, 2, 2)	3047920567708923264
(8, 7, 7, 3, 1)	761479183438470384	(8, 7, 7, 4, 0)	3881643757375656
(8, 8, 4, 3, 3)	135135632721772486224	(8, 8, 4, 4, 2)	49318322079952346112
(8, 8, 5, 3, 2)	20610476713078747200	(8, 8, 5, 4, 1)	1859138760210276768
(8, 8, 5, 5, 0)	3881643757375656	(8, 8, 6, 2, 2)	1329629977546611936
(8, 8, 6, 3, 1)	331824982853181696	(8, 8, 6, 4, 0)	1692511359568896
(8, 8, 7, 2, 1)	8359186921934400	(8, 8, 7, 3, 0)	126532108859856
(8, 8, 8, 1, 1)	15746747463456	(8, 8, 8, 2, 0)	1096632086784
(9, 5, 4, 4, 4)	3727698169135125498096	(9, 5, 5, 4, 3)	1597193840680108612560
(9, 5, 5, 5, 2)	250363560088071437904	(9, 6, 4, 4, 3)	719772958474534446912
(9, 6, 5, 3, 3)	305275969074406465728	(9, 6, 5, 4, 2)	111802499676622459032
(9, 6, 5, 5, 1)	4254301857683952288	(9, 6, 6, 3, 2)	20678681485694660736
(9, 6, 6, 4, 1)	1864566688856423904	(9, 6, 6, 5, 0)	3881643757375656
(9, 7, 4, 3, 3)	60590920000179493056	(9, 7, 4, 4, 2)	22012186784542835520
(9, 7, 5, 3, 2)	9117897040377080832	(9, 7, 5, 4, 1)	817746667654917168
(9, 7, 5, 5, 0)	1692511362069504	(9, 7, 6, 2, 2)	571128202199454336
(9, 7, 6, 3, 1)	142342287006477504	(9, 7, 6, 4, 0)	725912434085952
(9, 7, 7, 2, 1)	3377194221012096	(9, 7, 7, 3, 0)	51294957112992
(9, 8, 3, 3, 3)	2006276928131711424	(9, 8, 4, 3, 2)	713471511849776160

Continued on the following page

$\deg(I) = 26$, *continued*			
I	n_I	I	n_I
(9,8,4,4,1)	62675569121448240	(9,8,5,2,2)	100010833402440120
(9,8,5,3,1)	24840263013168672	(9,8,5,4,0)	126532108859856
(9,8,6,2,1)	1335301022489328	(9,8,6,3,0)	20350993239840
(9,8,7,1,1)	5601159429504	(9,8,7,2,0)	391409808576
(9,8,8,1,0)	347078520	(9,9,3,3,2)	7173870919736064
(9,9,4,2,2)	2422178398686816	(9,9,4,3,1)	594508678788528
(9,9,4,4,0)	2981800050480	(9,9,5,2,1)	71274491245200
(9,9,5,3,0)	1096632180480	(9,9,6,1,1)	639016897824
(9,9,6,2,0)	45007048752	(9,9,7,1,0)	88179456
(9,9,8,0,0)	24	(10,4,4,4,4)	69702170473826178048
(10,5,4,4,3)	28814753795787304128	(10,5,5,3,3)	11833136668383611040
(10,5,5,4,2)	4252005327651223776	(10,5,5,5,1)	152435152838866176
(10,6,4,3,3)	5023740750844977792	(10,6,4,4,2)	1795314514514344416
(10,6,5,3,2)	721163569257189312	(10,6,5,4,1)	63276065657309280
(10,6,5,5,0)	126532108859856	(10,6,6,2,2)	40767562975883520
(10,6,6,3,1)	10102374952223232	(10,6,6,4,0)	51294956593632
(10,7,3,3,3)	346896207708697296	(10,7,4,3,2)	121505012171479176
(10,7,4,4,1)	10462960782869952	(10,7,5,2,2)	16270300857476160
(10,7,5,3,1)	4021264698687264	(10,7,5,4,0)	20350993239840
(10,7,6,2,1)	194378107421760	(10,7,6,3,0)	2981800050480
(10,7,7,1,1)	639016897824	(10,7,7,2,0)	45007048752
(10,8,3,3,2)	2713101057421728	(10,8,4,2,2)	903893653068672
(10,8,4,3,1)	220840621188096	(10,8,4,4,0)	1096632086784
(10,8,5,2,1)	25377635878296	(10,8,5,3,0)	391409808576
(10,8,6,1,1)	203336907216	(10,8,6,2,0)	14386855920
(10,8,7,1,0)	20578560	(10,9,3,2,2)	4183230238656
(10,9,3,3,1)	980247769056	(10,9,4,2,1)	285207114048
(10,9,4,3,0)	4392333792	(10,9,5,1,1)	4826161680
(10,9,5,2,0)	347078520	(10,9,6,1,0)	795936
(10,10,2,2,2)	341681280	(10,10,3,2,1)	63576576
(10,10,3,3,0)	795936	(10,10,4,1,1)	1691856
(10,10,4,2,0)	122352	(10,10,5,1,0)	24
(11,4,4,4,3)	149583407202367176	(11,5,4,3,3)	57309129620711136
(11,5,4,4,2)	19680157760407104	(11,5,5,3,2)	7371081117191712
(11,5,5,4,1)	609209937409968	(11,5,5,5,0)	1096632180480
(11,6,3,3,3)	8236673292611808	(11,6,4,3,2)	2768640614245200
(11,6,4,4,1)	224917616990784	(11,6,5,2,2)	328447354833120
(11,6,5,3,1)	79804026346992	(11,6,5,4,0)	391409808576
(11,6,6,2,1)	2910089695872	(11,6,6,3,0)	45007048752
(11,7,3,3,2)	125365423769760	(11,7,4,2,2)	39692266181304
(11,7,4,3,1)	9502910875584	(11,7,4,4,0)	45007048752
(11,7,5,2,1)	931163905728	(11,7,5,3,0)	14386869840
(11,7,6,1,1)	4826161680	(11,7,6,2,0)	347078520
(11,7,7,1,0)	122448	(11,8,3,2,2)	377080188864
(11,8,3,3,1)	85495746528	(11,8,4,2,1)	22951602432
(11,8,4,3,0)	347078520	(11,8,5,1,1)	282674592
(11,8,5,2,0)	20578560	(11,8,6,1,0)	14496
(11,9,2,2,2)	63576576	(11,9,3,2,1)	10883712
(11,9,3,3,0)	122448	(11,9,4,1,1)	212880
(11,9,4,2,0)	14496	(12,4,4,3,3)	42411173392368
(12,4,4,4,2)	13138629854976	(12,5,3,3,3)	13834674726336
(12,5,4,3,2)	4183230238656	(12,5,4,4,1)	285207114048
(12,5,5,2,2)	366406656528	(12,5,5,3,1)	83099778720
(12,5,5,4,0)	347078520	(12,6,3,3,2)	377080188864
(12,6,4,2,2)	103492041648	(12,6,4,3,1)	22951602432
(12,6,4,4,0)	88177920	(12,6,5,2,1)	1426637712
(12,6,5,3,0)	20578560	(12,6,6,1,1)	1691856

Continued on the following page

deg(I) = 26, *continued*			
I	n_I	I	n_I
(12, 6, 6, 2, 0)	122352	(12, 7, 3, 2, 2)	1599622824
(12, 7, 3, 3, 1)	316997280	(12, 7, 4, 2, 1)	63576576
(12, 7, 4, 3, 0)	795936	(12, 7, 5, 1, 1)	212880
(12, 7, 5, 2, 0)	14496	(12, 8, 2, 2, 2)	122352
(12, 8, 3, 2, 1)	14496	(12, 8, 3, 3, 0)	24
(12, 8, 4, 1, 1)	24	(13, 4, 3, 3, 3)	88179456
(13, 4, 4, 3, 2)	20578560	(13, 4, 4, 4, 1)	795936
(13, 5, 3, 3, 2)	4326048	(13, 5, 4, 2, 2)	795936
(13, 5, 4, 3, 1)	122448	(13, 5, 4, 4, 0)	24
(13, 5, 5, 2, 1)	1104	(13, 6, 3, 2, 2)	14496
(13, 6, 3, 3, 1)	1104	(13, 6, 4, 2, 1)	24

deg(I) = 27			
I	n_I	I	n_I
(6, 6, 5, 5, 5)	28656849112544426796718608	(6, 6, 6, 5, 4)	13944411721206459640109952
(6, 6, 6, 6, 3)	3030705830464261116958752	(7, 5, 5, 5, 5)	14401495635309838652737536
(7, 6, 5, 5, 4)	6979017914791123565948416	(7, 6, 6, 4, 4)	3369981367793558156370720
(7, 6, 6, 5, 3)	1505729217469676504230592	(7, 6, 6, 6, 2)	120468636234042768002112
(7, 7, 5, 4, 4)	1670002775792759585584480	(7, 7, 5, 5, 3)	743754012075104160058368
(7, 7, 6, 4, 3)	355021932713338577724288	(7, 7, 6, 5, 2)	58904390293772710703920
(7, 7, 6, 6, 1)	1129438555365292906784	(7, 7, 7, 3, 3)	36027026438881932128256
(7, 7, 7, 4, 2)	13449577349429667122112	(7, 7, 7, 5, 1)	542448112625738749440
(7, 7, 7, 6, 0)	536474722655969280	(8, 5, 5, 5, 4)	1737004389084229283537184
(8, 6, 5, 4, 4)	830547124981920578846912	(8, 6, 5, 5, 3)	368512581418692060268192
(8, 6, 6, 4, 3)	174894056440451268167776	(8, 6, 6, 5, 2)	28841649917004887522080
(8, 6, 6, 6, 1)	546003716872720702848	(8, 7, 4, 4, 4)	192443720708788001680608
(8, 7, 5, 4, 3)	84779574976667980526368	(8, 7, 5, 5, 2)	13895992208920106396544
(8, 7, 6, 3, 3)	17421547466353577240096	(8, 7, 6, 4, 2)	6488505552963384035984
(8, 7, 6, 5, 1)	259867472483449630240	(8, 7, 6, 6, 0)	254791938658803840
(8, 7, 7, 3, 2)	612363334253114849568	(8, 7, 7, 4, 1)	56225444916409818816
(8, 7, 7, 5, 0)	119602242975339008	(8, 8, 4, 4, 3)	9090329485714422369312
(8, 8, 5, 3, 3)	3937816074403396325984	(8, 8, 5, 4, 2)	1458934381540619175680
(8, 8, 5, 5, 1)	57539715512784775920	(8, 8, 6, 3, 2)	285713812377367900976
(8, 8, 6, 4, 1)	26146648523244093888	(8, 8, 6, 5, 0)	55456767284050560
(8, 8, 7, 2, 2)	8999069221638485760	(8, 8, 7, 3, 1)	2251004857405139040
(8, 8, 7, 4, 0)	11461248223336400	(8, 8, 8, 2, 1)	27132798580132512
(8, 8, 8, 3, 0)	408865565088240	(9, 5, 5, 4, 4)	96959783616963943030208
(9, 5, 5, 5, 3)	42482450279496628079616	(9, 6, 4, 4, 4)	45328721421594382226880
(9, 6, 5, 4, 3)	19781742107681508906368	(9, 6, 5, 5, 2)	3194412266903770307840
(9, 6, 6, 3, 3)	3965695431311102575520	(9, 6, 6, 4, 2)	1468942204326701835456
(9, 6, 6, 5, 1)	57866098737278532384	(9, 6, 6, 6, 0)	55456767284050560
(9, 7, 4, 4, 3)	4302653199079323937920	(9, 7, 5, 3, 3)	1852914051293922601984
(9, 7, 5, 4, 2)	684339544738107833984	(9, 7, 5, 5, 1)	26729153888914424832
(9, 7, 6, 3, 2)	131895336423525168320	(9, 7, 6, 4, 1)	12023512502480326944
(9, 7, 6, 5, 0)	25383601647232320	(9, 7, 7, 2, 2)	4005072998835530048
(9, 7, 7, 3, 1)	1000944581364139008	(9, 7, 7, 4, 0)	5101035246140064
(9, 8, 4, 3, 3)	176048749037882981792	(9, 8, 4, 4, 2)	64342058578990962432
(9, 8, 5, 3, 2)	26964980823333578944	(9, 8, 5, 4, 1)	2436667741602374528
(9, 8, 5, 5, 0)	5101035246140064	(9, 8, 6, 2, 2)	1755969430220426496
(9, 8, 6, 3, 1)	438395891787545984	(9, 8, 6, 4, 0)	2235977596096128
(9, 8, 7, 2, 1)	11256418126498304	(9, 8, 7, 3, 0)	170193515484672
(9, 8, 8, 1, 1)	22072657897776	(9, 8, 8, 2, 0)	1535514818112
(9, 9, 3, 3, 3)	2670197361402514944	(9, 9, 4, 3, 2)	951716992566363648
(9, 9, 4, 4, 1)	83845604993916000	(9, 9, 5, 2, 2)	134329574207275104
(9, 9, 5, 3, 1)	33386278590988800	(9, 9, 5, 4, 0)	170193515484672

Continued on the following page

deg(I) = 27, *continued*			
I	n_I	I	n_I
(9,9,6,2,1)	1823776449179136	(9,9,6,3,0)	27765085214112
(9,9,7,1,1)	7933211814912	(9,9,7,2,0)	553728279360
(9,9,8,1,0)	539120544	(9,9,9,0,0)	112
(10,5,4,4,4)	2234583126440197477248	(10,5,5,4,3)	9533573062032279600000
(10,5,5,5,2)	1483888796284082873 28	(10,6,4,4,3)	4269924058137390537 60
(10,6,5,3,3)	180227194372605425904	(10,6,5,4,2)	65822715717348500960
(10,6,5,5,1)	2483294974158049312	(10,6,6,3,2)	12017787465197578008
(10,6,6,4,1)	1079565550915867008	(10,6,6,5,0)	2235977596096128
(10,7,4,3,3)	35230204567671156080	(10,7,4,4,2)	12756877670900976952
(10,7,5,3,2)	5251183397613765152	(10,7,5,4,1)	468967501173905952
(10,7,5,5,0)	964399018545152	(10,7,6,2,2)	322136481160659232
(10,7,6,3,1)	80203179581197904	(10,7,6,4,0)	408865565088240
(10,7,7,2,1)	1823829689450016	(10,7,7,3,0)	27765085214112
(10,8,3,3,3)	1126346577851592960	(10,8,4,3,2)	398660145480045856
(10,8,4,4,1)	34807080531177792	(10,8,5,2,2)	55084399995750144
(10,8,5,3,1)	13661798641742976	(10,8,5,4,0)	69468841810240
(10,8,6,2,1)	710229601026304	(10,8,6,3,0)	10848408360480
(10,8,7,1,1)	2760956499680	(10,8,7,2,0)	193411225936
(10,8,8,1,0)	140436672	(10,9,3,3,2)	3761948244770304
(10,9,4,2,2)	1259132047619264	(10,9,4,3,1)	308134225628128
(10,9,4,4,0)	1535514818112	(10,9,5,2,1)	35929933424832
(10,9,5,3,0)	553728279360	(10,9,6,1,1)	299302640864
(10,9,6,2,0)	21143067840	(10,9,7,1,0)	33777312
(10,10,3,2,2)	1912895782008	(10,10,3,3,1)	443961562528
(10,10,4,2,1)	126121309632	(10,10,4,3,0)	1935300720
(10,10,5,1,1)	1944767152	(10,10,5,2,0)	140436672
(10,10,6,1,0)	234048	(11,4,4,4,4)	17389206433621316832
(11,5,4,4,3)	70795671011094365 12	(11,5,5,3,3)	28600722896274447 36
(11,5,5,4,2)	1017289744237857120	(11,5,5,5,1)	35306571598392576
(11,6,4,3,3)	1187054464752608224	(11,6,4,4,2)	419478239436537264
(11,6,5,3,2)	165119843412344816	(11,6,5,4,1)	14258867760974432
(11,6,5,5,0)	27765085214112	(11,6,6,2,2)	8718347106041576
(11,6,6,3,1)	2150266975191936	(11,6,6,4,0)	10848408360480
(11,7,3,3,3)	75992812385562624	(11,7,4,3,2)	26217346711258048
(11,7,4,4,1)	2211223893638272	(11,7,5,2,2)	3356453655323136
(11,7,5,3,1)	824874647838720	(11,7,5,4,0)	4136092936448
(11,7,6,2,1)	35929933424832	(11,7,6,3,0)	553728279360
(11,7,7,1,1)	92396257280	(11,7,7,2,0)	6558863360
(11,8,3,3,2)	507096396665312	(11,8,4,2,2)	164605655104880
(11,8,4,3,1)	39821013536096	(11,8,4,4,0)	193411225936
(11,8,5,2,1)	4217701870608	(11,8,5,3,0)	65215603200
(11,8,6,1,1)	27120466144	(11,8,6,2,0)	1935300720
(11,8,7,1,0)	1423616	(11,9,3,2,2)	570360079168
(11,9,3,3,1)	130194945024	(11,9,4,2,1)	35487082592
(11,9,4,3,0)	539120544	(11,9,5,1,1)	464696832
(11,9,5,2,0)	33777312	(11,9,6,1,0)	30624
(11,10,2,2,2)	20299992	(11,10,3,2,1)	3222112
(11,10,3,3,0)	30624	(11,10,4,1,1)	45408
(11,10,4,2,0)	2800	(12,4,4,4,3)	11630106886504344
(12,5,4,3,3)	4272828104425920	(12,5,4,4,2)	1423524718242752
(12,5,5,3,2)	507096396665312	(12,5,5,4,1)	39821013536096
(12,5,5,5,0)	65215603200	(12,6,3,3,3)	552486590320032
(12,6,4,3,2)	178677828494464	(12,6,4,4,1)	13674852866304
(12,6,5,2,2)	18954386538304	(12,6,5,3,1)	4510722900128
(12,6,5,4,0)	21143067840	(12,6,6,2,1)	125948336640
(12,6,6,3,0)	1935300720	(12,7,3,3,2)	6446376071472
(12,7,4,2,2)	1912895782008	(12,7,4,3,1)	443961562528

Continued on the following page

deg(I) = 27, *continued*			
I	n_I	I	n_I
(12, 7, 4, 4, 0)	1935300720	(12, 7, 5, 2, 1)	35487082592
(12, 7, 5, 3, 0)	539120544	(12, 7, 6, 1, 1)	100919904
(12, 7, 6, 2, 0)	7371792	(12, 7, 7, 1, 0)	112
(12, 8, 3, 2, 2)	11032046624	(12, 8, 3, 3, 1)	2322325968
(12, 8, 4, 2, 1)	529392832	(12, 8, 4, 3, 0)	7371792
(12, 8, 5, 1, 1)	3222112	(12, 8, 5, 2, 0)	234048
(12, 9, 2, 2, 2)	234048	(12, 9, 3, 2, 1)	30624
(12, 9, 3, 3, 0)	112	(12, 9, 4, 1, 1)	112
(13, 4, 4, 3, 3)	443961562528	(13, 4, 4, 4, 2)	126121309632
(13, 5, 3, 3, 3)	130194945024	(13, 5, 4, 3, 2)	35487082592
(13, 5, 4, 4, 1)	1944767152	(13, 5, 5, 2, 2)	2306418848
(13, 5, 5, 3, 1)	464696832	(13, 5, 5, 4, 0)	1423616
(13, 6, 3, 3, 2)	2322325968	(13, 6, 4, 2, 2)	529392832
(13, 6, 4, 3, 1)	100919904	(13, 6, 4, 4, 0)	234048
(13, 6, 5, 2, 1)	3222112	(13, 6, 5, 3, 0)	30624
(13, 6, 6, 1, 1)	112	(13, 7, 3, 2, 2)	3222112
(13, 7, 3, 3, 1)	434688	(13, 7, 4, 2, 1)	45408
(13, 7, 4, 3, 0)	112		

deg(I) = 28			
I	n_I	I	n_I
(6, 6, 6, 5, 5)	3276846143873492999617387 68	(6, 6, 6, 6, 4)	1618236596168278920429466 56
(7, 6, 5, 5, 5)	1677836149066687612627167 84	(7, 6, 6, 5, 4)	8257904851047493278406012 8
(7, 6, 6, 6, 3)	1831206325001678542633245 6	(7, 7, 5, 5, 4)	4195299640711857910073251 2
(7, 7, 6, 4, 4)	2051235980277893410598282 4	(7, 7, 6, 5, 3)	9244971964576432583359680
(7, 7, 6, 6, 2)	7600895383068264317489 76	(7, 7, 7, 4, 3)	2248511154703665686900736
(7, 7, 7, 5, 2)	3786079520637243215323 20	(7, 7, 7, 6, 1)	748386373910305238486 4
(7, 7, 7, 7, 0)	3704581973944705776	(8, 5, 5, 5, 5)	4398122339157802802576731 2
(8, 6, 5, 5, 4)	2148006542468236992433060 8	(8, 6, 6, 4, 4)	1045675654920590535945830 4
(8, 6, 6, 5, 3)	4698818465684165361643776	(8, 6, 6, 6, 2)	382713191877285462148512
(8, 7, 5, 4, 4)	5236870677368742358091328	(8, 7, 5, 5, 3)	2346380263735039996887072
(8, 7, 6, 4, 3)	1130432594965350429319440	(8, 7, 6, 5, 2)	189389511540055226332992
(8, 7, 6, 6, 1)	37054220298001821884 16	(8, 7, 7, 3, 3)	118167437729899245359856
(8, 7, 7, 4, 2)	442767943100537804378 40	(8, 7, 7, 5, 1)	18053982905924607699 84
(8, 7, 7, 6, 0)	1810611871504105272	(8, 8, 4, 4, 4)	624198973773146079716784
(8, 8, 5, 4, 3)	2769575611407408176720 64	(8, 8, 5, 5, 2)	45903620873883847141776
(8, 8, 6, 3, 3)	579923346501905565704 88	(8, 8, 6, 4, 2)	21685078061596939766784
(8, 8, 6, 5, 1)	878968309413423252864	(8, 8, 6, 6, 0)	875827020273329664
(8, 8, 7, 3, 2)	212539570687167817641 6	(8, 8, 7, 4, 1)	1961139491827150521 60
(8, 8, 7, 5, 0)	419093788958668992	(8, 8, 8, 2, 2)	336417946109658624 00
(8, 8, 8, 3, 1)	8424735061429730304	(8, 8, 8, 4, 0)	42801528135993600
(9, 5, 5, 5, 4)	2754129399126265116578688	(9, 6, 5, 4, 4)	1321756163879149610446896
(9, 6, 5, 5, 3)	588017185467849919723392	(9, 6, 6, 4, 3)	280190865205345515202656
(9, 6, 6, 5, 2)	46407141063611686053552	(9, 6, 6, 6, 1)	886526887354737645072
(9, 7, 4, 4, 4)	308970407090480886905664	(9, 7, 5, 4, 3)	136517982401519160854256
(9, 7, 5, 5, 2)	22480410038381810846784	(9, 7, 6, 3, 3)	28271565861485256037152
(9, 7, 6, 4, 2)	10547216287172185866240	(9, 7, 6, 5, 1)	424552799703766464912
(9, 7, 6, 6, 0)	419093788958668992	(9, 7, 7, 3, 2)	1010995377658903433280
(9, 7, 7, 4, 1)	93022849378461001968	(9, 7, 7, 5, 0)	198280729061595552
(9, 8, 4, 4, 3)	14880953729756521482240	(9, 8, 5, 3, 3)	6470374366380782830464
(9, 8, 5, 4, 2)	2401953482064409297584	(9, 8, 5, 5, 1)	95308266595738550640
(9, 8, 6, 3, 2)	475191508986206197632	(9, 8, 6, 4, 1)	43589536992230477208
(9, 8, 6, 5, 0)	92700939550359360	(9, 8, 7, 2, 2)	15318198529776992064
(9, 8, 7, 3, 1)	3833600055583272480	(9, 8, 7, 4, 0)	195038206698768 00
(9, 8, 8, 2, 1)	48279338403693048	(9, 8, 8, 3, 0)	725912434085952

Continued on the following page

deg(I) = 28, *continued*

I	n_I	I	n_I
(9,9,4,3,3)	29752032637462684 6896	(9,9,4,4,2)	10903643930469194 8608
(9,9,5,3,2)	4594593158546907 2928	(9,9,5,4,1)	4165834565805729216
(9,9,5,5,0)	8765016259161504	(9,9,6,2,2)	3046779784226901072
(9,9,6,3,1)	761207182511922096	(9,9,6,4,0)	3881643757375656
(9,9,7,2,1)	20278800720533664	(9,9,7,3,0)	305922925426848
(9,9,8,1,1)	42951164308896	(9,9,8,2,0)	2981800050480
(9,9,9,1,0)	1272585120	(10,5,5,4,4)	7612190369826949 8879600
(10,5,5,5,3)	3329824202615699 8722144	(10,6,4,4,4)	3550494659194515 4063104
(10,6,5,4,3)	1546795097836666 3032576	(10,6,5,5,2)	2490930240945503131824
(10,6,6,3,3)	308710541054368 4178144	(10,6,6,4,2)	1142319760025546317200
(10,6,6,5,1)	44858935490060258472	(10,6,6,6,0)	42801528135993600
(10,7,4,4,3)	3345260435684206623648	(10,7,5,3,3)	1437664544143525963632
(10,7,5,4,2)	5303809508445988 02240	(10,7,5,5,1)	20644575230653895136
(10,7,6,3,2)	1016574859004340 92424	(10,7,6,4,1)	9254014284061822464
(10,7,6,5,0)	19503820669876800	(10,7,7,2,2)	3047920567708923264
(10,7,7,3,1)	761479183438470384	(10,7,7,4,0)	3881643757375656
(10,8,4,3,3)	1351356327217724 86224	(10,8,4,4,2)	493183220799523 46112
(10,8,5,3,2)	206104767130787 47200	(10,8,5,4,1)	1859138760210276768
(10,8,5,5,0)	3881643757375656	(10,8,6,2,2)	1329629977546611936
(10,8,6,3,1)	331824982853181696	(10,8,6,4,0)	1692511359568896
(10,8,7,2,1)	8359186921934400	(10,8,7,3,0)	126532108859856
(10,8,8,1,1)	15746747463456	(10,8,8,2,0)	1096632086784
(10,9,3,3,3)	2006276928131711424	(10,9,4,3,2)	713471511849776160
(10,9,4,4,1)	62675569121448240	(10,9,5,2,2)	100010833402440120
(10,9,5,3,1)	24840263013168672	(10,9,5,4,0)	126532108859856
(10,9,6,2,1)	1335301022489328	(10,9,6,3,0)	20350993239840
(10,9,7,1,1)	5601159429504	(10,9,7,2,0)	391409808576
(10,9,8,1,0)	347078520	(10,9,9,0,0)	24
(10,10,3,3,2)	2713101057421728	(10,10,4,2,2)	903893653068672
(10,10,4,3,1)	220840621188096	(10,10,4,4,0)	1096632086784
(10,10,5,2,1)	25377635878296	(10,10,5,3,0)	391409808576
(10,10,6,1,1)	203336907216	(10,10,6,2,0)	14386855920
(10,10,7,1,0)	20578560	(11,5,4,4,4)	7908639044437235 69376
(11,5,5,4,3)	3343708383918102 48432	(11,5,5,5,2)	512657796650180 65536
(11,6,4,4,3)	1478280497379971 20632	(11,6,5,3,3)	617575399438583 80704
(11,6,5,4,2)	2242049697702199 9680	(11,6,5,5,1)	830212985215356432
(11,6,6,3,2)	3982038442779651984	(11,6,6,4,1)	354725791310991552
(11,6,6,5,0)	725912434085952	(11,7,4,3,3)	1168959386362467 4656
(11,7,4,4,2)	4202606608677077184	(11,7,5,3,2)	1706830027589928192
(11,7,5,4,1)	150993571342096992	(11,7,5,5,0)	305922925426848
(11,7,6,2,2)	100108346194477248	(11,7,6,3,1)	24863416450991904
(11,7,6,4,0)	126532108859856	(11,7,7,2,1)	515881389602064
(11,7,7,3,0)	7888589144400	(11,8,3,3,3)	346896207708697296
(11,8,4,3,2)	121505012171479176	(11,8,4,4,1)	10462960782869952
(11,8,5,2,2)	16270300857476160	(11,8,5,3,1)	4021264698687264
(11,8,5,4,0)	20350993239840	(11,8,6,2,1)	194378107421760
(11,8,6,3,0)	2981800050480	(11,8,7,1,1)	639016897824
(11,8,7,2,0)	45007048752	(11,8,8,1,0)	20578560
(11,9,3,3,2)	1000740719949936	(11,9,4,2,2)	328447354833120
(11,9,4,3,1)	79804026346992	(11,9,4,4,0)	391409808576
(11,9,5,2,1)	8748592415904	(11,9,5,3,0)	135171775392
(11,9,6,1,1)	61773182400	(11,9,6,2,0)	4392333792
(11,9,7,1,0)	4326048	(11,10,3,2,2)	377080188864
(11,10,3,3,1)	85495746528	(11,10,4,2,1)	22951602432
(11,10,4,3,0)	347078520	(11,10,5,1,1)	282674592
(11,10,5,2,0)	20578560	(11,10,6,1,0)	14496
(11,11,2,2,2)	1691856	(11,11,3,2,1)	212880

Continued on the following page

$\deg(I) = 28$, *continued*			
I	n_I	I	n_I
(11, 11, 3, 3, 0)	1104	(11, 11, 4, 1, 1)	1104
(11, 11, 4, 2, 0)	24	(12, 4, 4, 4, 4)	2315758601706011520
(12, 5, 4, 4, 3)	920246692052672448	(12, 5, 5, 3, 3)	362176732991882256
(12, 5, 5, 4, 2)	126656377507736616	(12, 5, 5, 5, 1)	4162140562025760
(12, 6, 4, 3, 3)	145074948270672288	(12, 6, 4, 4, 2)	50310287851264512
(12, 6, 5, 3, 2)	19159936729163904	(12, 6, 5, 4, 1)	1608297381675072
(12, 6, 5, 5, 0)	2981800050480	(12, 6, 6, 2, 2)	903893653068672
(12, 6, 6, 3, 1)	220840621188096	(12, 6, 6, 4, 0)	1096632086784
(12, 7, 3, 3, 3)	8236673292611808	(12, 7, 4, 3, 2)	2768640614245200
(12, 7, 4, 4, 1)	224917616990784	(12, 7, 5, 2, 2)	328447354833120
(12, 7, 5, 3, 1)	79804026346992	(12, 7, 5, 4, 0)	391409808576
(12, 7, 6, 2, 1)	2910089695872	(12, 7, 6, 3, 0)	45007048752
(12, 7, 7, 1, 1)	4826161680	(12, 7, 7, 2, 0)	347078520
(12, 8, 3, 3, 2)	42411173392368	(12, 8, 4, 2, 2)	13138629854976
(12, 8, 4, 3, 1)	3114669545280	(12, 8, 4, 4, 0)	14386855920
(12, 8, 5, 2, 1)	285207114048	(12, 8, 5, 3, 0)	4392333792
(12, 8, 6, 1, 1)	1218252960	(12, 8, 6, 2, 0)	88177920
(12, 8, 7, 1, 0)	14496	(12, 9, 3, 2, 2)	27607031136
(12, 9, 3, 3, 1)	5950086192	(12, 9, 4, 2, 1)	1426637712
(12, 9, 4, 3, 0)	20578560	(12, 9, 5, 1, 1)	10883712
(12, 9, 5, 2, 0)	795936	(12, 9, 6, 1, 0)	24
(12, 10, 2, 2, 2)	122352	(12, 10, 3, 2, 1)	14496
(12, 10, 3, 3, 0)	24	(12, 10, 4, 1, 1)	24
(13, 4, 4, 4, 3)	363393804317664	(13, 5, 4, 3, 3)	125365423769760
(13, 5, 4, 4, 2)	39692266181304	(13, 5, 5, 3, 2)	13073262151968
(13, 5, 5, 4, 1)	931163905728	(13, 5, 5, 5, 0)	1272585120
(13, 6, 3, 3, 3)	13834674726336	(13, 6, 4, 3, 2)	4183230238656
(13, 6, 4, 4, 1)	285207114048	(13, 6, 5, 2, 2)	366406656528
(13, 6, 5, 3, 1)	83099778720	(13, 6, 5, 4, 0)	347078520
(13, 6, 6, 2, 1)	1426637712	(13, 6, 6, 3, 0)	20578560
(13, 7, 3, 3, 2)	105371446464	(13, 7, 4, 2, 2)	27607031136
(13, 7, 4, 3, 1)	5950086192	(13, 7, 4, 4, 0)	20578560
(13, 7, 5, 2, 1)	316997280	(13, 7, 5, 3, 0)	4326048
(13, 7, 6, 1, 1)	212880	(13, 7, 6, 2, 0)	14496
(13, 8, 3, 2, 2)	63576576	(13, 8, 3, 3, 1)	10883712
(13, 8, 4, 2, 1)	1691856	(13, 8, 4, 3, 0)	14496
(13, 8, 5, 1, 1)	1104	(13, 8, 5, 2, 0)	24
(14, 4, 4, 3, 3)	347078520	(14, 4, 4, 4, 2)	88177920
(14, 5, 3, 3, 3)	88179456	(14, 5, 4, 3, 2)	20578560
(14, 5, 4, 4, 1)	795936	(14, 5, 5, 2, 2)	795936
(14, 5, 5, 3, 1)	122448	(14, 5, 5, 4, 0)	24
(14, 6, 3, 3, 2)	795936	(14, 6, 4, 2, 2)	122352
(14, 6, 4, 3, 1)	14496	(14, 6, 5, 2, 1)	24
(14, 7, 3, 2, 2)	24		

$\deg(I) = 29$			
I	n_I	I	n_I
(6, 6, 6, 6, 5)	430077972107415124148088 4704	(7, 6, 6, 5, 5)	224081258977589558384415 6576
(7, 6, 6, 6, 4)	111859226244720841949470 0224	(7, 7, 5, 5, 5)	116339325247183686886178 6112
(7, 7, 6, 5, 4)	579083249672629145095030368	(7, 7, 6, 6, 3)	130956575256689626362575040
(7, 7, 7, 4, 4)	147818363038488716192722368	(7, 7, 7, 5, 3)	67179875933573491930042368
(7, 7, 7, 6, 2)	5668780293104872727438208	(7, 7, 7, 7, 1)	58444196223515692468224
(8, 6, 5, 5, 5)	611210312904590985126412992	(8, 6, 6, 5, 4)	303193957695295574167972368
(8, 6, 6, 6, 3)	68161114972295875241706048	(8, 7, 5, 5, 4)	155642883145653607449683376
(8, 7, 6, 4, 4)	76752707872006994882755104	(8, 7, 6, 5, 3)	34795170486545493534538224

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$\deg(I) = 29$, *continued*			
I	n_I	I	n_I
(8,7,6,6,2)	2913373653270128828358720	(8,7,7,4,3)	8644490285282936739547776
(8,7,7,5,2)	1469714544745529210370240	(8,7,7,6,1)	29632320884127789040512
(8,7,7,7,0)	15060587910821007264	(8,8,5,4,4)	20081002987269237834527544
(8,8,5,5,3)	9056138605664527122727872	(8,8,6,4,3)	4407669387277685669560512
(8,8,6,5,2)	746119098679547758175328	(8,8,6,6,1)	14913192160928967502416
(8,8,7,3,3)	475983733946283470184480	(8,8,7,4,2)	179029959278599331183904
(8,8,7,5,1)	7382355219180836928000	(8,8,7,6,0)	7505020393627384992
(8,8,8,3,2)	9106787701513392933312	(8,8,8,4,1)	844343421475958442384
(8,8,8,5,0)	1810611871504105272	(9,5,5,5,5)	85113274584443674676815872
(9,6,5,5,4)	41753519869402757126310048	(9,6,6,4,4)	20420265705357860466725736
(9,6,6,5,3)	9205492921536158089178400	(9,6,6,6,2)	757338998385448742259408
(9,7,5,4,4)	10288587378544353628959936	(9,7,5,5,3)	4625435952934481006642688
(9,7,6,4,3)	2240131001946976373134464	(9,7,6,5,2)	377337473299364349892128
(9,7,6,6,1)	7465381920406413655872	(9,7,7,3,3)	238119849054264183598080
(9,7,7,4,2)	89401881734473077280272	(9,7,7,5,1)	3667019460175374709248
(9,7,7,6,0)	3704581973944705776	(9,8,4,4,4)	1250148843664649970662352
(9,8,5,4,3)	556910170480793670482400	(9,8,5,5,2)	92873867581256305971840
(9,8,6,3,3)	117846752322559629255984	(9,8,6,4,2)	44162182140827544636144
(9,8,6,5,1)	1801675431551746250880	(9,8,6,6,0)	1810611871504105272
(9,8,7,3,2)	4421119882261082749248	(9,8,7,4,1)	408994002687679224816
(9,8,7,5,0)	875827020316064256	(9,8,8,2,2)	72939895256309497680
(9,8,8,3,1)	18275815076572138848	(9,8,8,4,0)	92700939550359360
(9,9,4,4,3)	30907191740331009622272	(9,9,5,3,3)	13510524682548173641728
(9,9,5,4,2)	5029248618061980310752	(9,9,5,5,1)	201260631020776796160
(9,9,6,3,2)	1009523697408684687360	(9,9,6,4,1)	92904071198229763248
(9,9,6,5,0)	198280729061595552	(9,9,7,2,2)	33639461017770972672
(9,9,7,3,1)	8424230964009345024	(9,9,7,4,0)	42801528146793216
(9,9,8,2,1)	112932475851555408	(9,9,8,3,0)	1692511362069504
(9,9,9,1,1)	113958894140160	(9,9,9,2,0)	7888589144400
(10,5,5,5,4)	2754129399126265116578688	(10,6,5,4,4)	1321756163879149610446896
(10,6,5,5,3)	588017185467849919723392	(10,6,6,4,3)	280190865205345515202656
(10,6,6,5,2)	46407141063611686053552	(10,6,6,6,1)	886526887354737645072
(10,7,4,4,4)	308970407090480886905664	(10,7,5,4,3)	136517982401519160854256
(10,7,5,5,2)	22480410038381810846784	(10,7,6,3,3)	28271565861485256037152
(10,7,6,4,2)	10547216287172185866240	(10,7,6,5,1)	424552799703766464912
(10,7,6,6,0)	419093788958668992	(10,7,7,3,2)	1010995377658903433280
(10,7,7,4,1)	93022849378461001968	(10,7,7,5,0)	198280729061595552
(10,8,4,4,3)	14880953729756521482240	(10,8,5,3,3)	6470374366380782830464
(10,8,5,4,2)	2401953482064409297584	(10,8,5,5,1)	95308266595738550640
(10,8,6,3,2)	475191508986206197632	(10,8,6,4,1)	43589536992230477208
(10,8,6,5,0)	92700939550359360	(10,8,7,2,2)	15318198529776992064
(10,8,7,3,1)	3833600055583272480	(10,8,7,4,0)	19503820669876800
(10,8,8,2,1)	48279338403693048	(10,8,8,3,0)	725912434085952
(10,9,4,3,3)	297520326374626846896	(10,9,4,4,2)	109036439304691948608
(10,9,5,3,2)	45945931585469072928	(10,9,5,4,1)	4165834565805729216
(10,9,5,5,0)	8765016259161504	(10,9,6,2,2)	3046779784226901072
(10,9,6,3,1)	761207182511922096	(10,9,6,4,0)	3881643757375656
(10,9,7,2,1)	20278800720533664	(10,9,7,3,0)	305922925426848
(10,9,8,1,1)	42951164308896	(10,9,8,2,0)	2981800050480
(10,9,9,1,0)	1272585120	(10,10,3,3,3)	2006276928131711424
(10,10,4,3,2)	713471511849776160	(10,10,4,4,1)	62675569121448240
(10,10,5,2,2)	100010833402440120	(10,10,5,3,1)	24840263013168672
(10,10,5,4,0)	126532108859856	(10,10,6,2,1)	1335301022489328
(10,10,6,3,0)	20350993239840	(10,10,7,1,1)	5601159429504
(10,10,7,2,0)	391409808576	(10,10,8,1,0)	347078520
(10,10,9,0,0)	24	(11,5,5,4,4)	36602428260502812573792
(11,5,5,5,3)	15930480413967177684480	(11,6,4,4,4)	16950676810888359150336

Continued on the following page

deg(I) = 29, *continued*			
I	n_I	I	n_I
(11, 6, 5, 4, 3)	7345251761305389562560	(11, 6, 5, 5, 2)	1172715223879828113648
(11, 6, 6, 3, 3)	1445782834458789325920	(11, 6, 6, 4, 2)	533243466879375407808
(11, 6, 6, 5, 1)	20734174826253969312	(11, 6, 6, 6, 0)	19503820669876800
(11, 7, 4, 4, 3)	1560763765722117846528	(11, 7, 5, 3, 3)	666467844013257615360
(11, 7, 5, 4, 2)	245004909605415502560	(11, 7, 5, 5, 1)	9433084896265973760
(11, 7, 6, 3, 2)	46154206945493038080	(11, 7, 6, 4, 1)	4182469007721935136
(11, 7, 6, 5, 0)	8765016259161504	(11, 7, 7, 2, 2)	1329872815417735680
(11, 7, 7, 3, 1)	331877990439469056	(11, 7, 7, 4, 0)	1692511362069504
(11, 8, 4, 3, 3)	60590920000179493056	(11, 8, 4, 4, 2)	22012186784542835520
(11, 8, 5, 3, 2)	9117897040377080832	(11, 8, 5, 4, 1)	817746667654917168
(11, 8, 5, 5, 0)	1692511362069504	(11, 8, 6, 2, 2)	571128202199454336
(11, 8, 6, 3, 1)	142342287006477504	(11, 8, 6, 4, 0)	725912434085952
(11, 8, 7, 2, 1)	3377194221012096	(11, 8, 7, 3, 0)	51294957112992
(11, 8, 8, 1, 1)	5601159429504	(11, 8, 8, 2, 0)	391409808576
(11, 9, 3, 3, 3)	841539378868429824	(11, 9, 4, 3, 2)	297115911452589936
(11, 9, 4, 4, 1)	25857038420140320	(11, 9, 5, 2, 2)	40746789567213888
(11, 9, 5, 3, 1)	10097809547695104	(11, 9, 5, 4, 0)	51294957112992
(11, 9, 6, 2, 1)	515881389602064	(11, 9, 6, 3, 0)	7888589144400
(11, 9, 7, 1, 1)	1927069671936	(11, 9, 7, 2, 0)	135171775392
(11, 9, 8, 1, 0)	88179456	(11, 10, 3, 3, 2)	1000740719949936
(11, 10, 4, 2, 2)	328447354833120	(11, 10, 4, 3, 1)	79804026346992
(11, 10, 4, 4, 0)	391409808576	(11, 10, 5, 2, 1)	8748592415904
(11, 10, 5, 3, 0)	135171775392	(11, 10, 6, 1, 1)	61773182400
(11, 10, 6, 2, 0)	4392333792	(11, 10, 7, 1, 0)	4326048
(11, 11, 3, 2, 2)	105371446464	(11, 11, 3, 3, 1)	23351460864
(11, 11, 4, 2, 1)	5950086192	(11, 11, 4, 3, 0)	88179456
(11, 11, 5, 1, 1)	59097600	(11, 11, 5, 2, 0)	4326048
(11, 11, 6, 1, 0)	1104	(12, 5, 4, 4, 4)	1598329602773986983 12
(12, 5, 5, 4, 3)	665734820653276694 40	(12, 5, 5, 5, 2)	99523700459152904 64
(12, 6, 4, 4, 3)	288147537957873041 28	(12, 6, 5, 3, 3)	118331366683836110 40
(12, 6, 5, 4, 2)	42520053276512237 76	(12, 6, 5, 5, 1)	1524351528388661 76
(12, 6, 6, 3, 2)	721163569257189312	(12, 6, 6, 4, 1)	63276065657309280
(12, 6, 6, 5, 0)	126532108859856	(12, 7, 4, 3, 3)	2124595552827372432
(12, 7, 4, 4, 2)	754387255278771840	(12, 7, 5, 3, 2)	299477728365291600
(12, 7, 5, 4, 1)	26040136828870752	(12, 7, 5, 5, 0)	51294957112992
(12, 7, 6, 2, 2)	16270300857476160	(12, 7, 6, 3, 1)	4021264698687264
(12, 7, 6, 4, 0)	20350993239840	(12, 7, 7, 2, 1)	71274491245200
(12, 7, 7, 3, 0)	1096632180480	(12, 8, 3, 3, 3)	55724768553096576
(12, 8, 4, 3, 2)	19159936729163904	(12, 8, 4, 4, 1)	1608297381675072
(12, 8, 5, 2, 2)	2428815576573408	(12, 8, 5, 3, 1)	596073535387056
(12, 8, 5, 4, 0)	2981800050480	(12, 8, 6, 2, 1)	25377635878296
(12, 8, 6, 3, 0)	391409808576	(12, 8, 7, 1, 1)	61773182400
(12, 8, 7, 2, 0)	4392333792	(12, 8, 8, 1, 0)	795936
(12, 9, 3, 3, 2)	125365423769760	(12, 9, 4, 2, 2)	39692266181304
(12, 9, 4, 3, 1)	9502910875584	(12, 9, 4, 4, 0)	45007048752
(12, 9, 5, 2, 1)	931163905728	(12, 9, 5, 3, 0)	14386869840
(12, 9, 6, 1, 1)	4826161680	(12, 9, 6, 2, 0)	347078520
(12, 9, 7, 1, 0)	122448	(12, 10, 3, 2, 2)	27607031136
(12, 10, 3, 3, 1)	5950086192	(12, 10, 4, 2, 1)	1426637712
(12, 10, 4, 3, 0)	20578560	(12, 10, 5, 1, 1)	10883712
(12, 10, 5, 2, 0)	795936	(12, 10, 6, 1, 0)	24
(12, 11, 2, 2, 2)	14496	(12, 11, 3, 2, 1)	1104
(13, 4, 4, 4, 4)	149583407202367176	(13, 5, 4, 4, 3)	57309129620711136
(13, 5, 5, 3, 3)	21671962905320448	(13, 5, 5, 4, 2)	7371081117191712
(13, 5, 5, 5, 1)	221145135246336	(13, 6, 4, 3, 3)	8236673292611808
(13, 6, 4, 4, 2)	2768640614245200	(13, 6, 5, 3, 2)	1000740719949936
(13, 6, 5, 4, 1)	79804026346992	(13, 6, 5, 5, 0)	135171775392

Continued on the following page

deg(I) = 29, *continued*			
I	n_I	I	n_I
(13, 6, 6, 2, 2)	39360165257928	(13, 6, 6, 3, 1)	9425697295296
(13, 6, 6, 4, 0)	45007048752	(13, 7, 3, 3, 3)	389973010495488
(13, 7, 4, 3, 2)	125365423769760	(13, 7, 4, 4, 1)	9502910875584
(13, 7, 5, 2, 2)	13073262151968	(13, 7, 5, 3, 1)	3100342138368
(13, 7, 5, 4, 0)	14386869840	(13, 7, 6, 2, 1)	83099778720
(13, 7, 6, 3, 0)	1272585120	(13, 7, 7, 1, 1)	59097600
(13, 7, 7, 2, 0)	4326048	(13, 8, 3, 3, 2)	1326841710624
(13, 8, 4, 2, 2)	377080188864	(13, 8, 4, 3, 1)	85495746528
(13, 8, 4, 4, 0)	347078520	(13, 8, 5, 2, 1)	5950086192
(13, 8, 5, 3, 0)	88179456	(13, 8, 6, 1, 1)	10883712
(13, 8, 6, 2, 0)	795936	(13, 9, 3, 2, 2)	316997280
(13, 9, 3, 3, 1)	59097600	(13, 9, 4, 2, 1)	10883712
(13, 9, 4, 3, 0)	122448	(13, 9, 5, 1, 1)	19200
(13, 9, 5, 2, 0)	1104	(14, 4, 4, 4, 3)	3114669545280
(14, 5, 4, 3, 3)	980247769056	(14, 5, 4, 4, 2)	285207114048
(14, 5, 5, 3, 2)	83099778720	(14, 5, 5, 4, 1)	4826161680
(14, 5, 5, 5, 0)	4326048	(14, 6, 3, 3, 3)	85495746528
(14, 6, 4, 3, 2)	22951602432	(14, 6, 4, 4, 1)	1218252960
(14, 6, 5, 2, 2)	1426637712	(14, 6, 5, 3, 1)	282674592
(14, 6, 5, 4, 0)	795936	(14, 6, 6, 2, 1)	1691856
(14, 6, 6, 3, 0)	14496	(14, 7, 3, 3, 2)	316997280
(14, 7, 4, 2, 2)	63576576	(14, 7, 4, 3, 1)	10883712
(14, 7, 4, 4, 0)	14496	(14, 7, 5, 2, 1)	212880
(14, 7, 5, 3, 0)	1104	(14, 8, 3, 2, 2)	14496
(14, 8, 3, 3, 1)	1104	(14, 8, 4, 2, 1)	24

E.2 Genus-1 instantons

Table 13: The genus-one instanton numbers d_I for deg(I) ⩽ 29.

deg(I) = 6			
I	d_I	I	d_I
(2, 2, 2, 0, 0)	4		

deg(I) = 7			
I	d_I	I	d_I
(2, 2, 2, 1, 0)	-48		

deg(I) = 8			
I	d_I	I	d_I
(2, 2, 2, 1, 1)	528	(2, 2, 2, 2, 0)	-2292
(3, 2, 2, 1, 0)	-48		

$\deg(I) = 9$			
I	d_I	I	d_I
$(2,2,2,2,1)$	29808	$(3,2,2,1,1)$	928
$(3,2,2,2,0)$	-5600	$(3,3,2,1,0)$	-224

$\deg(I) = 10$			
I	d_I	I	d_I
$(2,2,2,2,2)$	3666312	$(3,2,2,2,1)$	104352
$(3,3,2,1,1)$	4320	$(3,3,2,2,0)$	-29136
$(3,3,3,1,0)$	-2208	$(4,2,2,1,1)$	528
$(4,2,2,2,0)$	-2292	$(4,3,2,1,0)$	-48

$\deg(I) = 11$			
I	d_I	I	d_I
$(3,2,2,2,2)$	22958688	$(3,3,2,2,1)$	679968
$(3,3,3,1,1)$	30720	$(3,3,3,2,0)$	-251520
$(4,2,2,2,1)$	104352	$(4,3,2,1,1)$	4320
$(4,3,2,2,0)$	-29136	$(4,3,3,1,0)$	-2208
$(4,4,2,1,0)$	-48	$(5,2,2,2,0)$	-48

$\deg(I) = 12$			
I	d_I	I	d_I
$(3,3,2,2,2)$	230549312	$(3,3,3,2,1)$	6953664
$(3,3,3,3,0)$	-3031872	$(4,2,2,2,2)$	40083960
$(4,3,2,2,1)$	1194656	$(4,3,3,1,1)$	42560
$(4,3,3,2,0)$	-484896	$(4,4,2,1,1)$	10400
$(4,4,2,2,0)$	-61760	$(4,4,3,1,0)$	-5600
$(4,4,4,0,0)$	4	$(5,2,2,2,1)$	29808
$(5,3,2,1,1)$	928	$(5,3,2,2,0)$	-5600
$(5,3,3,1,0)$	-224	$(6,2,2,2,0)$	4

$\deg(I) = 13$			
I	d_I	I	d_I
$(3,3,3,2,2)$	3347625888	$(3,3,3,3,1)$	99761664
$(4,3,2,2,2)$	652777584	$(4,3,3,2,1)$	19494816
$(4,3,3,3,0)$	-9395616	$(4,4,2,2,1)$	3692400
$(4,4,3,1,1)$	73824	$(4,4,3,2,0)$	-1679040
$(4,4,4,1,0)$	-29136	$(5,2,2,2,2)$	22958688
$(5,3,2,2,1)$	679968	$(5,3,3,1,1)$	30720
$(5,3,3,2,0)$	-251520	$(5,4,2,1,1)$	4320
$(5,4,2,2,0)$	-29136	$(5,4,3,1,0)$	-2208
$(6,2,2,2,1)$	528	$(6,3,2,2,0)$	-48

deg(I) = 14

I	d_I	I	d_I
(3, 3, 3, 3, 2)	65707393920	(4, 3, 3, 2, 2)	14105356368
(4, 3, 3, 3, 1)	411633120	(4, 4, 2, 2, 2)	2937953580
(4, 4, 3, 2, 1)	86694528	(4, 4, 3, 3, 0)	-46049040
(4, 4, 4, 1, 1)	-317232	(4, 4, 4, 2, 0)	-9396672
(5, 3, 2, 2, 2)	652777584	(5, 3, 3, 2, 1)	19494816
(5, 3, 3, 3, 0)	-9395616	(5, 4, 2, 2, 1)	3692400
(5, 4, 3, 1, 1)	73824	(5, 4, 3, 2, 0)	-1679040
(5, 4, 4, 1, 0)	-29136	(5, 5, 2, 1, 1)	4320
(5, 5, 2, 2, 0)	-29136	(5, 5, 3, 1, 0)	-2208
(6, 2, 2, 2, 2)	3666312	(6, 3, 2, 2, 1)	104352
(6, 3, 3, 1, 1)	4320	(6, 3, 3, 2, 0)	-29136
(6, 4, 2, 1, 1)	528	(6, 4, 2, 2, 0)	-2292
(6, 4, 3, 1, 0)	-48	(7, 2, 2, 2, 1)	-48

deg(I) = 15

I	d_I	I	d_I
(3, 3, 3, 3, 3)	1668835805184	(4, 3, 3, 3, 2)	385951211712
(4, 4, 3, 2, 2)	87650018048	(4, 4, 3, 3, 1)	2496782816
(4, 4, 4, 2, 1)	561090816	(4, 4, 4, 3, 0)	-327015680
(5, 3, 3, 2, 2)	22327107072	(5, 3, 3, 3, 1)	646886400
(5, 4, 2, 2, 2)	4750051104	(5, 4, 3, 2, 1)	138982240
(5, 4, 3, 3, 0)	-76342880	(5, 4, 4, 1, 1)	-1114976
(5, 4, 4, 2, 0)	-16170272	(5, 5, 2, 2, 1)	6414464
(5, 5, 3, 1, 1)	75776	(5, 5, 3, 2, 0)	-3031872
(5, 5, 4, 1, 0)	-61920	(6, 3, 2, 2, 2)	230549312
(6, 3, 3, 2, 1)	6953664	(6, 3, 3, 3, 0)	-3031872
(6, 4, 2, 2, 1)	1194656	(6, 4, 3, 1, 1)	42560
(6, 4, 3, 2, 0)	-484896	(6, 4, 4, 1, 0)	-5600
(6, 5, 2, 1, 1)	928	(6, 5, 2, 2, 0)	-5600
(6, 5, 3, 1, 0)	-224	(7, 2, 2, 2, 2)	29808
(7, 3, 2, 2, 1)	928	(7, 3, 3, 2, 0)	-224

deg(I) = 16

I	d_I	I	d_I
(4, 3, 3, 3, 3)	13029814091424	(4, 4, 3, 3, 2)	3154648420512
(4, 4, 4, 2, 2)	755118268080	(4, 4, 4, 3, 1)	20875131744
(4, 4, 4, 4, 0)	-3110590260	(5, 3, 3, 3, 2)	894337855968
(5, 4, 3, 2, 2)	208350582720	(5, 4, 3, 3, 1)	5848333440
(5, 4, 4, 2, 1)	1342319904	(5, 4, 4, 3, 0)	-824199120
(5, 5, 2, 2, 2)	12168742800	(5, 5, 3, 2, 1)	351706176
(5, 5, 3, 3, 0)	-203310240	(5, 5, 4, 1, 1)	-6126048
(5, 5, 4, 2, 0)	-46049040	(5, 5, 5, 1, 0)	-251520
(6, 3, 3, 2, 2)	14105356368	(6, 3, 3, 3, 1)	411633120
(6, 4, 2, 2, 2)	2937953580	(6, 4, 3, 2, 1)	86694528
(6, 4, 3, 3, 0)	-46049040	(6, 4, 4, 1, 1)	-317232
(6, 4, 4, 2, 0)	-9396672	(6, 5, 2, 2, 1)	3692400
(6, 5, 3, 1, 1)	73824	(6, 5, 3, 2, 0)	-1679040
(6, 5, 4, 1, 0)	-29136	(6, 6, 2, 1, 1)	528
(6, 6, 2, 2, 0)	-2292	(6, 6, 3, 1, 0)	-48
(7, 3, 2, 2, 2)	22958688	(7, 3, 3, 2, 1)	679968
(7, 3, 3, 3, 0)	-251520	(7, 4, 2, 2, 1)	104352

Continued on the following page

deg(I) = 16, *continued*			
I	d_I	I	d_I
(7,4,3,1,1)	4320	(7,4,3,2,0)	-29136
(7,4,4,1,0)	-48	(7,5,2,2,0)	-48
(8,2,2,2,2)	-2292	(8,3,2,2,1)	-48

deg(I) = 17			
I	d_I	I	d_I
(4,4,3,3,3)	135453779066496	(4,4,4,3,2)	34155140507184
(4,4,4,4,1)	228415121472	(5,3,3,3,3)	41704406393856
(5,4,3,3,2)	10348372749216	(5,4,4,2,2)	2545705442112
(5,4,4,3,1)	68863079616	(5,4,4,4,0)	-11043084816
(5,5,3,2,2)	733831612704	(5,5,3,3,1)	20194851840
(5,5,4,2,1)	4741984896	(5,5,4,3,0)	-3110582880
(5,5,5,1,1)	-46978560	(5,5,5,2,0)	-203310240
(6,3,3,3,2)	894337855968	(6,4,3,2,2)	208350582720
(6,4,3,3,1)	5848333440	(6,4,4,2,1)	1342319904
(6,4,4,3,0)	-824199120	(6,5,2,2,2)	12168742800
(6,5,3,2,1)	351706176	(6,5,3,3,0)	-203310240
(6,5,4,1,1)	-6126048	(6,5,4,2,0)	-46049040
(6,5,5,1,0)	-251520	(6,6,2,2,1)	3692400
(6,6,3,1,1)	73824	(6,6,3,2,0)	-1679040
(6,6,4,1,0)	-29136	(7,3,3,2,2)	3347625888
(7,3,3,3,1)	99761664	(7,4,2,2,2)	652777584
(7,4,3,2,1)	19494816	(7,4,3,3,0)	-9395616
(7,4,4,1,1)	73824	(7,4,4,2,0)	-1679040
(7,5,2,2,1)	679968	(7,5,3,1,1)	30720
(7,5,3,2,0)	-251520	(7,5,4,1,0)	-2208
(7,6,2,2,0)	-48	(8,3,2,2,2)	104352
(8,3,3,2,1)	4320	(8,3,3,3,0)	-2208
(8,4,2,2,1)	528	(8,4,3,2,0)	-48

deg(I) = 18			
I	d_I	I	d_I
(4,4,4,3,3)	1803381971700144	(4,4,4,4,2)	470537427014352
(5,4,3,3,3)	586171325733792	(5,4,4,3,2)	151342528026688
(5,4,4,4,1)	1011188967744	(5,5,3,3,2)	47693058783296
(5,5,4,2,2)	12091316695232	(5,5,4,3,1)	318330381792
(5,5,4,4,0)	-55127514240	(5,5,5,2,1)	23350187616
(5,5,5,3,0)	-16642969280	(6,3,3,3,3)	60862991224384
(6,4,3,3,2)	15219924472416	(6,4,4,2,2)	3775716012840
(6,4,4,3,1)	101366312448	(6,4,4,4,0)	-16642956928
(6,5,3,2,2)	1103727042528	(6,5,3,3,1)	30154035584
(6,5,4,2,1)	7112117856	(6,5,4,3,0)	-4775506080
(6,5,5,1,1)	-87015936	(6,5,5,2,0)	-327015680
(6,6,2,2,2)	19270744144	(6,6,3,2,1)	554248640
(6,6,3,3,0)	-327015680	(6,6,4,1,1)	-12495360
(6,6,4,2,0)	-76341160	(6,6,5,1,0)	-484896
(6,6,6,0,0)	4	(7,3,3,3,2)	385951211712
(7,4,3,2,2)	87650018048	(7,4,3,3,1)	2496782816
(7,4,4,2,1)	561090816	(7,4,4,3,0)	-327015680
(7,5,2,2,2)	4750051104	(7,5,3,2,1)	138982240
(7,5,3,3,0)	-76342880	(7,5,4,1,1)	-1114976
(7,5,4,2,0)	-16170272	(7,5,5,1,0)	-61920

Continued on the following page

deg(I) = 18, *continued*

I	d_I	I	d_I
(7,6,2,2,1)	1194656	(7,6,3,1,1)	42560
(7,6,3,2,0)	-484896	(7,6,4,1,0)	-5600
(8,3,3,2,2)	230549312	(8,3,3,3,1)	6953664
(8,4,2,2,2)	40083960	(8,4,3,2,1)	1194656
(8,4,3,3,0)	-484896	(8,4,4,1,1)	10400
(8,4,4,2,0)	-61760	(8,5,2,2,1)	29808
(8,5,3,1,1)	928	(8,5,3,2,0)	-5600
(8,6,2,2,0)	4	(9,3,2,2,2)	-5600
(9,3,3,2,1)	-224		

deg(I) = 19

I	d_I	I	d_I
(4,4,4,4,3)	29809312235610960	(5,4,4,3,3)	10159668608774304
(5,4,4,4,2)	2707370108500416	(5,5,3,3,3)	3417190702574592
(5,5,4,3,2)	903625742797728	(5,5,4,4,1)	6007581031968
(5,5,5,2,2)	77522333436960	(5,5,5,3,1)	1974181959168
(5,5,5,4,0)	-368134832160	(6,4,3,3,3)	1190848151512512
(6,4,4,3,2)	310831260169488	(6,4,4,4,1)	2072265197088
(6,5,3,3,2)	99761061359136	(6,5,4,2,2)	25630803734064
(6,5,4,3,1)	665062141248	(6,5,4,4,0)	-119442727776
(6,5,5,2,1)	49806889344	(6,5,5,3,0)	-37176746592
(6,6,3,2,2)	2461712752416	(6,6,3,3,1)	66382892544
(6,6,4,2,1)	15766948032	(6,6,4,3,0)	-11043084816
(6,6,5,1,1)	-273996960	(6,6,5,2,0)	-824199120
(6,6,6,1,0)	-1679040	(7,3,3,3,3)	41704406393856
(7,4,3,3,2)	10348372749216	(7,4,4,2,2)	2545705442112
(7,4,4,3,1)	68863079616	(7,4,4,4,0)	-11043084816
(7,5,3,2,2)	733831612704	(7,5,3,3,1)	20194851840
(7,5,4,2,1)	4741984896	(7,5,4,3,0)	-3110582880
(7,5,5,1,1)	-46978560	(7,5,5,2,0)	-203310240
(7,6,2,2,2)	12168742800	(7,6,3,2,1)	351706176
(7,6,3,3,0)	-203310240	(7,6,4,1,1)	-6126048
(7,6,4,2,0)	-46049040	(7,6,5,1,0)	-251520
(7,7,2,2,1)	679968	(7,7,3,1,1)	30720
(7,7,3,2,0)	-251520	(7,7,4,1,0)	-2208
(8,3,3,3,2)	65707393920	(8,4,3,2,2)	14105356368
(8,4,3,3,1)	411633120	(8,4,4,2,1)	86694528
(8,4,4,3,0)	-46049040	(8,5,2,2,2)	652777584
(8,5,3,2,1)	19494816	(8,5,3,3,0)	-9395616
(8,5,4,1,1)	73824	(8,5,4,2,0)	-1679040
(8,5,5,1,0)	-2208	(8,6,2,2,1)	104352
(8,6,3,1,1)	4320	(8,6,3,2,0)	-29136
(8,6,4,1,0)	-48	(9,3,3,2,2)	679968
(9,3,3,3,1)	30720	(9,4,2,2,2)	104352
(9,4,3,2,1)	4320	(9,4,3,3,0)	-2208
(9,4,4,2,0)	-48		

deg(I) = 20

I	d_I	I	d_I
(4,4,4,4,4)	597237294763420872	(5,4,4,4,3)	211913083229294304
(5,5,4,3,3)	74503089764268384	(5,5,4,4,2)	20254395759934128
(5,5,5,3,2)	7011987726247008	(5,5,5,4,1)	46131775979616

Continued on the following page

deg(I) = 20, *continued*			
I	d_I	I	d_I
(5,5,5,5,0)	-3138370134624	(6,4,4,3,3)	27569906261747088
(6,4,4,4,2)	7431936473155680	(6,5,3,3,3)	9449678610817056
(6,5,4,3,2)	2530955656217280	(6,5,4,4,1)	16731712682064
(6,5,5,2,2)	225463832566752	(6,5,5,3,1)	5619611350656
(6,5,5,4,0)	-1093125957120	(6,6,3,3,2)	294058742512224
(6,6,4,2,2)	76906534439280	(6,6,4,3,1)	1954674541824
(6,6,4,4,0)	-368134868160	(6,6,5,2,1)	149497953456
(6,6,5,3,0)	-119442727776	(6,6,6,1,1)	-1334560224
(6,6,6,2,0)	-3110590260	(7,4,3,3,3)	1190848151512512
(7,4,4,3,2)	310831260169488	(7,4,4,4,1)	2072265197088
(7,5,3,3,2)	99761061359136	(7,5,4,2,2)	25630803734064
(7,5,4,3,1)	665062141248	(7,5,4,4,0)	-119442727776
(7,5,5,2,1)	49806889344	(7,5,5,3,0)	-37176746592
(7,6,3,2,2)	2461712752416	(7,6,3,3,1)	66382892544
(7,6,4,2,1)	15766948032	(7,6,4,3,0)	-11043084816
(7,6,5,1,1)	-273996960	(7,6,5,2,0)	-824199120
(7,6,6,1,0)	-1679040	(7,7,2,2,2)	12168742800
(7,7,3,2,1)	351706176	(7,7,3,3,0)	-203310240
(7,7,4,1,1)	-6126048	(7,7,4,2,0)	-46049040
(7,7,5,1,0)	-251520	(8,3,3,3,3)	13029814091424
(8,4,3,3,2)	3154648420512	(8,4,4,2,2)	755118268080
(8,4,4,3,1)	20875131744	(8,4,4,4,0)	-3110590260
(8,5,3,2,2)	208350582720	(8,5,3,3,1)	5848333440
(8,5,4,2,1)	1342319904	(8,5,4,3,0)	-824199120
(8,5,5,1,1)	-6126048	(8,5,5,2,0)	-46049040
(8,6,2,2,2)	2937953580	(8,6,3,2,1)	86694528
(8,6,3,3,0)	-46049040	(8,6,4,1,1)	-317232
(8,6,4,2,0)	-9396672	(8,6,5,1,0)	-29136
(8,7,2,2,1)	104352	(8,7,3,1,1)	4320
(8,7,3,2,0)	-29136	(8,7,4,1,0)	-48
(9,3,3,3,2)	3347625888	(9,4,3,2,2)	652777584
(9,4,3,3,1)	19494816	(9,4,4,2,1)	3692400
(9,4,4,3,0)	-1679040	(9,5,2,2,2)	22958688
(9,5,3,2,1)	679968	(9,5,3,3,0)	-251520
(9,5,4,1,1)	4320	(9,5,4,2,0)	-29136
(9,6,2,2,1)	528	(9,6,3,2,0)	-48
(10,3,3,2,2)	-29136	(10,3,3,3,1)	-2208
(10,4,2,2,2)	-2292	(10,4,3,2,1)	-48

deg(I) = 21			
I	d_I	I	d_I
(5,4,4,4,4)	5224733955268106112	(5,5,4,4,3)	1905129808949968992
(5,5,5,3,3)	689915910456635392	(5,5,5,4,2)	190939831236687552
(5,5,5,5,1)	442883019280896	(6,4,4,4,3)	742174782726416480
(6,5,4,3,3)	265869597857942752	(6,5,4,4,2)	73138223025414832
(6,5,5,3,2)	25872896752378400	(6,5,5,4,1)	168513518883456
(6,5,5,5,0)	-12215408263200	(6,6,3,3,3)	35367733329831456
(6,6,4,3,2)	9613082109166896	(6,6,4,4,1)	63004921310816
(6,6,5,2,2)	894838196834976	(6,6,5,3,1)	21688029832256
(6,6,5,4,0)	-4429601736480	(6,6,6,2,1)	604322817696
(6,6,6,3,0)	-531223501536	(7,4,4,3,3)	38231916995852064
(7,4,4,4,2)	10343926869883680	(7,5,3,3,3)	13183353406838784
(7,5,4,3,2)	3545177906512576	(7,5,4,4,1)	23383915823040
(7,5,5,2,2)	319546789488192	(7,5,5,3,1)	7905805097984
(7,5,5,4,0)	-1559208760288	(7,6,3,3,2)	418651558115072

Continued on the following page

deg(I) = 21, *continued*			
I	d_I	I	d_I
(7,6,4,2,2)	110090172437152	(7,6,4,3,1)	2778335397408
(7,6,4,4,0)	-531223501536	(7,6,5,2,1)	213298398784
(7,6,5,3,0)	-174588053440	(7,6,6,1,1)	-2202805408
(7,6,6,2,0)	-4775506080	(7,7,3,2,2)	3650629114944
(7,7,3,3,1)	97849986048	(7,7,4,2,1)	23296907136
(7,7,4,3,0)	-16642969280	(7,7,5,1,1)	-471020544
(7,7,5,2,0)	-1292723968	(7,7,6,1,0)	-3031872
(8,4,3,3,3)	586171325733792	(8,4,4,3,2)	151342528026688
(8,4,4,4,1)	1011188967744	(8,5,3,3,2)	47693058783296
(8,5,4,2,2)	12091316695232	(8,5,4,3,1)	318330381792
(8,5,4,4,0)	-55127514240	(8,5,5,2,1)	23350187616
(8,5,5,3,0)	-16642969280	(8,6,3,2,2)	1103727042528
(8,6,3,3,1)	30154035584	(8,6,4,2,1)	7112117856
(8,6,4,3,0)	-4775506080	(8,6,5,1,1)	-87015936
(8,6,5,2,0)	-327015680	(8,6,6,1,0)	-484896
(8,7,2,2,2)	4750051104	(8,7,3,2,1)	138982240
(8,7,3,3,0)	-76342880	(8,7,4,1,1)	-1114976
(8,7,4,2,0)	-16170272	(8,7,5,1,0)	-61920
(8,8,2,2,1)	29808	(8,8,3,1,1)	928
(8,8,3,2,0)	-5600	(9,3,3,3,3)	1668835805184
(9,4,3,3,2)	385951211712	(9,4,4,2,2)	87650018048
(9,4,4,3,1)	2496782816	(9,4,4,4,0)	-327015680
(9,5,3,2,2)	22327107072	(9,5,3,3,1)	646886400
(9,5,4,2,1)	138982240	(9,5,4,3,0)	-76342880
(9,5,5,1,1)	75776	(9,5,5,2,0)	-3031872
(9,6,2,2,2)	230549312	(9,6,3,2,1)	6953664
(9,6,3,3,0)	-3031872	(9,6,4,1,1)	42560
(9,6,4,2,0)	-484896	(9,6,5,1,0)	-224
(9,7,2,2,1)	928	(9,7,3,2,0)	-224
(10,3,3,3,2)	6953664	(10,4,3,2,2)	1194656
(10,4,3,3,1)	42560	(10,4,4,2,1)	10400
(10,4,4,3,0)	-5600	(10,5,2,2,2)	29808
(10,5,3,2,1)	928	(10,5,3,3,0)	-224

deg(I) = 22			
I	d_I	I	d_I
(5,5,4,4,4)	56439747241501122192	(5,5,5,4,3)	21104661093843211008
(5,5,5,5,2)	2206547301748229184	(6,4,4,4,4)	22962634839334473072
(6,5,4,4,3)	8521174156401735360	(6,5,5,3,3)	3144478089605250720
(6,5,5,4,2)	879997438952410320	(6,5,5,5,1)	2055320972401920
(6,6,4,3,3)	1243841228095131744	(6,6,4,4,2)	346443574026127704
(6,6,5,3,2)	125469245902996512	(6,6,5,4,1)	805991883499728
(6,6,5,5,0)	-62415555336480	(6,6,6,2,2)	4694772196282128
(6,6,6,3,1)	109867633989312	(6,6,6,4,0)	-23657222126952
(7,4,4,4,3)	1369210879561818480	(7,5,4,3,3)	494845153306899264
(7,5,4,4,2)	136879891485939456	(7,5,5,3,2)	48909204818311680
(7,5,5,4,1)	316717033197408	(7,5,5,5,0)	-23657221999872
(7,6,3,3,3)	67323025901888832	(7,6,4,3,2)	18421518301369920
(7,6,4,4,1)	120114507109632	(7,6,5,2,2)	1749317561578944
(7,6,5,3,1)	41786034116160	(7,6,5,4,0)	-8738280013680
(7,6,6,2,1)	1179552933984	(7,6,6,3,0)	-1093125957120
(7,7,3,3,2)	840536568752160	(7,7,4,2,2)	223306712255904
(7,7,4,3,1)	5559537648384	(7,7,4,4,0)	-1093125957120
(7,7,5,2,1)	428753909184	(7,7,5,3,0)	-368134832160
(7,7,6,1,1)	-5776067616	(7,7,6,2,0)	-11043084816

Continued on the following page

deg(I) = 22, *continued*			
I	d_I	I	d_I
(7,7,7,1,0)	-9395616	(8,4,4,3,3)	27569906261747088
(8,4,4,4,2)	7431936473155680	(8,5,3,3,3)	9449678610817056
(8,5,4,3,2)	2530955656217280	(8,5,4,4,1)	16731712682064
(8,5,5,2,2)	225463832566752	(8,5,5,3,1)	5619611350656
(8,5,5,4,0)	-1093125957120	(8,6,3,3,2)	294058742512224
(8,6,4,2,2)	76906534439280	(8,6,4,3,1)	1954674541824
(8,6,4,4,0)	-368134868160	(8,6,5,2,1)	149497953456
(8,6,5,3,0)	-119442727776	(8,6,6,1,1)	-1334560224
(8,6,6,2,0)	-3110590260	(8,7,3,2,2)	2461712752416
(8,7,3,3,1)	66382892544	(8,7,4,2,1)	15766948032
(8,7,4,3,0)	-11043084816	(8,7,5,1,1)	-273996960
(8,7,5,2,0)	-824199120	(8,7,6,1,0)	-1679040
(8,8,2,2,2)	2937953580	(8,8,3,2,1)	86694528
(8,8,3,3,0)	-46049040	(8,8,4,1,1)	-317232
(8,8,4,2,0)	-9396672	(8,8,5,1,0)	-29136
(9,4,3,3,3)	135453779066496	(9,4,4,3,2)	34155140507184
(9,4,4,4,1)	228415121472	(9,5,3,3,2)	10348372749216
(9,5,4,2,2)	2545705442112	(9,5,4,3,1)	68863079616
(9,5,4,4,0)	-11043084816	(9,5,5,2,1)	4741984896
(9,5,5,3,0)	-3110582880	(9,6,3,2,2)	208350582720
(9,6,3,3,1)	5848333440	(9,6,4,2,1)	1342319904
(9,6,4,3,0)	-824199120	(9,6,5,1,1)	-6126048
(9,6,5,2,0)	-46049040	(9,6,6,1,0)	-29136
(9,7,2,2,2)	652777584	(9,7,3,2,1)	19494816
(9,7,3,3,0)	-9395616	(9,7,4,1,1)	73824
(9,7,4,2,0)	-1679040	(9,7,5,1,0)	-2208
(9,8,2,2,1)	528	(9,8,3,2,0)	-48
(10,3,3,3,3)	65707393920	(10,4,3,3,2)	14105356368
(10,4,4,2,2)	2937953580	(10,4,4,3,1)	86694528
(10,4,4,4,0)	-9396672	(10,5,3,2,2)	652777584
(10,5,3,3,1)	19494816	(10,5,4,2,1)	3692400
(10,5,4,3,0)	-1679040	(10,5,5,1,1)	4320
(10,5,5,2,0)	-29136	(10,6,2,2,2)	3666312
(10,6,3,2,1)	104352	(10,6,3,3,0)	-29136
(10,6,4,1,1)	528	(10,6,4,2,0)	-2292
(10,7,2,2,1)	-48	(11,3,3,3,2)	-251520
(11,4,3,2,2)	-29136	(11,4,3,3,1)	-2208
(11,4,4,2,1)	-48	(11,5,2,2,2)	-48

deg(I) = 23			
I	d_I	I	d_I
(5,5,5,4,4)	736557199694836925664	(5,5,5,5,3)	281693674984303028736
(6,5,4,4,4)	309132463141878510000	(6,5,5,4,3)	117552717975524482368
(6,5,5,5,2)	12636043517074729152	(6,6,4,4,3)	48619672420408782672
(6,6,5,3,3)	18293419124197908384	(6,6,5,4,2)	5175740190127316160
(6,6,5,5,1)	12132995150259168	(6,6,6,3,2)	777895843534310448
(6,6,6,4,1)	4906985498846880	(6,6,6,5,0)	-405156007308576
(7,4,4,4,4)	54505767240269122368	(7,5,4,4,3)	20425869092209172544
(7,5,5,3,3)	7616709702249560064	(7,5,5,4,2)	2144674430360819712
(7,5,5,5,1)	5019686189816832	(7,6,4,3,3)	3057298493530985568
(7,6,4,4,2)	857299960689895392	(7,6,5,3,2)	314475153857119104
(7,6,5,4,1)	2001252354617280	(7,6,5,5,0)	-160791639748800
(7,6,6,2,2)	12276819589743504	(7,6,6,3,1)	281059260470880
(7,6,6,4,0)	-62415555336480	(7,7,3,3,3)	173735334568551936
(7,7,4,3,2)	47974716780673152	(7,7,4,4,1)	310320577356576

Continued on the following page

deg(I) = 23, *continued*

I	d_I	I	d_I
(7,7,5,2,2)	4682747289739200	(7,7,5,3,1)	109488862752768
(7,7,5,4,0)	-23657221999872	(7,7,6,2,1)	3117411980160
(7,7,6,3,0)	-3138370134624	(7,7,7,1,1)	-22739284992
(7,7,7,2,0)	-37176746592	(8,4,4,4,3)	1369210879561818480
(8,5,4,3,3)	494845153306899264	(8,5,4,4,2)	136879891485939456
(8,5,5,3,2)	48909204818311680	(8,5,5,4,1)	316717033197408
(8,5,5,5,0)	-23657221999872	(8,6,3,3,3)	6732302590188832
(8,6,4,3,2)	18421518301369920	(8,6,4,4,1)	120114507109632
(8,6,5,2,2)	1749317561578944	(8,6,5,3,1)	41786034116160
(8,6,5,4,0)	-8738280013680	(8,6,6,2,1)	1179552933984
(8,6,6,3,0)	-1093125957120	(8,7,3,3,2)	840536568752160
(8,7,4,2,2)	223306712255904	(8,7,4,3,1)	5559537648384
(8,7,4,4,0)	-1093125957120	(8,7,5,2,1)	428753909184
(8,7,5,3,0)	-368134832160	(8,7,6,1,1)	-5776067616
(8,7,6,2,0)	-11043084816	(8,7,7,1,0)	-9395616
(8,8,3,2,2)	2461712752416	(8,8,3,3,1)	66382892544
(8,8,4,2,1)	15766948032	(8,8,4,3,0)	-11043084816
(8,8,5,1,1)	-273996960	(8,8,5,2,0)	-824199120
(8,8,6,1,0)	-1679040	(9,4,4,3,3)	10159668608774304
(9,4,4,4,2)	2707370108500416	(9,5,3,3,3)	3417190702574592
(9,5,4,3,2)	903625742797728	(9,5,4,4,1)	6007581031968
(9,5,5,2,2)	77522333436960	(9,5,5,3,1)	1974181959168
(9,5,5,4,0)	-368134832160	(9,6,3,3,2)	99761061359136
(9,6,4,2,2)	25630803734064	(9,6,4,3,1)	665062141248
(9,6,4,4,0)	-119442727776	(9,6,5,2,1)	49806889344
(9,6,5,3,0)	-37176746592	(9,6,6,1,1)	-273996960
(9,6,6,2,0)	-824199120	(9,7,3,2,2)	733831612704
(9,7,3,3,1)	20194851840	(9,7,4,2,1)	4741984896
(9,7,4,3,0)	-3110582880	(9,7,5,1,1)	-46978560
(9,7,5,2,0)	-203310240	(9,7,6,1,0)	-251520
(9,8,2,2,2)	652777584	(9,8,3,2,1)	19494816
(9,8,3,3,0)	-9395616	(9,8,4,1,1)	73824
(9,8,4,2,0)	-1679040	(9,8,5,1,0)	-2208
(10,4,3,3,3)	13029814091424	(10,4,4,3,2)	3154648420512
(10,4,4,4,1)	20875131744	(10,5,3,3,2)	894337855968
(10,5,4,2,2)	208350582720	(10,5,4,3,1)	5848333440
(10,5,4,4,0)	-824199120	(10,5,5,2,1)	351706176
(10,5,5,3,0)	-203310240	(10,6,3,2,2)	14105356368
(10,6,3,3,1)	411633120	(10,6,4,2,1)	86694528
(10,6,4,3,0)	-46049040	(10,6,5,1,1)	73824
(10,6,5,2,0)	-1679040	(10,6,6,1,0)	-48
(10,7,2,2,2)	22958688	(10,7,3,2,1)	679968
(10,7,3,3,0)	-251520	(10,7,4,1,1)	4320
(10,7,4,2,0)	-29136	(10,8,2,2,1)	-48
(11,3,3,3,3)	99761664	(11,4,3,3,2)	19494816
(11,4,4,2,2)	3692400	(11,4,4,3,1)	73824
(11,4,4,4,0)	-29136	(11,5,3,2,2)	679968
(11,5,3,3,1)	30720	(11,5,4,2,1)	4320
(11,5,4,3,0)	-2208	(11,6,2,2,2)	528

deg(I) = 24

I	d_I	I	d_I
(5,5,5,5,4)	113989024543595926131184	(6,5,5,4,4)	49208142659321800005216
(6,5,5,5,3)	1911288345346635589568	(6,6,4,4,4)	2110546919582910040272
(6,6,5,4,3)	816025913316248333120	(6,6,5,5,2)	90095498612893458272

Continued on the following page

deg(I) = 24, *continued*			
I	d_I	I	d_I
(6,6,6,3,3)	132628671085592097024	(6,6,6,4,2)	37906436889576198396
(6,6,6,5,1)	88798104825584352	(6,6,6,6,0)	-3238317070364520
(7,5,4,4,4)	928035242813679386880	(7,5,5,4,3)	356555435715410957376
(7,5,5,5,2)	38978014973683179264	(7,6,4,4,3)	149677043087457971504
(7,6,5,3,3)	56981868551941098752	(7,6,5,4,2)	16227227828645916976
(7,6,5,5,1)	38035145900721056	(7,6,6,3,2)	2517326301790872576
(7,6,6,4,1)	15666901809220000	(7,6,6,5,0)	-1345692401785920
(7,7,4,3,3)	9881551554602229408	(7,7,4,4,2)	2792965386085320880
(7,7,5,3,2)	1040501263302552608	(7,7,5,4,1)	6536958768444736
(7,7,5,5,0)	-548768158982912	(7,7,6,2,2)	42751365570060704
(7,7,6,3,1)	950472340606656	(7,7,6,4,0)	-219322647849280
(7,7,7,2,1)	10748457996288	(7,7,7,3,0)	-12215408263200
(8,4,4,4,4)	72441168290052535416	(8,5,4,4,3)	27232968315848722400
(8,5,5,3,3)	10189105935322611904	(8,5,5,4,2)	2874629165830386272
(8,5,5,5,1)	6730910671190400	(8,6,4,3,3)	4109159918573469824
(8,6,4,4,2)	1154720984430059840	(8,6,5,3,2)	425294962053433344
(8,6,5,4,1)	2697580082412768	(8,6,5,5,0)	-219322647849280
(8,6,6,2,2)	16827059517736904	(8,6,6,3,1)	382345702776672
(8,6,6,4,0)	-85781801535360	(8,7,3,3,3)	237188563748902272
(8,7,4,3,2)	65682988009064448	(8,7,4,4,1)	423655401709600
(8,7,5,2,2)	6466615047655104	(8,7,5,3,1)	150108480756288
(8,7,5,4,0)	-32779917518560	(8,7,6,2,1)	4272120540288
(8,7,6,3,0)	-4429601736480	(8,7,7,1,1)	-35309984768
(8,7,7,2,0)	-55127514240	(8,8,3,3,2)	1185430621063936
(8,8,4,2,2)	316457581635168	(8,8,4,3,1)	7826902863232
(8,8,4,4,0)	-1559208686912	(8,8,5,2,1)	604016711936
(8,8,5,3,0)	-531223501536	(8,8,6,1,1)	-9200679520
(8,8,6,2,0)	-16642956928	(8,8,7,1,0)	-16170272
(8,8,8,0,0)	4	(9,4,4,4,3)	742174782726416480
(9,5,4,3,3)	265869597857942752	(9,5,4,4,2)	73138223025414832
(9,5,5,3,2)	25872896752378400	(9,5,5,4,1)	168513518883456
(9,5,5,5,0)	-12215408263200	(9,6,3,3,3)	35367733329831456
(9,6,4,3,2)	9613082109166896	(9,6,4,4,1)	63004921310816
(9,6,5,2,2)	894838196834976	(9,6,5,3,1)	21688029832256
(9,6,5,4,0)	-4429601736480	(9,6,6,2,1)	604322817696
(9,6,6,3,0)	-531223501536	(9,7,3,3,2)	418651558115072
(9,7,4,2,2)	110090172437152	(9,7,4,3,1)	2778335397408
(9,7,4,4,0)	-531223501536	(9,7,5,2,1)	213298398784
(9,7,5,3,0)	-174588053440	(9,7,6,1,1)	-2202805408
(9,7,6,2,0)	-4775506080	(9,7,7,1,0)	-3031872
(9,8,3,2,2)	1103727042528	(9,8,3,3,1)	30154035584
(9,8,4,2,1)	7112117856	(9,8,4,3,0)	-4775506080
(9,8,5,1,1)	-87015936	(9,8,5,2,0)	-327015680
(9,8,6,1,0)	-484896	(9,9,2,2,2)	230549312
(9,9,3,2,1)	6953664	(9,9,3,3,0)	-3031872
(9,9,4,1,1)	42560	(9,9,4,2,0)	-484896
(9,9,5,1,0)	-224	(10,4,4,3,3)	1803381971700144
(10,4,4,4,2)	470537427014352	(10,5,3,3,3)	586171325733792
(10,5,4,3,2)	151342528026688	(10,5,4,4,1)	1011188967744
(10,5,5,2,2)	12091316695232	(10,5,5,3,1)	318330381792
(10,5,5,4,0)	-55127514240	(10,6,3,3,2)	15219924472416
(10,6,4,2,2)	3775716012840	(10,6,4,3,1)	101366312448
(10,6,4,4,0)	-16642956928	(10,6,5,2,1)	7112117856
(10,6,5,3,0)	-4775506080	(10,6,6,1,1)	-12495360
(10,6,6,2,0)	-76341160	(10,7,3,2,2)	87650018048
(10,7,3,3,1)	2496782816	(10,7,4,2,1)	561090816
(10,7,4,3,0)	-327015680	(10,7,5,1,1)	-1114976

Continued on the following page

$\deg(I) = 24$, *continued*			
I	d_I	I	d_I
(10, 7, 5, 2, 0)	-16170272	(10, 7, 6, 1, 0)	-5600
(10, 8, 2, 2, 2)	40083960	(10, 8, 3, 2, 1)	1194656
(10, 8, 3, 3, 0)	-484896	(10, 8, 4, 1, 1)	10400
(10, 8, 4, 2, 0)	-61760	(11, 4, 3, 3, 3)	385951211712
(11, 4, 4, 3, 2)	87650018048	(11, 4, 4, 4, 1)	561090816
(11, 5, 3, 3, 2)	22327107072	(11, 5, 4, 2, 2)	4750051104
(11, 5, 4, 3, 1)	138982240	(11, 5, 4, 4, 0)	-16170272
(11, 5, 5, 2, 1)	6414464	(11, 5, 5, 3, 0)	-3031872
(11, 6, 3, 2, 2)	230549312	(11, 6, 3, 3, 1)	6953664
(11, 6, 4, 2, 1)	1194656	(11, 6, 4, 3, 0)	-484896
(11, 6, 5, 1, 1)	928	(11, 6, 5, 2, 0)	-5600
(11, 7, 2, 2, 2)	29808	(11, 7, 3, 2, 1)	928
(11, 7, 3, 3, 0)	-224	(12, 3, 3, 3, 3)	-3031872
(12, 4, 3, 3, 2)	-484896	(12, 4, 4, 2, 2)	-61760
(12, 4, 4, 3, 1)	-5600	(12, 4, 4, 4, 0)	4
(12, 5, 3, 2, 2)	-5600	(12, 5, 3, 3, 1)	-224
(12, 6, 2, 2, 2)	4		

$\deg(I) = 25$			
I	d_I	I	d_I
(5, 5, 5, 5, 5)	205994740015586336392704	(6, 5, 5, 5, 4)	91183351722137909496000
(6, 6, 5, 4, 4)	40163166616613493587568	(6, 6, 5, 5, 3)	15832086999292901742336
(6, 6, 6, 4, 3)	6912402731596987607520	(6, 6, 6, 5, 2)	782286842060773523040
(6, 6, 6, 6, 1)	786907500135989952	(7, 5, 5, 4, 4)	18266693407737182177184
(7, 5, 5, 5, 3)	7166819934960635255808	(7, 6, 4, 4, 4)	7947181852777331886432
(7, 6, 5, 4, 3)	3105932030883925870176	(7, 6, 5, 5, 2)	348836588057363657472
(7, 6, 6, 3, 3)	519156967290582783648	(7, 6, 6, 4, 2)	149323799470083244320
(7, 6, 6, 5, 1)	348635537576206944	(7, 6, 6, 6, 0)	-1346196999093600
(7, 7, 4, 4, 3)	590236080316778614944	(7, 7, 5, 3, 3)	227671001933190288384
(7, 7, 5, 4, 2)	65291193554279696544	(7, 7, 5, 5, 1)	152724917794335744
(7, 7, 6, 3, 2)	10495212981639920256	(7, 7, 6, 4, 1)	64196301394125984
(7, 7, 6, 5, 0)	-5758034709276000	(7, 7, 7, 2, 2)	193409949629545344
(7, 7, 7, 3, 1)	4142934152742912	(7, 7, 7, 4, 0)	-1000067627051904
(8, 5, 4, 4, 4)	1592870480807489129616	(8, 5, 5, 4, 3)	614995894410428607648
(8, 5, 5, 5, 2)	67768604147499442944	(8, 6, 4, 4, 3)	260001810225111467808
(8, 6, 5, 3, 3)	99531632175997196352	(8, 6, 5, 4, 2)	28431306071376793872
(8, 6, 5, 5, 1)	66591254167125120	(8, 6, 6, 3, 2)	4476560442767249184
(8, 6, 6, 4, 1)	27663244132912416	(8, 6, 6, 5, 0)	-2421429008571216
(8, 7, 4, 3, 3)	17566723688872737408	(8, 7, 4, 4, 2)	4983325664622060480
(8, 7, 5, 3, 2)	1869909171436016640	(8, 7, 5, 4, 1)	11668379661795360
(8, 7, 5, 5, 0)	-1000067627051904	(8, 7, 6, 2, 2)	78667271480859264
(8, 7, 6, 3, 1)	1722815728418688	(8, 7, 6, 4, 0)	-405156007308576
(8, 7, 7, 2, 1)	19460158766688	(8, 7, 7, 3, 0)	-23657221999872
(8, 8, 3, 3, 3)	439198792163267520	(8, 8, 4, 3, 2)	122281608111146832
(8, 8, 4, 4, 1)	784225690196688	(8, 8, 5, 2, 2)	12238194838809408
(8, 8, 5, 3, 1)	280061917894368	(8, 8, 5, 4, 0)	-62415555336480
(8, 8, 6, 2, 1)	7932337626384	(8, 8, 6, 3, 0)	-8738280013680
(8, 8, 7, 1, 1)	-83145266112	(8, 8, 7, 2, 0)	-119442727776
(8, 8, 8, 1, 0)	-46049040	(9, 4, 4, 4, 4)	5450576724026912236 8
(9, 5, 4, 4, 3)	2042586909220917254 4	(9, 5, 5, 3, 3)	7616709702249560064
(9, 5, 5, 4, 2)	2144674430360819712	(9, 5, 5, 5, 1)	5019686189816832
(9, 6, 4, 3, 3)	3057298493530985568	(9, 6, 4, 4, 2)	857299960689895392
(9, 6, 5, 3, 2)	314475153857119104	(9, 6, 5, 4, 1)	2001252354617280
(9, 6, 5, 5, 0)	-160791639748800	(9, 6, 6, 2, 2)	12276819589743504
(9, 6, 6, 3, 1)	281059260470880	(9, 6, 6, 4, 0)	-62415555336480

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deg(I) = 25, *continued*			
I	d_I	I	d_I
(9,7,3,3,3)	173735334568551936	(9,7,4,3,2)	47974716780673152
(9,7,4,4,1)	310320577356576	(9,7,5,2,2)	4682747289739200
(9,7,5,3,1)	109488862752768	(9,7,5,4,0)	-23657221999872
(9,7,6,2,1)	3117411980160	(9,7,6,3,0)	-3138370134624
(9,7,7,1,1)	-22739284992	(9,7,7,2,0)	-37176746592
(9,8,3,3,2)	840536568752160	(9,8,4,2,2)	223306712255904
(9,8,4,3,1)	5559537648384	(9,8,4,4,0)	-1093125957120
(9,8,5,2,1)	428753909184	(9,8,5,3,0)	-368134832160
(9,8,6,1,1)	-5776067616	(9,8,6,2,0)	-11043084816
(9,8,7,1,0)	-9395616	(9,9,3,2,2)	733831612704
(9,9,3,3,1)	20194851840	(9,9,4,2,1)	4741984896
(9,9,4,3,0)	-3110582880	(9,9,5,1,1)	-46978560
(9,9,5,2,0)	-203310240	(9,9,6,1,0)	-251520
(10,4,4,4,3)	211913083229294304	(10,5,4,3,3)	74503089764268384
(10,5,4,4,2)	20254395759934128	(10,5,5,3,2)	7011987726247008
(10,5,5,4,1)	46131775979616	(10,5,5,5,0)	-3138370134624
(10,6,3,3,3)	9449678610817056	(10,6,4,3,2)	2530955656217280
(10,6,4,4,1)	16731712682064	(10,6,5,2,2)	225463832566752
(10,6,5,3,1)	5619611350656	(10,6,5,4,0)	-1093125957120
(10,6,6,2,1)	149497953456	(10,6,6,3,0)	-119442727776
(10,7,3,3,2)	99761061359136	(10,7,4,2,2)	25630803734064
(10,7,4,3,1)	665062141248	(10,7,4,4,0)	-119442727776
(10,7,5,2,1)	49806889344	(10,7,5,3,0)	-37176746592
(10,7,6,1,1)	-273996960	(10,7,6,2,0)	-824199120
(10,7,7,1,0)	-251520	(10,8,3,2,2)	208350582720
(10,8,3,3,1)	5848333440	(10,8,4,2,1)	1342319904
(10,8,4,3,0)	-824199120	(10,8,5,1,1)	-6126048
(10,8,5,2,0)	-46049040	(10,8,6,1,0)	-29136
(10,9,2,2,2)	22958688	(10,9,3,2,1)	679968
(10,9,3,3,0)	-251520	(10,9,4,1,1)	4320
(10,9,4,2,0)	-29136	(11,4,4,3,3)	135453779066496
(11,4,4,4,2)	34155140507184	(11,5,3,3,3)	41704406393856
(11,5,4,3,2)	10348372749216	(11,5,4,4,1)	68863079616
(11,5,5,2,2)	733831612704	(11,5,5,3,1)	20194851840
(11,5,5,4,0)	-3110582880	(11,6,3,3,2)	894337855968
(11,6,4,2,2)	208350582720	(11,6,4,3,1)	5848333440
(11,6,4,4,0)	-824199120	(11,6,5,2,1)	351706176
(11,6,5,3,0)	-203310240	(11,6,6,1,1)	73824
(11,6,6,2,0)	-1679040	(11,7,3,2,2)	3347625888
(11,7,3,3,1)	99761664	(11,7,4,2,1)	19494816
(11,7,4,3,0)	-9395616	(11,7,5,1,1)	30720
(11,7,5,2,0)	-251520	(11,8,2,2,2)	104352
(11,8,3,2,1)	4320	(11,8,3,3,0)	-2208
(11,8,4,2,0)	-48	(12,4,3,3,3)	411633120
(12,4,4,3,2)	86694528	(12,4,4,4,1)	-317232
(12,5,3,3,2)	19494816	(12,5,4,2,2)	3692400
(12,5,4,3,1)	73824	(12,5,4,4,0)	-29136
(12,5,5,2,1)	4320	(12,5,5,3,0)	-2208
(12,6,3,2,2)	104352	(12,6,3,3,1)	4320
(12,6,4,2,1)	528	(12,6,4,3,0)	-48
(12,7,2,2,2)	-48		

$\deg(I) = 26$			
I	d_I	I	d_I
(6,5,5,5,5)	1942248611676865375886400	(6,6,5,5,4)	8755904998208130914966888
(6,6,6,4,4)	3931690063036520950722272	(6,6,6,5,3)	1571200062089302681789444
(6,6,6,6,2)	81064530082626311365444	(7,5,5,5,4)	4104263789406502230111136
(7,6,5,4,4)	18329089286737704916704	(7,6,5,5,3)	72977409472143740469600
(7,6,6,4,3)	3234973665410396756924	(7,6,6,5,2)	37215250892747358899952
(7,6,6,6,1)	378252678543507840	(7,7,4,4,4)	37482832972502804145504
(7,7,5,4,3)	148166987359764353029444	(7,7,5,5,2)	16939050423445666178888
(7,7,6,3,3)	25517788707978107690888	(7,7,6,4,2)	73864812380865247905 6
(7,7,6,5,1)	1714200026193282432	(7,7,6,6,0)	-70199768003592720
(7,7,7,3,2)	55156658249311147008	(7,7,7,4,1)	330197068437575904
(7,7,7,5,0)	-30974226462689184	(8,5,5,4,4)	3948041810698146653352 0
(8,5,5,5,3)	15578338962013218600864	(8,6,4,4,4)	17316166253014236259776
(8,6,5,4,3)	68085485198144158800 00	(8,6,5,5,2)	7720324750827831167 52
(8,6,6,3,3)	11560706118776618222 88	(8,6,6,4,2)	3336852765078324988 68
(8,6,6,5,1)	776764446454911264	(8,6,6,6,0)	-30974226462442944
(8,7,4,4,3)	13200635340105590777 76	(8,7,5,3,3)	5129036254585041408 00
(8,7,5,4,2)	1476539371021004673 60	(8,7,5,5,1)	344594558995733856
(8,7,6,3,2)	2420567745952147987 2	(8,7,6,4,1)	146449762016579712
(8,7,6,5,0)	-13461969999093600	(8,7,7,2,2)	466497292363869360
(8,7,7,3,1)	9764259633947136	(8,7,7,4,0)	-2421429008571216
(8,8,4,3,3)	41111092665529983696	(8,8,4,4,2)	11721366876475799760
(8,8,5,3,2)	4443062729668252800	(8,8,5,4,1)	27442949686074432
(8,8,5,5,0)	-2421429008571216	(8,8,6,2,2)	193232919723748536
(8,8,6,3,1)	4137353643774720	(8,8,6,4,0)	-1000067627325696
(8,8,7,2,1)	46040552189760	(8,8,7,3,0)	-62415555336480
(8,8,8,1,1)	-285485916336	(8,8,8,2,0)	-368134868160
(9,5,4,4,4)	15928704808074891296 16	(9,5,5,4,3)	6149958944104286076 48
(9,5,5,5,2)	6776860414749944294 4	(9,6,4,4,3)	26000181022511146780 8
(9,6,5,3,3)	9953163217599719635 2	(9,6,5,4,2)	2843130607137679387 2
(9,6,5,5,1)	665912541671251 20	(9,6,6,3,2)	4476560442767249184
(9,6,6,4,1)	27663244132912416	(9,6,6,5,0)	-2421429008571216
(9,7,4,3,3)	1756672368887273740 8	(9,7,4,4,2)	4983325664622060480
(9,7,5,3,2)	1869909171436016640	(9,7,5,4,1)	11668379661795360
(9,7,5,5,0)	-1000067627051904	(9,7,6,2,2)	78667271480859264
(9,7,6,3,1)	1722815728418688	(9,7,6,4,0)	-405156007308576
(9,7,7,2,1)	19460158766688	(9,7,7,3,0)	-23657221999872
(9,8,3,3,3)	439198792163267520	(9,8,4,3,2)	122281608111146832
(9,8,4,4,1)	784225690196688	(9,8,5,2,2)	12238194838809408
(9,8,5,3,1)	280061917894368	(9,8,5,4,0)	-62415555336480
(9,8,6,2,1)	7932337626384	(9,8,6,3,0)	-8738280013680
(9,8,7,1,1)	-83145266112	(9,8,7,2,0)	-119442727776
(9,8,8,1,0)	-46049040	(9,9,3,3,2)	840536568752160
(9,9,4,2,2)	223306712255904	(9,9,4,3,1)	5559537648384
(9,9,4,4,0)	-1093125957120	(9,9,5,2,1)	428753909184
(9,9,5,3,0)	-368134832160	(9,9,6,1,1)	-5776067616
(9,9,6,2,0)	-11043084816	(9,9,7,1,0)	-9395616
(10,4,4,4,4)	229626348393344730 72	(10,5,4,4,3)	852117415640173536 0
(10,5,5,3,3)	314447808960525072 0	(10,5,5,4,2)	879997438952410320
(10,5,5,5,1)	2055320972401920	(10,6,4,3,3)	1243841228095131744
(10,6,4,4,2)	346443574026127704	(10,6,5,3,2)	125469245902996512
(10,6,5,4,1)	805991883499728	(10,6,5,5,0)	-62415555336480
(10,6,6,2,2)	4694772196282128	(10,6,6,3,1)	109867633989312
(10,6,6,4,0)	-23657222126952	(10,7,3,3,3)	67323025901888832
(10,7,4,3,2)	18421518301369920	(10,7,4,4,1)	120114507109632
(10,7,5,2,2)	1749317561578944	(10,7,5,3,1)	41786034116160
(10,7,5,4,0)	-8738280013680	(10,7,6,2,1)	1179552933984

Continued on the following page

$\deg(I) = 26$, *continued*			
I	d_I	I	d_I
(10,7,6,3,0)	-1093125957120	(10,7,7,1,1)	-5776067616
(10,7,7,2,0)	-11043084816	(10,8,3,3,2)	294058742512224
(10,8,4,2,2)	76906534439280	(10,8,4,3,1)	1954674541824
(10,8,4,4,0)	-368134868160	(10,8,5,2,1)	149497953456
(10,8,5,3,0)	-119442727776	(10,8,6,1,1)	-1334560224
(10,8,6,2,0)	-3110590260	(10,8,7,1,0)	-1679040
(10,9,3,2,2)	208350582720	(10,9,3,3,1)	5848333440
(10,9,4,2,1)	1342319904	(10,9,4,3,0)	-824199120
(10,9,5,1,1)	-6126048	(10,9,5,2,0)	-46049040
(10,9,6,1,0)	-29136	(10,10,2,2,2)	3666312
(10,10,3,2,1)	104352	(10,10,3,3,0)	-29136
(10,10,4,1,1)	528	(10,10,4,2,0)	-2292
(11,4,4,4,3)	29809312235610960	(11,5,4,3,3)	10159668608774304
(11,5,4,4,2)	2707370108500416	(11,5,5,3,2)	903625742797728
(11,5,5,4,1)	6007581031968	(11,5,5,5,0)	-368134832160
(11,6,3,3,3)	1190848151512512	(11,6,4,3,2)	310831260169488
(11,6,4,4,1)	2072265197088	(11,6,5,2,2)	25630803734064
(11,6,5,3,1)	665062141248	(11,6,5,4,0)	-119442727776
(11,6,6,2,1)	15766948032	(11,6,6,3,0)	-11043084816
(11,7,3,3,2)	10348372749216	(11,7,4,2,2)	2545705442112
(11,7,4,3,1)	68863079616	(11,7,4,4,0)	-11043084816
(11,7,5,2,1)	4741984896	(11,7,5,3,0)	-3110582880
(11,7,6,1,1)	-6126048	(11,7,6,2,0)	-46049040
(11,7,7,1,0)	-2208	(11,8,3,2,2)	14105356368
(11,8,3,3,1)	411633120	(11,8,4,2,1)	86694528
(11,8,4,3,0)	-46049040	(11,8,5,1,1)	73824
(11,8,5,2,0)	-1679040	(11,8,6,1,0)	-48
(11,9,2,2,2)	104352	(11,9,3,2,1)	4320
(11,9,3,3,0)	-2208	(11,9,4,2,0)	-48
(12,4,4,3,3)	3154648420512	(12,4,4,4,2)	755118268080
(12,5,3,3,3)	894337855968	(12,5,4,3,2)	208350582720
(12,5,4,4,1)	1342319904	(12,5,5,2,2)	12168742800
(12,5,5,3,1)	351706176	(12,5,5,4,0)	-46049040
(12,6,3,3,2)	14105356368	(12,6,4,2,2)	2937953580
(12,6,4,3,1)	86694528	(12,6,4,4,0)	-9396672
(12,6,5,2,1)	3692400	(12,6,5,3,0)	-1679040
(12,6,6,1,1)	528	(12,6,6,2,0)	-2292
(12,7,3,2,2)	22958688	(12,7,3,3,1)	679968
(12,7,4,2,1)	104352	(12,7,4,3,0)	-29136
(12,7,5,2,0)	-48	(12,8,2,2,2)	-2292
(12,8,3,2,1)	-48	(13,4,3,3,3)	-9395616
(13,4,4,3,2)	-1679040	(13,4,4,4,1)	-29136
(13,5,3,3,2)	-251520	(13,5,4,2,2)	-29136
(13,5,4,3,1)	-2208	(13,6,3,2,2)	-48

$\deg(I) = 27$			
I	d_I	I	d_I
(6,6,5,5,5)	21652833697465345825473216	(6,6,6,5,4)	9930103217664737498362752
(6,6,6,6,3)	1836744316255728093377504	(7,5,5,5,5)	10424016378695267249799168
(7,6,5,5,4)	4760519298349779631098496	(7,6,6,4,4)	2166849385502855883360624
(7,6,6,5,3)	874186334599890564570432	(7,6,6,6,2)	46465382392154451939072
(7,7,5,4,4)	1028056808615702374843808	(7,7,5,5,3)	413486815490189461567488
(7,7,6,4,3)	186191937088342721822720	(7,7,6,5,2)	21768872364464612982848
(7,7,6,6,1)	22298898608487186016	(7,7,7,3,3)	15448516510046832328704
(7,7,7,4,2)	4498860807385583308224	(7,7,7,5,1)	10343645286248254464

Continued on the following page

deg(I) = 27, *continued*			
I	d_I	I	d_I
(7,7,7,6,0)	-448609384225666560	(8,5,5,5,4)	1092905783425309905030912
(8,6,5,4,4)	492306078671771466820064	(8,6,5,5,3)	197228805113267715996416
(8,6,6,4,3)	88255562432100365628448	(8,6,6,5,2)	10255257716817787654800
(8,6,6,6,1)	10474506505204294080	(8,7,4,4,4)	102771309648644620264992
(8,7,5,4,3)	40909136655697997550496	(8,7,5,5,2)	4727639000530609605248
(8,7,6,3,3)	7175765870665787614560	(8,7,6,4,2)	2085054959697204491664
(8,7,6,5,1)	4813453384531950048	(8,7,6,6,0)	-204417944244944160
(8,7,7,3,2)	161547529688023582240	(8,7,7,4,1)	952690031556127200
(8,7,7,5,0)	-91914770340089280	(8,8,4,4,3)	3788700251223612606992
(8,8,5,3,3)	1485322356047272900832	(8,8,5,4,2)	429548694784566103072
(8,8,5,5,1)	998655920868343392	(8,8,6,3,2)	72150775131428807216
(8,8,6,4,1)	430155649709142688	(8,8,6,5,0)	-40752668020556480
(8,8,7,2,2)	1469239641997479152	(8,8,7,3,1)	29808201189352224
(8,8,7,4,0)	-7656245352109120	(8,8,8,2,1)	137150695673232
(8,8,8,3,0)	-219322647849280	(9,5,5,4,4)	50915057095808531822560
(9,5,5,5,3)	20127314721021911676928	(9,6,4,4,4)	22390120482647854754272
(9,6,5,4,3)	8820756832744533115968	(9,6,5,5,2)	1003282043598184664064
(9,6,6,3,3)	1505336844006318854880	(9,6,6,4,2)	434985047971986769440
(9,6,6,5,1)	1011424106940266496	(9,6,6,6,0)	-40752668020556480
(9,7,4,4,3)	1721332405243795563776	(9,7,5,3,3)	670374726927803695104
(9,7,5,4,2)	193221962896538278144	(9,7,5,5,1)	450524230700139520
(9,7,6,3,2)	31875712216477298048	(9,7,6,4,1)	192137049043395264
(9,7,6,5,0)	-17802901998922240	(9,7,7,2,2)	623145599186074560
(9,7,7,3,1)	12940864211175424	(9,7,7,4,0)	-3238317072282240
(9,8,4,3,3)	54398198798074388800	(9,8,4,4,2)	15534320958543602432
(9,8,5,3,2)	5907389929476004224	(9,8,5,4,1)	36360838742410016
(9,8,5,5,0)	-3238317072282240	(9,8,6,2,2)	259636655223086848
(9,8,6,3,1)	5516943176462464	(9,8,6,4,0)	-1345692401785920
(9,8,7,2,1)	60842853741632	(9,8,7,3,0)	-85781801925696
(9,8,8,1,1)	-425570551040	(9,8,8,2,0)	-531223501536
(9,9,3,3,3)	595724740895599616	(9,9,4,3,2)	166286435635301824
(9,9,4,4,1)	1063377379790976	(9,9,5,2,2)	16773356868339520
(9,9,5,3,1)	381144946134016	(9,9,5,4,0)	-85781801925696
(9,9,6,2,1)	10747142102848	(9,9,6,3,0)	-12215408263200
(9,9,7,1,1)	-126220459008	(9,9,7,2,0)	-174588053440
(9,9,8,1,0)	-76342880	(10,5,4,4,4)	928035242813679386880
(10,5,5,4,3)	356555435715410957376	(10,5,5,5,2)	38978014973683179264
(10,6,4,4,3)	149677043087457971504	(10,6,5,3,3)	56981868551941098752
(10,6,5,4,2)	16227227828645916976	(10,6,5,5,1)	38035145900721056
(10,6,6,3,2)	2517326301790872576	(10,6,6,4,1)	15666901809220000
(10,6,6,5,0)	-1345692401785920	(10,7,4,3,3)	9881551554602229408
(10,7,4,4,2)	2792965386085320880	(10,7,5,3,2)	1040501263302552608
(10,7,5,4,1)	6536958768444736	(10,7,5,5,0)	-548768158982912
(10,7,6,2,2)	42751365570060704	(10,7,6,3,1)	950472340606656
(10,7,6,4,0)	-219322647849280	(10,7,7,2,1)	10748457996288
(10,7,7,3,0)	-12215408263200	(10,8,3,3,3)	237188563748902272
(10,8,4,3,2)	65682988009064448	(10,8,4,4,1)	423655401709600
(10,8,5,2,2)	6466615047655104	(10,8,5,3,1)	150108480756288
(10,8,5,4,0)	-32779917518560	(10,8,6,2,1)	4272120540288
(10,8,6,3,0)	-4429601736480	(10,8,7,1,1)	-35309984768
(10,8,7,2,0)	-55127514240	(10,8,8,1,0)	-16170272
(10,9,3,3,2)	418651558115072	(10,9,4,2,2)	110090172437152
(10,9,4,3,1)	2778335397408	(10,9,4,4,0)	-531223501536
(10,9,5,2,1)	213298398784	(10,9,5,3,0)	-174588053440
(10,9,6,1,1)	-2202805408	(10,9,6,2,0)	-4775506080
(10,9,7,1,0)	-3031872	(10,10,3,2,2)	87650018048
(10,10,3,3,1)	2496782816	(10,10,4,2,1)	561090816

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deg(I) = 27, *continued*

I	d_I	I	d_I
(10, 10, 4, 3, 0)	-327015680	(10, 10, 5, 1, 1)	-1114976
(10, 10, 5, 2, 0)	-16170272	(10, 10, 6, 1, 0)	-5600
(11, 4, 4, 4, 4)	5224733955268106112	(11, 5, 4, 4, 3)	1905129808949968992
(11, 5, 5, 3, 3)	689915910456635392	(11, 5, 5, 4, 2)	190939831236687552
(11, 5, 5, 5, 1)	442883019280896	(11, 6, 4, 3, 3)	265869597857942752
(11, 6, 4, 4, 2)	73138223025414832	(11, 6, 5, 3, 2)	25872896752378400
(11, 6, 5, 4, 1)	168513518883456	(11, 6, 5, 5, 0)	-12215408263200
(11, 6, 6, 2, 2)	894838196834976	(11, 6, 6, 3, 1)	21688029832256
(11, 6, 6, 4, 0)	-4429601736480	(11, 7, 3, 3, 3)	13183353406838784
(11, 7, 4, 3, 2)	3545177906512576	(11, 7, 4, 4, 1)	23383915823040
(11, 7, 5, 2, 2)	319546789488192	(11, 7, 5, 3, 1)	7905805097984
(11, 7, 5, 4, 0)	-1559208760288	(11, 7, 6, 2, 1)	213298398784
(11, 7, 6, 3, 0)	-174588053440	(11, 7, 7, 1, 1)	-471020544
(11, 7, 7, 2, 0)	-1292723968	(11, 8, 3, 3, 2)	47693058783296
(11, 8, 4, 2, 2)	12091316695232	(11, 8, 4, 3, 1)	318330381792
(11, 8, 4, 4, 0)	-55127514240	(11, 8, 5, 2, 1)	23350187616
(11, 8, 5, 3, 0)	-16642969280	(11, 8, 6, 1, 1)	-87015936
(11, 8, 6, 2, 0)	-327015680	(11, 8, 7, 1, 0)	-61920
(11, 9, 3, 2, 2)	22327107072	(11, 9, 3, 3, 1)	646886400
(11, 9, 4, 2, 1)	138982240	(11, 9, 4, 3, 0)	-76342880
(11, 9, 5, 1, 1)	75776	(11, 9, 5, 2, 0)	-3031872
(11, 9, 6, 1, 0)	-224	(11, 10, 2, 2, 2)	29808
(11, 10, 3, 2, 1)	928	(11, 10, 3, 3, 0)	-224
(12, 4, 4, 4, 3)	1803381971700144	(12, 5, 4, 3, 3)	586171325733792
(12, 5, 4, 4, 2)	151342528026688	(12, 5, 5, 3, 2)	47693058783296
(12, 5, 5, 4, 1)	318330381792	(12, 5, 5, 5, 0)	-16642969280
(12, 6, 3, 3, 3)	60862991224384	(12, 6, 4, 3, 2)	15219924472416
(12, 6, 4, 4, 1)	101366312448	(12, 6, 5, 2, 2)	1103727042528
(12, 6, 5, 3, 1)	30154035584	(12, 6, 5, 4, 0)	-4775506080
(12, 6, 6, 2, 1)	554248640	(12, 6, 6, 3, 0)	-327015680
(12, 7, 3, 3, 2)	385951211712	(12, 7, 4, 2, 2)	87650018048
(12, 7, 4, 3, 1)	2496782816	(12, 7, 4, 4, 0)	-327015680
(12, 7, 5, 2, 1)	138982240	(12, 7, 5, 3, 0)	-76342880
(12, 7, 6, 1, 1)	42560	(12, 7, 6, 2, 0)	-484896
(12, 8, 3, 2, 2)	230549312	(12, 8, 3, 3, 1)	6953664
(12, 8, 4, 2, 1)	1194656	(12, 8, 4, 3, 0)	-484896
(12, 8, 5, 1, 1)	928	(12, 8, 5, 2, 0)	-5600
(12, 9, 2, 2, 2)	-5600	(12, 9, 3, 2, 1)	-224
(13, 4, 4, 3, 3)	2496782816	(13, 4, 4, 4, 2)	561090816
(13, 5, 3, 3, 3)	646886400	(13, 5, 4, 3, 2)	138982240
(13, 5, 4, 4, 1)	-1114976	(13, 5, 5, 2, 2)	6414464
(13, 5, 5, 3, 1)	75776	(13, 5, 5, 4, 0)	-61920
(13, 6, 3, 3, 2)	6953664	(13, 6, 4, 2, 2)	1194656
(13, 6, 4, 3, 1)	42560	(13, 6, 4, 4, 0)	-5600
(13, 6, 5, 2, 1)	928	(13, 6, 5, 3, 0)	-224
(13, 7, 3, 2, 2)	928		

deg(I) = 28

I	d_I	I	d_I
(6, 6, 6, 5, 5)	281396105446038829764570336	(6, 6, 6, 6, 4)	131098620977119640532677640
(7, 6, 5, 5, 5)	138237049377631956406444992	(7, 6, 6, 5, 4)	64187540696210944731760560
(7, 6, 6, 6, 3)	12132928161031878393585456	(7, 7, 5, 5, 4)	31280567724121492258032672
(7, 7, 6, 4, 4)	14433618175802344857159648	(7, 7, 6, 5, 3)	5877368216672968441640544
(7, 7, 6, 6, 2)	321577464094704294681456	(7, 7, 7, 4, 3)	1295336936098611323052960
(7, 7, 7, 5, 2)	153782147365152593438592	(7, 7, 7, 6, 1)	158207445568417102656

Continued on the following page

deg(I) = 28, *continued*

I	d_I	I	d_I
(7,7,7,7,0)	-3435204329200397376	(8,5,5,5,5)	33565543023690642272679360
(8,6,5,5,4)	15461693507386297016774064	(8,6,6,4,4)	7101369488031789966243504
(8,6,6,5,3)	2882884240831835131741344	(8,6,6,6,2)	156242327334663646224888
(8,7,5,4,4)	3408941234035601841250272	(8,7,5,5,3)	1380187929485720284723200
(8,7,6,4,3)	627928503566826840282528	(8,7,6,5,2)	74184255759899839436832
(8,7,6,6,1)	76233400110735695520	(8,7,7,3,3)	53848312436074659256320
(8,7,7,4,2)	15741021131132583500640	(8,7,7,5,1)	35900356719080933088
(8,7,7,6,0)	-1616941505273075616	(8,8,4,4,4)	353812311322644431070252
(8,8,5,4,3)	141959405252146760964096	(8,8,5,5,2)	16604759404714367599872
(8,8,6,3,3)	25428538397714935697520	(8,8,6,4,2)	7419452787535790690304
(8,8,6,5,1)	17001413171806593504	(8,8,6,6,0)	-752251254607854720
(8,8,7,3,2)	599695012064780172192	(8,8,7,4,1)	3469894328671355136
(8,8,7,5,0)	-345708325307878560	(8,8,8,2,2)	5923901626220768640
(8,8,8,3,1)	115480640308379520	(8,8,8,4,0)	-30974226462442944
(9,5,5,5,4)	1771748988189311827248096	(9,6,5,4,4)	801414379844189789048640
(9,6,5,5,3)	322017472455505977946944	(9,6,6,4,3)	144746838738025811348976
(9,6,6,5,2)	16899949831877607148848	(9,6,6,6,1)	17294122346233404288
(9,7,4,4,4)	168965147904886287828768	(9,7,5,4,3)	67481808424198752531456
(9,7,5,5,2)	7838457970390311440448	(9,7,6,3,3)	11940345167441877911232
(9,7,6,4,2)	3475713613337883829632	(9,7,6,5,1)	8000113898763198720
(9,7,6,6,0)	-345708325307878560	(9,7,7,3,2)	274043841200077763232
(9,7,7,4,1)	1603689039494806080	(9,7,7,5,0)	-156843054827785632
(9,8,4,4,3)	6366888695682697842144	(9,8,5,3,3)	2506621507185094907424
(9,8,5,4,2)	726440453016521093136	(9,8,5,5,1)	1685025369541284480
(9,8,6,3,2)	123424186729923646080	(9,8,6,4,1)	730324510146863712
(9,8,6,5,0)	-70199768003592720	(9,8,7,2,2)	2579059445631070176
(9,8,7,3,1)	51494523118178400	(9,8,7,4,0)	-13461969999093600
(9,8,8,2,1)	230627260020528	(9,8,8,3,0)	-405156007308576
(9,9,4,3,3)	94786453063647970848	(9,9,4,4,2)	27149794915803480576
(9,9,5,3,2)	10388937656332753536	(9,9,5,4,1)	63502596678612960
(9,9,5,5,0)	-5758034709276000	(9,9,6,2,2)	465966091929471216
(9,9,6,3,1)	9751729607816928	(9,9,6,4,0)	-2421429008571216
(9,9,7,2,1)	105019516952544	(9,9,7,3,0)	-160791639748800
(9,9,8,1,1)	-929847901440	(9,9,8,2,0)	-1093125957120
(9,9,9,1,0)	-203310240	(10,5,5,4,4)	39480418106981466533520
(10,5,5,5,3)	15578338962013218600864	(10,6,4,4,4)	17316166253014236259776
(10,6,5,4,3)	6808548519814415880000	(10,6,5,5,2)	772032475082783116752
(10,6,6,3,3)	1156070611877661822288	(10,6,6,4,2)	333685276507832498868
(10,6,6,5,1)	776764446454911264	(10,6,6,6,0)	-30974226462442944
(10,7,4,4,3)	1320063534010559077776	(10,7,5,3,3)	512903625458504140800
(10,7,5,4,2)	147653937102100467360	(10,7,5,5,1)	344594558995733856
(10,7,6,3,2)	24205677459521479872	(10,7,6,4,1)	146449762016579712
(10,7,6,5,0)	-13461969999093600	(10,7,7,2,2)	466497292363869360
(10,7,7,3,1)	9764259633947136	(10,7,7,4,0)	-2421429008571216
(10,8,4,3,3)	41111092665529983696	(10,8,4,4,2)	11721366876475799760
(10,8,5,3,2)	4443062729668252800	(10,8,5,4,1)	27442949686074432
(10,8,5,5,0)	-2421429008571216	(10,8,6,2,2)	193232919723748536
(10,8,6,3,1)	4137353643774720	(10,8,6,4,0)	-1000067627325696
(10,8,7,2,1)	46040552189760	(10,8,7,3,0)	-62415555336480
(10,8,8,1,1)	-285485916336	(10,8,8,2,0)	-368134868160
(10,9,3,3,3)	439198792163267520	(10,9,4,3,2)	122281608111146832
(10,9,4,4,1)	784225690196688	(10,9,5,2,2)	12238194838809408
(10,9,5,3,1)	280061917894368	(10,9,5,4,0)	-62415555336480
(10,9,6,2,1)	7932337626384	(10,9,6,3,0)	-8738280013680
(10,9,7,1,1)	-83145266112	(10,9,7,2,0)	-119442727776
(10,9,8,1,0)	-46049040	(10,10,3,3,2)	294058742512224
(10,10,4,2,2)	76906534439280	(10,10,4,3,1)	1954674541824

Continued on the following page

$\deg(I) = 28$, *continued*

I	d_I	I	d_I
(10, 10, 4, 4, 0)	-368134868160	(10, 10, 5, 2, 1)	149497953456
(10, 10, 5, 3, 0)	-119442727776	(10, 10, 6, 1, 1)	-1334560224
(10, 10, 6, 2, 0)	-3110590260	(10, 10, 7, 1, 0)	-1679040
(11, 5, 4, 4, 4)	309132463141878510000	(11, 5, 5, 4, 3)	117552717975524482368
(11, 5, 5, 5, 2)	12636043517074729152	(11, 6, 4, 4, 3)	48619672420408782672
(11, 6, 5, 3, 3)	18293419124197908384	(11, 6, 5, 4, 2)	5175740190127316160
(11, 6, 5, 5, 1)	12132995150259168	(11, 6, 6, 3, 2)	777895843534310448
(11, 6, 6, 4, 1)	4906985498846880	(11, 6, 6, 5, 0)	-405156007308576
(11, 7, 4, 3, 3)	3057298493530985568	(11, 7, 4, 4, 2)	857299960689895392
(11, 7, 5, 3, 2)	314475153857119104	(11, 7, 5, 4, 1)	2001252354617280
(11, 7, 5, 5, 0)	-160791639748800	(11, 7, 6, 2, 2)	12276819589743504
(11, 7, 6, 3, 1)	281059260470880	(11, 7, 6, 4, 0)	-62415555336480
(11, 7, 7, 2, 1)	3117411980160	(11, 7, 7, 3, 0)	-3138370134624
(11, 8, 3, 3, 3)	67323025901888832	(11, 8, 4, 3, 2)	18421518301369920
(11, 8, 4, 4, 1)	120114507109632	(11, 8, 5, 2, 2)	1749317561578944
(11, 8, 5, 3, 1)	41786034116160	(11, 8, 5, 4, 0)	-8738280013680
(11, 8, 6, 2, 1)	1179552933984	(11, 8, 6, 3, 0)	-1093125957120
(11, 8, 7, 1, 1)	-5776067616	(11, 8, 7, 2, 0)	-11043084816
(11, 8, 8, 1, 0)	-1679040	(11, 9, 3, 3, 2)	99761061359136
(11, 9, 4, 2, 2)	25630803734064	(11, 9, 4, 3, 1)	665062141248
(11, 9, 4, 4, 0)	-119442727776	(11, 9, 5, 2, 1)	49806889344
(11, 9, 5, 3, 0)	-37176746592	(11, 9, 6, 1, 1)	-273996960
(11, 9, 6, 2, 0)	-824199120	(11, 9, 7, 1, 0)	-251520
(11, 10, 3, 2, 2)	14105356368	(11, 10, 3, 3, 1)	411633120
(11, 10, 4, 2, 1)	86694528	(11, 10, 4, 3, 0)	-46049040
(11, 10, 5, 1, 1)	73824	(11, 10, 5, 2, 0)	-1679040
(11, 10, 6, 1, 0)	-48	(11, 11, 2, 2, 2)	528
(12, 4, 4, 4, 4)	597237294763420872	(12, 5, 4, 4, 3)	211913083229294304
(12, 5, 5, 3, 3)	74503089764268384	(12, 5, 5, 4, 2)	20254395759934128
(12, 5, 5, 5, 1)	46131775979616	(12, 6, 4, 3, 3)	27569906261747088
(12, 6, 4, 4, 2)	7431936473155680	(12, 6, 5, 3, 2)	2530955656217280
(12, 6, 5, 4, 1)	16731712682064	(12, 6, 5, 5, 0)	-1093125957120
(12, 6, 6, 2, 2)	76906534439280	(12, 6, 6, 3, 1)	1954674541824
(12, 6, 6, 4, 0)	-368134868160	(12, 7, 3, 3, 3)	1190848151512512
(12, 7, 4, 3, 2)	310831260169488	(12, 7, 4, 4, 1)	2072265197088
(12, 7, 5, 2, 2)	25630803734064	(12, 7, 5, 3, 1)	665062141248
(12, 7, 5, 4, 0)	-119442727776	(12, 7, 6, 2, 1)	15766948032
(12, 7, 6, 3, 0)	-11043084816	(12, 7, 7, 1, 1)	-6126048
(12, 7, 7, 2, 0)	-46049040	(12, 8, 3, 3, 2)	3154648420512
(12, 8, 4, 2, 2)	755118268080	(12, 8, 4, 3, 1)	20875131744
(12, 8, 4, 4, 0)	-3110590260	(12, 8, 5, 2, 1)	1342319904
(12, 8, 5, 3, 0)	-824199120	(12, 8, 6, 1, 1)	-317232
(12, 8, 6, 2, 0)	-9396672	(12, 8, 7, 1, 0)	-48
(12, 9, 3, 2, 2)	652777584	(12, 9, 3, 3, 1)	19494816
(12, 9, 4, 2, 1)	3692400	(12, 9, 4, 3, 0)	-1679040
(12, 9, 5, 1, 1)	4320	(12, 9, 5, 2, 0)	-29136
(12, 10, 2, 2, 2)	-2292	(12, 10, 3, 2, 1)	-48
(13, 4, 4, 4, 3)	34155140507184	(13, 5, 4, 3, 3)	10348372749216
(13, 5, 4, 4, 2)	2545705442112	(13, 5, 5, 3, 2)	733831612704
(13, 5, 5, 4, 1)	4741984896	(13, 5, 5, 5, 0)	-203310240
(13, 6, 3, 3, 3)	894337855968	(13, 6, 4, 3, 2)	208350582720
(13, 6, 4, 4, 1)	1342319904	(13, 6, 5, 2, 2)	12168742800
(13, 6, 5, 3, 1)	351706176	(13, 6, 5, 4, 0)	-46049040
(13, 6, 6, 2, 1)	3692400	(13, 6, 6, 3, 0)	-1679040
(13, 7, 3, 3, 2)	3347625888	(13, 7, 4, 2, 2)	652777584
(13, 7, 4, 3, 1)	19494816	(13, 7, 4, 4, 0)	-1679040
(13, 7, 5, 2, 1)	679968	(13, 7, 5, 3, 0)	-251520

Continued on the following page

$\deg(I) = 28$, *continued*			
I	d_I	I	d_I
$(13,7,6,2,0)$	-48	$(13,8,3,2,2)$	104352
$(13,8,3,3,1)$	4320	$(13,8,4,2,1)$	528
$(13,8,4,3,0)$	-48	$(14,4,4,3,3)$	-46049040
$(14,4,4,4,2)$	-9396672	$(14,5,3,3,3)$	-9395616
$(14,5,4,3,2)$	-1679040	$(14,5,4,4,1)$	-29136
$(14,5,5,2,2)$	-29136	$(14,5,5,3,1)$	-2208
$(14,6,3,3,2)$	-29136	$(14,6,4,2,2)$	-2292
$(14,6,4,3,1)$	-48		

$\deg(I) = 29$			
I	d_I	I	d_I
$(6,6,6,6,5)$	42102606163818323057770680000	$(7,6,6,5,5)$	210739248385457951852149238 4
$(7,6,6,6,4)$	9932682642142516589750659 20	$(7,7,5,5,5)$	105099921320394553093215436 8
$(7,7,6,5,4)$	4939848168668063162138378 88	$(7,7,6,6,3)$	95352673153203400660517280
$(7,7,7,4,4)$	1144246182733128923665203 84	$(7,7,7,5,3)$	47005257591061770949294080
$(7,7,7,6,2)$	26429106519158442844905 60	$(7,7,7,7,1)$	1327111985411944965120
$(8,6,5,5,5)$	5341679801843636520236788 80	$(8,6,6,5,4)$	2501740290241521326129418 72
$(8,6,6,6,3)$	480002824491734628588480 96	$(8,7,5,5,4)$	1233201718025271090678611 52
$(8,7,6,4,4)$	574402103925219755313023 52	$(8,7,6,5,3)$	235380327878273174404308 48
$(8,7,6,6,2)$	13133824881638585905833 60	$(8,7,7,4,3)$	531082413995027918641305 6
$(8,7,7,5,2)$	6369660418385971820723 52	$(8,7,7,6,1)$	655738370875420162368
$(8,7,7,7,0)$	-1500207605159734 3200	$(8,8,5,4,4)$	139373394786670001192364 00
$(8,8,5,5,3)$	5683140308380169857515 456	$(8,8,6,4,3)$	2614878610961089413438 336
$(8,8,6,5,2)$	3123342946473617912295 36	$(8,8,6,6,1)$	321538657422443256528
$(8,8,7,3,3)$	2324640843882285058512 96	$(8,8,7,4,2)$	682166551503065106694 56
$(8,8,7,5,1)$	15393093592710681024 0	$(8,8,7,6,0)$	-7217177104026255792
$(8,8,8,3,2)$	277203631658172875904 0	$(8,8,8,4,1)$	15670595020056748992
$(8,8,8,5,0)$	-1616941505273075616	$(9,5,5,5,5)$	669369163519938856522168 32
$(9,6,5,5,4)$	3098653392387423060521136 0	$(9,6,6,4,4)$	1430511103306682980896427 2
$(9,6,6,5,3)$	5827961507523161595876576	$(9,6,6,6,2)$	319351886711731639092048
$(9,7,5,4,4)$	6913304732636292162876864	$(9,7,5,5,3)$	2809517795490346324727808
$(9,7,6,4,3)$	1285699665017929239033504	$(9,7,6,5,2)$	152785612691569068751872
$(9,7,6,6,1)$	15717408345510855590 4	$(9,7,7,3,3)$	112332804021170119948800
$(9,7,7,4,2)$	32906148327728452809888	$(9,7,7,5,1)$	7464891127545554841 6
$(9,7,7,6,0)$	-3435204329200397376	$(9,8,4,4,4)$	733085308674475362988800
$(9,8,5,4,3)$	2954439570827262347109 12	$(9,8,5,5,2)$	347905209762138935504 64
$(9,8,6,3,3)$	53548008570062637417696	$(9,8,6,4,2)$	15659653975155880831536
$(9,8,6,5,1)$	357054554862821233 92	$(9,8,6,6,0)$	-1616941505273075616
$(9,8,7,3,2)$	12960691031566169527 68	$(9,8,7,4,1)$	741179588639898662 4
$(9,8,7,5,0)$	-752251254614400960	$(9,8,8,2,2)$	1340420997576916939 2
$(9,8,8,3,1)$	254920513927767648	$(9,8,8,4,0)$	-70199768003592720
$(9,9,4,4,3)$	13740363716226750753600	$(9,9,5,3,3)$	5442063622506471346176
$(9,9,5,4,2)$	1581805356183712228896	$(9,9,5,5,1)$	365538475425584179 2
$(9,9,6,3,2)$	2731792530144274440 00	$(9,9,6,4,1)$	1598269886739062400
$(9,9,6,5,0)$	-156843054827785632	$(9,9,7,2,2)$	592179744190913606 4
$(9,9,7,3,1)$	115413671534048256	$(9,9,7,4,0)$	-30974226462689184
$(9,9,8,2,1)$	486039827347776	$(9,9,8,3,0)$	-1000067627051904
$(9,9,9,1,1)$	-2891432289792	$(9,9,9,2,0)$	-3138370134624
$(10,5,5,5,4)$	177174898818931182724809 6	$(10,6,5,4,4)$	80141437984418978904864 0
$(10,6,5,5,3)$	3220174724555059779469 44	$(10,6,6,4,3)$	144746838738025811348976
$(10,6,6,5,2)$	16899949831877607148848	$(10,6,6,6,1)$	1729412234623340428 8
$(10,7,4,4,4)$	168965147904886287828768	$(10,7,5,4,3)$	67481808424198752531456
$(10,7,5,5,2)$	7838457970390311440448	$(10,7,6,3,3)$	11940345167441877911232
$(10,7,6,4,2)$	3475713613337883829632	$(10,7,6,5,1)$	800011389876319872 0
$(10,7,6,6,0)$	-345708325307878560	$(10,7,7,3,2)$	27404384120007776323 2

Continued on the following page

deg(I) = 29, *continued*			
I	d_I	I	d_I
(10,7,7,4,1)	1603689039494806080	(10,7,7,5,0)	-156843054827785632
(10,8,4,4,3)	6366888695682697842144	(10,8,5,3,3)	2506621507185094907424
(10,8,5,4,2)	726440453016521093136	(10,8,5,5,1)	1685025369541284480
(10,8,6,3,2)	123424186729923646080	(10,8,6,4,1)	730324510146863712
(10,8,6,5,0)	-70199768003592720	(10,8,7,2,2)	2579059445631070176
(10,8,7,3,1)	51494523118178400	(10,8,7,4,0)	-13461969999093600
(10,8,8,2,1)	230627260020528	(10,8,8,3,0)	-405156007308576
(10,9,4,3,3)	94786453063647970848	(10,9,4,4,2)	27149794915803480576
(10,9,5,3,2)	10388937656332753536	(10,9,5,4,1)	63502596678612960
(10,9,5,5,0)	-5758034709276000	(10,9,6,2,2)	465966091929471216
(10,9,6,3,1)	9751729607816928	(10,9,6,4,0)	-2421429008571216
(10,9,7,2,1)	105019516952544	(10,9,7,3,0)	-160791639748800
(10,9,8,1,1)	-929847901440	(10,9,8,2,0)	-1093125957120
(10,9,9,1,0)	-203310240	(10,10,3,3,3)	439198792163267520
(10,10,4,3,2)	122281608111146832	(10,10,4,4,1)	784225690196688
(10,10,5,2,2)	12238194838809408	(10,10,5,3,1)	280061917894368
(10,10,5,4,0)	-62415555336480	(10,10,6,2,1)	7932337626384
(10,10,6,3,0)	-8738280013680	(10,10,7,1,1)	-83145266112
(10,10,7,2,0)	-119442727776	(10,10,8,1,0)	-46049040
(11,5,5,4,4)	18266693407737182177184	(11,5,5,5,3)	7166819934960635255808
(11,6,4,4,4)	7947181852777331886432	(11,6,5,4,3)	3105932030883925870176
(11,6,5,5,2)	348836588057363657472	(11,6,6,3,3)	519156967290582783648
(11,6,6,4,2)	149323799470083244320	(11,6,6,5,1)	348635537576206944
(11,6,6,6,0)	-13461969999093600	(11,7,4,4,3)	590236080316778614944
(11,7,5,3,3)	227671001933190288384	(11,7,5,4,2)	65291193554279696544
(11,7,5,5,1)	152724917794335744	(11,7,6,3,2)	10495212981639920256
(11,7,6,4,1)	64196301394125984	(11,7,6,5,0)	-5758034709276000
(11,7,7,2,2)	193409949629545344	(11,7,7,3,1)	4142934152742912
(11,7,7,4,0)	-1000067627051904	(11,8,4,3,3)	17566723688872737408
(11,8,4,4,2)	4983325664622060480	(11,8,5,3,2)	1869909171436016640
(11,8,5,4,1)	11668379661795360	(11,8,5,5,0)	-1000067627051904
(11,8,6,2,2)	78667271480859264	(11,8,6,3,1)	1722815728418688
(11,8,6,4,0)	-405156007308576	(11,8,7,2,1)	19460158766688
(11,8,7,3,0)	-23657221999872	(11,8,8,1,1)	-83145266112
(11,8,8,2,0)	-119442727776	(11,9,3,3,3)	173735334568551936
(11,9,4,3,2)	47974716780673152	(11,9,4,4,1)	310320577356576
(11,9,5,2,2)	4682747289739200	(11,9,5,3,1)	109488862752768
(11,9,5,4,0)	-23657221999872	(11,9,6,2,1)	3117411980160
(11,9,6,3,0)	-3138370134624	(11,9,7,1,1)	-22739284992
(11,9,7,2,0)	-37176746592	(11,9,8,1,0)	-9395616
(11,10,3,3,2)	99761061359136	(11,10,4,2,2)	25630803734064
(11,10,4,3,1)	665062141248	(11,10,4,4,0)	-119442727776
(11,10,5,2,1)	49806889344	(11,10,5,3,0)	-37176746592
(11,10,6,1,1)	-273996960	(11,10,6,2,0)	-824199120
(11,10,7,1,0)	-251520	(11,11,3,2,2)	3347625888
(11,11,3,3,1)	99761664	(11,11,4,2,1)	19494816
(11,11,4,3,0)	-9395616	(11,11,5,1,1)	30720
(11,11,5,2,0)	-251520	(12,5,4,4,4)	56439747241501122192
(12,5,5,4,3)	21104661093843211008	(12,5,5,5,2)	2206547301748229184
(12,6,4,4,3)	8521174156401735360	(12,6,5,3,3)	3144478089605250720
(12,6,5,4,2)	879997438952410320	(12,6,5,5,1)	2055320972401920
(12,6,6,3,2)	125469245902996512	(12,6,6,4,1)	805991883499728
(12,6,6,5,0)	-62415555336480	(12,7,4,3,3)	494845153306899264
(12,7,4,4,2)	136879891485939456	(12,7,5,3,2)	48909204818311680
(12,7,5,4,1)	316717033197408	(12,7,5,5,0)	-23657221999872
(12,7,6,2,2)	1749317561578944	(12,7,6,3,1)	41786034116160
(12,7,6,4,0)	-8738280013680	(12,7,7,2,1)	428753909184

Continued on the following page

deg(I) = 29, *continued*			
I	d_I	I	d_I
(12,7,7,3,0)	-368134832160	(12,8,3,3,3)	9449678610817056
(12,8,4,3,2)	2530955656217280	(12,8,4,4,1)	16731712682064
(12,8,5,2,2)	225463832566752	(12,8,5,3,1)	5619611350656
(12,8,5,4,0)	-1093125957120	(12,8,6,2,1)	149497953456
(12,8,6,3,0)	-119442727776	(12,8,7,1,1)	-273996960
(12,8,7,2,0)	-824199120	(12,8,8,1,0)	-29136
(12,9,3,3,2)	10348372749216	(12,9,4,2,2)	2545705442112
(12,9,4,3,1)	68863079616	(12,9,4,4,0)	-11043084816
(12,9,5,2,1)	4741984896	(12,9,5,3,0)	-3110582880
(12,9,6,1,1)	-6126048	(12,9,6,2,0)	-46049040
(12,9,7,1,0)	-2208	(12,10,3,2,2)	652777584
(12,10,3,3,1)	19494816	(12,10,4,2,1)	3692400
(12,10,4,3,0)	-1679040	(12,10,5,1,1)	4320
(12,10,5,2,0)	-29136	(12,11,2,2,2)	-48
(13,4,4,4,4)	29809312235610960	(13,5,4,4,3)	10159668608774304
(13,5,5,3,3)	3417190702574592	(13,5,5,4,2)	903625742797728
(13,5,5,5,1)	1974181959168	(13,6,4,3,3)	1190848151512512
(13,6,4,4,2)	310831260169488	(13,6,5,3,2)	99761061359136
(13,6,5,4,1)	665062141248	(13,6,5,5,0)	-37176746592
(13,6,6,2,2)	2461712752416	(13,6,6,3,1)	66382892544
(13,6,6,4,0)	-11043084816	(13,7,3,3,3)	41704406393856
(13,7,4,3,2)	10348372749216	(13,7,4,4,1)	68863079616
(13,7,5,2,2)	733831612704	(13,7,5,3,1)	20194851840
(13,7,5,4,0)	-3110582880	(13,7,6,2,1)	351706176
(13,7,6,3,0)	-203310240	(13,7,7,1,1)	30720
(13,7,7,2,0)	-251520	(13,8,3,3,2)	65707393920
(13,8,4,2,2)	14105356368	(13,8,4,3,1)	411633120
(13,8,4,4,0)	-46049040	(13,8,5,2,1)	19494816
(13,8,5,3,0)	-9395616	(13,8,6,1,1)	4320
(13,8,6,2,0)	-29136	(13,9,3,2,2)	679968
(13,9,3,3,1)	30720	(13,9,4,2,1)	4320
(13,9,4,3,0)	-2208	(14,4,4,4,3)	20875131744
(14,5,4,3,3)	5848333440	(14,5,4,4,2)	1342319904
(14,5,5,3,2)	351706176	(14,5,5,4,1)	-6126048
(14,5,5,5,0)	-251520	(14,6,3,3,3)	411633120
(14,6,4,3,2)	86694528	(14,6,4,4,1)	-317232
(14,6,5,2,2)	3692400	(14,6,5,3,1)	73824
(14,6,5,4,0)	-29136	(14,6,6,2,1)	528
(14,6,6,3,0)	-48	(14,7,3,3,2)	679968
(14,7,4,2,2)	104352	(14,7,4,3,1)	4320
(14,7,4,4,0)	-48	(14,8,3,2,2)	-48

E.3 The list of webs

Table 14: The positive webs, together with $\mathbb{W}_+$, and the instanton numbers n_W and d_W, for source vectors up to degree 19. deg(W) denotes the degree of the source vector $\mathring{I}$ of the corresponding web, and $h(W)$ is defined as $h(W) = h(\mathring{I})$.

Web	deg(W)	$h(W)$	n_W	d_W
$\mathbb{W}_+$[(1,0,0,0,0)]	1	1	24	0
W[(1,1,1,0,0)]	3	0	112	0
W[(1,1,1,1,0)]	4	-2	1104	0
W[(1,1,1,1,1)]	5	-5	19200	0
W[(2,1,1,1,1)]	6	-6	45408	0
W[(2,2,1,1,0)]	6	-3	2800	0
W[(2,2,1,1,1)]	7	-8	212880	0
W[(2,2,2,0,0)]	6	0	80	4
W[(2,2,2,1,0)]	7	-5	14496	-48
W[(2,2,2,1,1)]	8	-11	1691856	528
W[(2,2,2,2,0)]	8	-8	122352	-2292
W[(2,2,2,2,1)]	9	-15	20299992	29808
W[(2,2,2,2,2)]	10	-20	341681280	3666312
W[(3,2,2,1,1)]	9	-12	3222112	928
W[(3,2,2,2,0)]	9	-9	234048	-5600
W[(3,2,2,2,1)]	10	-17	63576576	104352
W[(3,2,2,2,2)]	11	-23	1599622824	22958688
W[(3,3,1,1,1)]	9	-9	434688	0
W[(3,3,2,1,0)]	9	-6	30624	-224
W[(3,3,2,1,1)]	10	-14	10883712	4320
W[(3,3,2,2,0)]	10	-11	795936	-29136
W[(3,3,2,2,1)]	11	-20	316997280	679968
W[(3,3,2,2,2)]	12	-27	11032046624	230549312
W[(3,3,3,0,0)]	9	0	112	0
W[(3,3,3,1,0)]	10	-8	122448	-2208
W[(3,3,3,1,1)]	11	-17	59097600	30720
W[(3,3,3,2,0)]	11	-14	4326048	-251520
W[(3,3,3,2,1)]	12	-24	2322325968	6953664
W[(3,3,3,2,2)]	13	-32	105371446464	3347625888
W[(3,3,3,3,0)]	12	-18	33777312	-3031872
W[(3,3,3,3,1)]	13	-29	23351460864	99761664
W[(3,3,3,3,2)]	14	-38	1326841710624	65707393920
W[(3,3,3,3,3)]	15	-45	21228933784320	1668835805184
W[(4,2,2,2,2)]	12	-24	2624447520	40083960
W[(4,3,2,2,1)]	12	-21	529392832	1194656
W[(4,3,2,2,2)]	13	-29	27607031136	652777584
W[(4,3,3,1,1)]	12	-18	100919904	42560
W[(4,3,3,2,0)]	12	-15	7371792	-484896
W[(4,3,3,2,1)]	13	-26	5950086192	19494816
W[(4,3,3,2,2)]	14	-35	377080188864	14105356368
W[(4,3,3,3,0)]	13	-20	88179456	-9395616
W[(4,3,3,3,1)]	14	-32	85495746528	411633120
W[(4,3,3,3,2)]	15	-42	6446376071472	385951211712
W[(4,3,3,3,3)]	16	-50	134508124418928	13029814091424
W[(4,4,2,1,1)]	12	-15	19420400	10400
W[(4,4,2,2,0)]	12	-12	1423104	-61760
W[(4,4,2,2,1)]	13	-23	1426637712	3692400
W[(4,4,2,2,2)]	14	-32	103492041648	2937953580

Continued on next page

Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(4,4,3,1,0)]	12	-9	234048	-5600
W[(4,4,3,1,1)]	13	-20	282674592	73824
W[(4,4,3,2,0)]	13	-17	20578560	-1679040
W[(4,4,3,2,1)]	14	-29	22951602432	86694528
W[(4,4,3,2,2)]	15	-39	1912895782008	87650018048
W[(4,4,3,3,0)]	14	-23	347078520	-46049040
W[(4,4,3,3,1)]	15	-36	443961562528	2496782816
W[(4,4,3,3,2)]	16	-47	42411173392368	3154648420512
W[(4,4,3,3,3)]	17	-56	1112487680575968	135453779066496
W[(4,4,4,0,0)]	12	0	80	4
W[(4,4,4,1,0)]	13	-11	795936	-29136
W[(4,4,4,1,1)]	14	-23	1218252960	-317232
W[(4,4,4,2,0)]	14	-20	88177920	-9396672
W[(4,4,4,2,1)]	15	-33	126121309632	561090816
W[(4,4,4,2,2)]	16	-44	13138629854976	755118268080
W[(4,4,4,3,0)]	15	-27	1935300720	-327015680
W[(4,4,4,3,1)]	16	-41	3114669545280	20875131744
W[(4,4,4,3,2)]	17	-53	363393804317664	34155140507184
W[(4,4,4,3,3)]	18	-63	11630106886504344	1803381971700144
W[(4,4,4,4,0)]	16	-32	14386855920	-3110590260
W[(4,4,4,4,1)]	17	-47	28258960027296	228415121472
W[(4,4,4,4,2)]	18	-60	3920585033699328	470537427014352
W[(4,4,4,4,3)]	19	-71	149583407202367176	29809312235610960
W[(4,4,4,4,4)]	20	-80	2315758601706011520	597237294763420872
W[(5,3,3,2,2)]	15	-36	570360079168	22327107072
W[(5,3,3,3,1)]	15	-33	130194945024	646886400
W[(5,3,3,3,2)]	16	-44	13834674726336	894337855968
W[(5,3,3,3,3)]	17	-53	389973010495488	41704406393856
W[(5,4,2,2,2)]	15	-33	158730945984	4750051104
W[(5,4,3,2,1)]	15	-30	35487082592	138982240
W[(5,4,3,2,2)]	16	-41	4183230238656	208350582720
W[(5,4,3,3,0)]	15	-24	539120544	-76342880
W[(5,4,3,3,1)]	16	-38	980247769056	5848333440
W[(5,4,3,3,2)]	17	-50	125365423769760	10348372749216
W[(5,4,3,3,3)]	18	-60	4272828104425920	586171325733792
W[(5,4,4,1,1)]	15	-24	1944767152	-1114976
W[(5,4,4,2,0)]	15	-21	140436672	-16170272
W[(5,4,4,2,1)]	16	-35	285207114048	1342319904
W[(5,4,4,2,2)]	17	-47	39692266181304	2545705442112
W[(5,4,4,3,0)]	16	-29	4392333792	-824199120
W[(5,4,4,3,1)]	17	-44	9502910875584	68863079616
W[(5,4,4,3,2)]	18	-57	1423524718242752	151342528026688
W[(5,4,4,3,3)]	19	-68	57309129620711136	10159668608774304
W[(5,4,4,4,0)]	17	-35	45007048752	-11043084816
W[(5,4,4,4,1)]	18	-51	114110495895360	1011188967744
W[(5,4,4,4,2)]	19	-65	19680157760407104	2707370108500416
W[(5,4,4,4,3)]	20	-77	920246692052672448	211913083229294304
W[(5,4,4,4,4)]	21	-87	17389206433621316832	5224733955268106112
W[(5,5,2,2,1)]	15	-24	2306418848	6414464
W[(5,5,2,2,2)]	16	-35	366406656528	12168742800
W[(5,5,3,1,1)]	15	-21	464696832	75776
W[(5,5,3,2,0)]	15	-18	33777312	-3031872
W[(5,5,3,2,1)]	16	-32	83099778720	351706176
W[(5,5,3,2,2)]	17	-44	13073262151968	733831612704
W[(5,5,3,3,0)]	16	-26	1272585120	-203310240
W[(5,5,3,3,1)]	17	-41	3100342138368	20194851840
W[(5,5,3,3,2)]	18	-54	507096396665312	47693058783296

Continued on next page

Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(5,5,3,3,3)]	19	-65	21671962905320448	3417190702574592
W[(5,5,4,1,0)]	15	-12	1423616	-61920
W[(5,5,4,1,1)]	16	-26	4826161680	-6126048
W[(5,5,4,2,0)]	16	-23	347078520	-46049040
W[(5,5,4,2,1)]	17	-38	931163905728	4741984896
W[(5,5,4,2,2)]	18	-51	164605655104880	12091316695232
W[(5,5,4,3,0)]	17	-32	14386869840	-3110582880
W[(5,5,4,3,1)]	18	-48	39821013536096	318330381792
W[(5,5,4,3,2)]	19	-62	7371081117191712	903625742797728
W[(5,5,4,3,3)]	20	-74	362176732991882256	74503089764268384
W[(5,5,4,4,0)]	18	-39	193411225936	-55127514240
W[(5,5,4,4,1)]	19	-56	609209937409968	6007581031968
W[(5,5,4,4,2)]	20	-71	126656377507736616	20254395759934128
W[(5,5,4,4,3)]	21	-84	7079567101109436512	1905129808949968992
W[(5,5,4,4,4)]	22	-95	159832960277398698312	56439747241501122192
W[(5,5,5,0,0)]	15	0	112	0
W[(5,5,5,1,0)]	16	-14	4326048	-251520
W[(5,5,5,1,1)]	17	-29	17798444544	-46978560
W[(5,5,5,2,0)]	17	-26	1272585120	-203310240
W[(5,5,5,2,1)]	18	-42	4217701870608	23350187616
W[(5,5,5,2,2)]	19	-56	905275754212800	77522333436960
W[(5,5,5,3,0)]	18	-36	65215603200	-16642969280
W[(5,5,5,3,1)]	19	-53	221145135246336	1974181959168
W[(5,5,5,3,2)]	20	-68	48949713376347552	7011987726247008
W[(5,5,5,3,3)]	21	-81	2860072289627444736	689915910456635392
W[(5,5,5,4,0)]	19	-44	1096632180480	-368134832160
W[(5,5,5,4,1)]	20	-62	4162140562025760	46131775979616
W[(5,5,5,4,2)]	21	-78	1017289744237857120	190939831236687552
W[(5,5,5,4,3)]	22	-92	66573482065327669440	21104661093843211008
W[(5,5,5,4,4)]	23	-104	1763388068567027864736	736557199694836925664
W[(5,5,5,5,0)]	20	-50	7888589144400	-3138370134624
W[(5,5,5,5,1)]	21	-69	35306571598392576	442883019280896
W[(5,5,5,5,2)]	22	-86	9952370045915290464	2206547301748229184
W[(5,5,5,5,3)]	23	-101	749974117128947225088	281693674984303028736
W[(5,5,5,5,4)]	24	-114	22958958469178899286112	11398902454359592613184
W[(5,5,5,5,5)]	25	-125	347718598088041789328640	205994740015586336392704
W[(6,3,3,3,3)]	18	-54	552486590320032	60862991224384
W[(6,4,3,3,2)]	18	-51	178677828494464	15219924472416
W[(6,4,3,3,3)]	19	-62	8236673292611808	1190848151512512
W[(6,4,4,2,2)]	18	-48	56949598227232	3775716012840
W[(6,4,4,3,1)]	18	-45	13674852866304	101366312448
W[(6,4,4,3,2)]	19	-59	2768640614245200	310831260169488
W[(6,4,4,3,3)]	20	-71	145074948270672288	27569906261747088
W[(6,4,4,4,0)]	18	-36	65215569408	-16642956928
W[(6,4,4,4,1)]	19	-53	224917616990784	2072265197088
W[(6,4,4,4,2)]	20	-68	50310287851264512	7431936473155680
W[(6,4,4,4,3)]	21	-81	2968386268852263168	742174782726416480
W[(6,4,4,4,4)]	22	-92	69702170473826178048	22962634839334473072
W[(6,5,3,2,2)]	18	-45	18954386538304	1103727042528
W[(6,5,3,3,1)]	18	-42	4510722900128	30154035584
W[(6,5,3,3,2)]	19	-56	1000740719949936	99761061359136
W[(6,5,3,3,3)]	20	-68	55724768553096576	9449678610817056
W[(6,5,4,2,1)]	18	-39	1367836823744	7112117856
W[(6,5,4,2,2)]	19	-53	328447354833120	25630803734064
W[(6,5,4,3,0)]	18	-33	21143067840	-4775506080
W[(6,5,4,3,1)]	19	-50	79804026346992	665062141248
W[(6,5,4,3,2)]	20	-65	19159936729163904	2530955656217280

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Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(6,5,4,3,3)]	21	-78	1187054464752608224	265869597857942752
W[(6,5,4,4,0)]	19	-41	391409808576	-119442727776
W[(6,5,4,4,1)]	20	-59	1608297381675072	16731712682064
W[(6,5,4,4,2)]	21	-75	419478239436537264	73138223025414832
W[(6,5,4,4,3)]	22	-89	28814753795787304128	8521174156401735360
W[(6,5,4,4,4)]	23	-101	790863904443723569376	309132463141878510000
W[(6,5,5,1,1)]	18	-30	27120466144	-87015936
W[(6,5,5,2,0)]	18	-27	1935300720	-327015680
W[(6,5,5,2,1)]	19	-44	8748592415904	49806889344
W[(6,5,5,2,2)]	20	-59	2428815576573408	225463832566752
W[(6,5,5,3,0)]	19	-38	135171775392	-37176746592
W[(6,5,5,3,1)]	20	-56	596073535387056	5619611350656
W[(6,5,5,3,2)]	21	-72	165119843412344816	25872896752378400
W[(6,5,5,3,3)]	22	-86	11833136668383611040	3144478089605250720
W[(6,5,5,4,0)]	20	-47	2981800050480	-1093125957120
W[(6,5,5,4,1)]	21	-66	14258867760974432	168513518883456
W[(6,5,5,4,2)]	22	-83	4252005327651223776	879997438952410320
W[(6,5,5,4,3)]	23	-98	334370838391810248432	117552717975524482368
W[(6,5,5,4,4)]	24	-111	10565118218002014469248	4920814265932180005216
W[(6,5,5,5,0)]	21	-54	27765085214112	-12215408263200
W[(6,5,5,5,1)]	22	-74	152435152838866176	2055320972401920
W[(6,5,5,5,2)]	23	-92	51265779665018065536	12636043517074729152
W[(6,5,5,5,3)]	24	-108	4556958521329222612288	1911288345346635589568
W[(6,5,5,5,4)]	25	-122	163766423699355653551056	91183351722137909496000
W[(6,5,5,5,5)]	26	-134	2910174233830401416162688	1942248611676865375886400
W[(6,6,2,2,2)]	18	-36	551803842816	19270744144
W[(6,6,3,2,1)]	18	-33	125948336640	554248640
W[(6,6,3,2,2)]	19	-47	39360165257928	2461712752416
W[(6,6,3,3,0)]	18	-27	1935300720	-327015680
W[(6,6,3,3,1)]	19	-44	9425697295296	66382892544
W[(6,6,3,3,2)]	20	-59	2713101057421728	294058742512224
W[(6,6,3,3,3)]	21	-72	190193228131870512	35367733329831456
W[(6,6,4,1,1)]	18	-27	7510615200	-12495360
W[(6,6,4,2,0)]	18	-24	539115744	-76341160
W[(6,6,4,2,1)]	19	-41	2910089695872	15766948032
W[(6,6,4,2,2)]	20	-56	903893653068672	76906534439280
W[(6,6,4,3,0)]	19	-35	45007048752	-11043084816
W[(6,6,4,3,1)]	20	-53	220840621188096	1954674541824
W[(6,6,4,3,2)]	21	-69	66233922634330080	9613082109166896
W[(6,6,4,3,3)]	22	-83	5023740750844977792	1243841228095131744
W[(6,6,4,4,0)]	20	-44	1096632086784	-368134868160
W[(6,6,4,4,1)]	21	-63	5658979212554128	63004921310816
W[(6,6,4,4,2)]	22	-80	1795314514514344416	346443574026127704
W[(6,6,4,4,3)]	23	-95	147828049737997120632	48619672420408782672
W[(6,6,4,4,4)]	24	-108	4831190355131709036288	2110546919582910040272
W[(6,6,5,1,0)]	18	-15	7371792	-484896
W[(6,6,5,1,1)]	19	-32	61773182400	-273996960
W[(6,6,5,2,0)]	19	-29	4392333792	-824199120
W[(6,6,5,2,1)]	20	-47	25377635878296	149497953456
W[(6,6,5,2,2)]	21	-63	8718347106041576	894838196834976
W[(6,6,5,3,0)]	20	-41	391409808576	-119442727776
W[(6,6,5,3,1)]	21	-60	2150266975191936	21688029832256
W[(6,6,5,3,2)]	22	-77	721163569257189312	125469245902996512
W[(6,6,5,3,3)]	23	-92	61757539943858380704	18293419124197908384
W[(6,6,5,4,0)]	21	-51	10848408360480	-4429601736480
W[(6,6,5,4,1)]	22	-71	63276065657309280	805991883499728
W[(6,6,5,4,2)]	23	-89	22420496977021999680	5175740190127316160

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Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(6,6,5,4,3)]	24	-105	2073506802039240412736	816025913316248333120
W[(6,6,5,4,4)]	25	-119	76746430278444036385392	40163166616613493587568
W[(6,6,5,5,0)]	22	-59	126532108859856	-62415555336480
W[(6,6,5,5,1)]	23	-80	830212985215356432	12132995150259168
W[(6,6,5,5,2)]	24	-99	325919053252224299168	90095498612893458272
W[(6,6,5,5,3)]	25	-116	33557221088952835248384	15832086999292901742336
W[(6,6,5,5,4)]	26	-131	1393776642755701910391504	875590499820813091496688
W[(6,6,5,5,5)]	27	-144	28656849112544426796718608	21652833697465345825473216
W[(6,6,6,0,0)]	18	0	80	4
W[(6,6,6,1,0)]	19	-17	20578560	-1679040
W[(6,6,6,1,1)]	20	-35	203336907216	-1334560224
W[(6,6,6,2,0)]	20	-32	14386855920	-3110590260
W[(6,6,6,2,1)]	21	-51	99894446151552	604322817696
W[(6,6,6,2,2)]	22	-68	40767562975883520	4694772196282128
W[(6,6,6,3,0)]	21	-45	1535514818112	-531223501536
W[(6,6,6,3,1)]	22	-65	10102374952223232	109867633989312
W[(6,6,6,3,2)]	23	-83	3982038442779651984	777895843534310448
W[(6,6,6,3,3)]	24	-99	398117530652334602832	132628671085592097024
W[(6,6,6,4,0)]	22	-56	51294956593632	-23657222126952
W[(6,6,6,4,1)]	23	-77	354725791310991552	4906985498846880
W[(6,6,6,4,2)]	24	-96	145958872012447992432	37906436889576198396
W[(6,6,6,4,3)]	25	-113	15579801166584314831616	6912402731596987607520
W[(6,6,6,4,4)]	26	-128	664776534906643820467776	393169006303652095072272
W[(6,6,6,5,0)]	23	-65	725912434085952	-405156007308576
W[(6,6,6,5,1)]	24	-87	5571858588504821712	88798104825584352
W[(6,6,6,5,2)]	25	-107	2507158978553441682912	782286842060773523040
W[(6,6,6,5,3)]	26	-125	294433074567120966718080	157120006208930268178944
W[(6,6,6,5,4)]	27	-141	13944411721206459640109952	9930103217664737498362752
W[(6,6,6,5,5)]	28	-155	327684614387349299961738768	281396105446038829764570336
W[(6,6,6,6,0)]	24	-72	5101035241706976	-3238317070364520
W[(6,6,6,6,1)]	25	-95	45075021198059982144	786907500135989952
W[(6,6,6,6,2)]	26	-116	22914149837439123291648	8106453008262631136544
W[(6,6,6,6,3)]	27	-135	3030705830464261116958752	1836744316255728093377504
W[(6,6,6,6,4)]	28	-152	161823659616827892042946656	131098620977119640532677640
W[(6,6,6,6,5)]	29	-167	4300779721074151241480884704	4210260616381832305777068000
W[(7,4,4,3,3)]	21	-72	196866216448867200	38231916995852064
W[(7,4,4,4,2)]	21	-69	68481669752665152	10343926869883680
W[(7,4,4,4,3)]	22	-83	5273427759409817952	1369210879561818480
W[(7,4,4,4,4)]	23	-95	157391952685989052728	54505767240269122368
W[(7,5,3,3,3)]	21	-69	75992812385562624	13183353406838784
W[(7,5,4,3,2)]	21	-66	26217346711258048	3545177906512576
W[(7,5,4,3,3)]	22	-80	2124595552827372432	494845153306899264
W[(7,5,4,4,1)]	21	-60	2211223893638272	23383915823040
W[(7,5,4,4,2)]	22	-77	754387255278771840	136879891485939456
W[(7,5,4,4,3)]	23	-92	65617907601886711296	20425869092209172544
W[(7,5,4,4,4)]	24	-105	2234583126440197477248	928035242813679386880
W[(7,5,5,2,2)]	21	-60	3356453655323136	319546789488192
W[(7,5,5,3,1)]	21	-57	824874647838720	7905805097984
W[(7,5,5,3,2)]	22	-74	299477728365291600	48909204818311680
W[(7,5,5,3,3)]	23	-89	27190620887571766272	7616709702249560064
W[(7,5,5,4,0)]	21	-48	4136092936448	-1559208760288
W[(7,5,5,4,1)]	22	-68	26040136828870752	316717033197408
W[(7,5,5,4,2)]	23	-86	9824161857371476896	2144674430360819712
W[(7,5,5,4,3)]	24	-102	953357306203227960000	356555435715410957376
W[(7,5,5,4,4)]	25	-116	36602428260502812573792	18266693407737182177184
W[(7,5,5,5,0)]	22	-56	51294957112992	-23657221999872
W[(7,5,5,5,1)]	23	-77	358330187751266304	5019686189816832

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Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(7,5,5,5,2)]	24	-96	148388879628408287328	38978014973683179264
W[(7,5,5,5,3)]	25	-113	15930480413967177684480	7166819934960635255808
W[(7,5,5,5,4)]	26	-128	683358195482651060173200	410426378940650223011136
W[(7,5,5,5,5)]	27	-141	14401495635309838652737536	10424016378695267249799168
W[(7,6,3,3,2)]	21	-60	3761948244770304	418651558115072
W[(7,6,3,3,3)]	22	-74	346896207708697296	67323025901888832
W[(7,6,4,2,2)]	21	-57	1259132047619264	110090172437152
W[(7,6,4,3,1)]	21	-54	308134225628128	2778335397408
W[(7,6,4,3,2)]	22	-71	121505012171479176	18421518301369920
W[(7,6,4,3,3)]	23	-86	11689593863624674656	3057298493530985568
W[(7,6,4,4,0)]	21	-45	1535514818112	-531223501536
W[(7,6,4,4,1)]	22	-65	10462960782869952	120114507109632
W[(7,6,4,4,2)]	23	-83	4202606608677077184	857299960689895392
W[(7,6,4,4,3)]	24	-99	426992405813739053760	149677043087457971504
W[(7,6,4,4,4)]	25	-113	16950676810888359150336	7947181852777331886432
W[(7,6,5,2,1)]	21	-48	35929933424832	213298398784
W[(7,6,5,2,2)]	22	-65	16270300857476160	1749317561578944
W[(7,6,5,3,0)]	21	-42	553728279360	-174588053440
W[(7,6,5,3,1)]	22	-62	4021264698687264	41786034116160
W[(7,6,5,3,2)]	23	-80	1706830027589928192	314475153857119104
W[(7,6,5,3,3)]	24	-96	180227194372605425904	56981868551941098752
W[(7,6,5,4,0)]	22	-53	20350993239840	-8738280013680
W[(7,6,5,4,1)]	23	-74	150993571342096992	2001252354617280
W[(7,6,5,4,2)]	24	-93	65822715717348500960	16227227828645916976
W[(7,6,5,4,3)]	25	-110	7345251761305389562560	3105932030883925870176
W[(7,6,5,4,4)]	26	-125	324239422338282700223616	183290892867377049167040
W[(7,6,5,5,0)]	23	-62	305922925426848	-160791639748800
W[(7,6,5,5,1)]	24	-84	2483294974158049312	38035145900721056
W[(7,6,5,5,2)]	25	-104	1172715223879828113648	348836588057363657472
W[(7,6,5,5,3)]	26	-122	143065987059929651882064	72977409472143740469600
W[(7,6,5,5,4)]	27	-138	6979017914791123565948416	4760519298349779631098496
W[(7,6,5,5,5)]	28	-152	167783614906668761262716784	138237049377631956406444992
W[(7,6,6,1,1)]	21	-36	299302640864	-2202805408
W[(7,6,6,2,0)]	21	-33	21143067840	-4775506080
W[(7,6,6,2,1)]	22	-53	194378107421760	1179552933984
W[(7,6,6,2,2)]	23	-71	100108346194477248	12276819589743504
W[(7,6,6,3,0)]	22	-47	2981800050480	-1093125957120
W[(7,6,6,3,1)]	23	-68	24863416450991904	281059260470880
W[(7,6,6,3,2)]	24	-87	12017787465197578008	2517326301790872576
W[(7,6,6,3,3)]	25	-104	1445782834458789325920	519156967290582783648
W[(7,6,6,4,0)]	23	-59	126532108859856	-62415555336480
W[(7,6,6,4,1)]	24	-81	1079565550915867008	15666901809220000
W[(7,6,6,4,2)]	25	-101	533243466879375407808	149323799470083244320
W[(7,6,6,4,3)]	26	-119	67330722644161744497600	32349736654103967569424
W[(7,6,6,4,4)]	27	-135	3369981367793558156370720	2166849385502855883360624
W[(7,6,6,5,0)]	24	-69	2235977596096128	-1345692401785920
W[(7,6,6,5,1)]	25	-92	20734174826253969312	348635537576206944
W[(7,6,6,5,2)]	26	-113	11001712604766772877568	3721525089274735889952
W[(7,6,6,5,3)]	27	-132	1505729217469676504230592	874186334599890564570432
W[(7,6,6,5,4)]	28	-149	82579048510474932784060128	64187540696210944731760560
W[(7,6,6,5,5)]	29	-164	2240812589775895583844156576	2107392483854579518521492384
W[(7,6,6,6,0)]	25	-77	19503820669876800	-13461969999093600
W[(7,6,6,6,1)]	26	-101	204289830851585811840	3782526785435078400
W[(7,6,6,6,2)]	27	-123	120468636234042768002112	46465382392154451939072
W[(7,6,6,6,3)]	28	-143	18312063250016785426332456	12132928161031878393585456
W[(7,6,6,6,4)]	29	-161	1118592262447208419494700224	993268264214251658975065920
W[(7,7,3,2,2)]	21	-48	56389985840000	3650629114944

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Table 15 – continued

Web	deg(W)	h(W)	n_W	d_W
W[(7,7,3,3,1)]	21	-45	13542066341888	97849986048
W[(7,7,3,3,2)]	22	-62	7173870919736064	840536568752160
W[(7,7,3,3,3)]	23	-77	841539378868429824	173735334568551936
W[(7,7,4,2,1)]	21	-42	4216297529824	23296907136
W[(7,7,4,2,2)]	22	-59	2422178398686816	223306712255904
W[(7,7,4,3,0)]	21	-36	65215603200	-16642969280
W[(7,7,4,3,1)]	22	-56	594508678788528	5559537648384
W[(7,7,4,3,2)]	23	-74	297115911452589936	47974716780673152
W[(7,7,4,3,3)]	24	-90	35230204567671156080	9881551554602229408
W[(7,7,4,4,0)]	22	-47	2981800050480	-1093125957120
W[(7,7,4,4,1)]	23	-68	25857038420140320	310320577356576
W[(7,7,4,4,2)]	24	-87	12756877670900976952	2792965386085320880
W[(7,7,4,4,3)]	25	-104	1560763765722117846528	590236080316778614944
W[(7,7,4,4,4)]	26	-119	73760592549207990341160	37482832972502804145504
W[(7,7,5,1,1)]	21	-33	92396257280	-471020544
W[(7,7,5,2,0)]	21	-30	6558863360	-1292723968
W[(7,7,5,2,1)]	22	-50	71274491245200	428753909184
W[(7,7,5,2,2)]	23	-68	40746789567213888	4682747289739200
W[(7,7,5,3,0)]	22	-44	1096632180480	-368134832160
W[(7,7,5,3,1)]	23	-65	10097809547695104	109488862752768
W[(7,7,5,3,2)]	24	-84	5251183397613765152	1040501263302552608
W[(7,7,5,3,3)]	25	-101	666467844013257615360	227671001933190288384
W[(7,7,5,4,0)]	23	-56	51294957112992	-23657221999872
W[(7,7,5,4,1)]	24	-78	468967501173905952	6536958768444736
W[(7,7,5,4,2)]	25	-98	245004909605415502560	65291193554279696544
W[(7,7,5,4,3)]	26	-116	32292308644789613776992	14816698735976435302944
W[(7,7,5,4,4)]	27	-132	1670002775792759585584480	1028056808615702374843808
W[(7,7,5,5,0)]	24	-66	964399018545152	-548768158982912
W[(7,7,5,5,1)]	25	-89	9433084896265973760	152724917794335744
W[(7,7,5,5,2)]	26	-110	5240983985031424336512	1693905042344566617888
W[(7,7,5,5,3)]	27	-129	743754012075104160058368	413486815490189461567488
W[(7,7,5,5,4)]	28	-146	41952996407118579100732512	31280567724121492258032672
W[(7,7,5,5,5)]	29	-161	1163393252471836868861786112	1050999213203945530932154368
W[(7,7,6,1,0)]	21	-18	33777312	-3031872
W[(7,7,6,1,1)]	22	-38	639016897824	-5776067616
W[(7,7,6,2,0)]	22	-35	45007048752	-11043084816
W[(7,7,6,2,1)]	23	-56	515881389602064	3117411980160
W[(7,7,6,2,2)]	24	-75	322136481160659232	42751365570060704
W[(7,7,6,3,0)]	23	-50	7888589144400	-3138370134624
W[(7,7,6,3,1)]	24	-72	80203179581197904	950472340606656
W[(7,7,6,3,2)]	25	-92	46154206945493038080	10495212981639920256
W[(7,7,6,3,3)]	26	-110	6528373563454253739936	2551778870797810769088
W[(7,7,6,4,0)]	24	-63	408865565088240	-219322647849280
W[(7,7,6,4,1)]	25	-86	4182469007721935136	64196301394125984
W[(7,7,6,4,2)]	26	-107	2422666442072438912352	738648123808652479056
W[(7,7,6,4,3)]	27	-126	355021932713338577724288	186191937088342721822720
W[(7,7,6,4,4)]	28	-143	20512359802778934105982824	14433618175802344857159648
W[(7,7,6,5,0)]	25	-74	8765016259161504	-5758034709276000
W[(7,7,6,5,1)]	26	-98	95977233617823957552	1714200026193282432
W[(7,7,6,5,2)]	27	-120	58904390293772710703920	21768872364464612982848
W[(7,7,6,5,3)]	28	-140	9244971964576432583359680	5877368216672968441640544
W[(7,7,6,5,4)]	29	-158	579083249672629145095030368	493984816866806316213837888
W[(7,7,6,6,0)]	26	-83	92700939550359360	-70199768003592720
W[(7,7,6,6,1)]	27	-108	1129438555365292906784	22298898608487186016
W[(7,7,6,6,2)]	28	-131	760089538306826431748976	321577464094704294681456
W[(7,7,6,6,3)]	29	-152	130956575256689626362575040	95352673153203400660517280
W[(7,7,7,0,0)]	21	0	112	0

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Table 15 – continued

Web	deg(*W*)	*h*(*W*)	n_W	d_W
W[(7,7,7,1,0)]	22	-20	88179456	-9395616
W[(7,7,7,1,1)]	23	-41	1927069671936	-22739284992
W[(7,7,7,2,0)]	23	-38	135171775392	-37176746592
W[(7,7,7,2,1)]	24	-60	1823829689450016	10748457996288
W[(7,7,7,2,2)]	25	-80	1329872815417735680	193409949629545344
W[(7,7,7,3,0)]	24	-54	27765085214112	-12215408263200
W[(7,7,7,3,1)]	25	-77	331877990439469056	4142934152742912
W[(7,7,7,3,2)]	26	-98	221273208968851435344	55156658249311147008
W[(7,7,7,3,3)]	27	-117	36027026438881932128256	15448516510046832328704
W[(7,7,7,4,0)]	25	-68	1692511362069504	-1000067627051904
W[(7,7,7,4,1)]	26	-92	20221514209776438144	330197068437575904
W[(7,7,7,4,2)]	27	-114	13449577349429667122112	4498860807385583308224
W[(7,7,7,4,3)]	28	-134	2248511154703665686900736	1295336936098611323052960
W[(7,7,7,4,4)]	29	-152	147818363038488716192722368	114424618273312892366520384
W[(7,7,7,5,0)]	26	-80	42801528146793216	-30974226462689184
W[(7,7,7,5,1)]	27	-105	542448112625738749440	10343645286248254464
W[(7,7,7,5,2)]	28	-128	378607952063724321532320	153782147365152593438592
W[(7,7,7,5,3)]	29	-149	67179875933573491930042368	47005257591061770949294080
W[(7,7,7,6,0)]	27	-90	536474722655969280	-448609384225666560
W[(7,7,7,6,1)]	28	-116	7483863739103052384864	158207445568417102656
W[(7,7,7,6,2)]	29	-140	5668780293104872727438208	2642910651915844284490560
W[(7,7,7,7,0)]	28	-98	3704581973944705776	-3435204329200397376
W[(7,7,7,7,1)]	29	-125	58444196223515692468224	1327111985411944965120
W[(8,4,4,4,4)]	24	-96	205889395932163617312	72441168290052535416
W[(8,5,4,4,3)]	24	-93	86069413996832124352	27232968315848722400
W[(8,5,4,4,4)]	25	-107	3727698169135125498096	1592870480807489129616
W[(8,5,5,3,3)]	24	-90	35767925041629127584	10189105935322611904
W[(8,5,5,4,2)]	24	-87	12945658352928829152	2874629165830386272
W[(8,5,5,4,3)]	25	-104	1597193840680108612560	614995894410428607648
W[(8,5,5,4,4)]	26	-119	76121903698269498879600	39480418106981466533520
W[(8,5,5,5,1)]	24	-78	474753531842519136	6730910671190400
W[(8,5,5,5,2)]	25	-98	250363560088071437904	67768604147499442944
W[(8,5,5,5,3)]	26	-116	33298242026156998722144	15578338962013218600864
W[(8,5,5,5,4)]	27	-132	1737004389084229283537184	1092905783425309905030912
W[(8,5,5,5,5)]	28	-146	43981223391578028025767312	33565543023690642272679360
W[(8,6,4,3,3)]	24	-87	15439042985968145360	4109159918573469824
W[(8,6,4,4,2)]	24	-84	5561099164846002688	1154720984430059840
W[(8,6,4,4,3)]	25	-101	719772958474534446912	260001810225111467808
W[(8,6,4,4,4)]	26	-116	35504946591945154063104	17316166253014236259776
W[(8,6,5,3,2)]	24	-81	2266457718090172160	425294962053433344
W[(8,6,5,3,3)]	25	-98	305275969074406465728	99531632175997196352
W[(8,6,5,4,1)]	24	-75	201010148340707968	2697580082412768
W[(8,6,5,4,2)]	25	-95	111802499676622459032	28431306071376793872
W[(8,6,5,4,3)]	26	-113	15467950978366663032576	6808548519814415880000
W[(8,6,5,4,4)]	27	-129	830547124981920578846912	492306078671771466820064
W[(8,6,5,5,0)]	24	-63	408865565088240	-219322647849280
W[(8,6,5,5,1)]	25	-86	4254301857683952288	66591254167125120
W[(8,6,5,5,2)]	26	-107	2490930240945503131824	772032475082783116752
W[(8,6,5,5,3)]	27	-126	368512581418692060268192	197228805113267715996416
W[(8,6,5,5,4)]	28	-143	21480065424682369924330608	15461693507386297016774064
W[(8,6,5,5,5)]	29	-158	611210312904590985126412992	534167980184363652023678880
W[(8,6,6,2,2)]	24	-72	134490191192202528	16827059517736904
W[(8,6,6,3,1)]	24	-69	33425310221788864	382345702776672
W[(8,6,6,3,2)]	25	-89	20678681485694660736	4476560442767249184
W[(8,6,6,3,3)]	26	-107	3087105410543684178144	1156070611877661822288
W[(8,6,6,4,0)]	24	-60	170193514498560	-85781801535360
W[(8,6,6,4,1)]	25	-83	1864566688856423904	27663244132912416

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Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(8,6,6,4,2)]	26	-104	1142319760025546317200	333685276507832498868
W[(8,6,6,4,3)]	27	-123	174894056440451268167776	88255562432100365628448
W[(8,6,6,4,4)]	28	-140	10456756549205905359458304	7101369488031789966243504
W[(8,6,6,5,0)]	25	-71	3881643757375656	-2421429008571216
W[(8,6,6,5,1)]	26	-95	44858935490060258472	776764446454911264
W[(8,6,6,5,2)]	27	-117	28841649917004887522080	10255257716817787654800
W[(8,6,6,5,3)]	28	-137	4698818465684165361643776	2882884240831835131741344
W[(8,6,6,5,4)]	29	-155	303193957695295574167972368	250174029024152132612941872
W[(8,6,6,6,0)]	26	-80	42801528135993600	-30974226462442944
W[(8,6,6,6,1)]	27	-105	546003716872720702848	10474506505204294080
W[(8,6,6,6,2)]	28	-128	382713191877285462148512	156242327334663646224888
W[(8,6,6,6,3)]	29	-149	68161114972295875241706048	48000282449173462858848096
W[(8,7,3,3,3)]	24	-78	1126346577851592960	237188563748902272
W[(8,7,4,3,2)]	24	-75	398660145480045856	65682988009064448
W[(8,7,4,3,3)]	25	-92	60590920000179493056	17566723688872737408
W[(8,7,4,4,1)]	24	-69	34807080531177792	423655401709600
W[(8,7,4,4,2)]	25	-89	22012186784542835520	4983325664622060480
W[(8,7,4,4,3)]	26	-107	3345260435684206623648	1320063534010559077776
W[(8,7,4,4,4)]	27	-123	192443720708788001680608	102771309648644620264992
W[(8,7,5,2,2)]	24	-69	55084399995750144	6466615047655104
W[(8,7,5,3,1)]	24	-66	13661798641742976	150108480756288
W[(8,7,5,3,2)]	25	-86	9117897040377080832	1869909171436016640
W[(8,7,5,3,3)]	26	-104	1437664544143525963632	512903625458504140800
W[(8,7,5,4,0)]	24	-57	69468841810240	-32779917518560
W[(8,7,5,4,1)]	25	-80	817746667654917168	11668379661795360
W[(8,7,5,4,2)]	26	-101	530380950844598802240	147653937102100467360
W[(8,7,5,4,3)]	27	-120	84779574976667980526368	40909136655697997550496
W[(8,7,5,4,4)]	28	-137	5236870677368742358091328	3408941234035601841250272
W[(8,7,5,5,0)]	25	-68	1692511362069504	-1000067627051904
W[(8,7,5,5,1)]	26	-92	20644575230653895136	344594558995733856
W[(8,7,5,5,2)]	27	-114	13895992208920106396544	4727639000530609605248
W[(8,7,5,5,3)]	28	-134	2346380263735039996887072	1380187929485720284723200
W[(8,7,5,5,4)]	29	-152	155642883145653607449683376	123320171802527109067861152
W[(8,7,6,2,1)]	24	-57	710229601026304	4272120540288
W[(8,7,6,2,2)]	25	-77	571128202199454336	78667271480859264
W[(8,7,6,3,0)]	24	-51	10848408360480	-4429601736480
W[(8,7,6,3,1)]	25	-74	142342287006477504	1722815728418688
W[(8,7,6,3,2)]	26	-95	101657485900434092424	24205677459521479872
W[(8,7,6,3,3)]	27	-114	17421547466353577240096	7175765870665787614560
W[(8,7,6,4,0)]	25	-65	725912434085952	-405156007308576
W[(8,7,6,4,1)]	26	-89	9254014284061822464	146449762016579712
W[(8,7,6,4,2)]	27	-111	6488505552963384035984	2085054959697204491664
W[(8,7,6,4,3)]	28	-131	1130432594965350429319440	627928503566826840282528
W[(8,7,6,4,4)]	29	-149	76752707872006994882755104	57440210392521975531302352
W[(8,7,6,5,0)]	26	-77	19503820669876800	-13461969999093600
W[(8,7,6,5,1)]	27	-102	259867472483449630240	4813453384531950048
W[(8,7,6,5,2)]	28	-125	189389511540055226332992	74184255759899839436832
W[(8,7,6,5,3)]	29	-146	34795170486545493534538224	23538032787827317440430848
W[(8,7,6,6,0)]	27	-87	254791938658803840	-204417944244944160
W[(8,7,6,6,1)]	28	-113	3705422029800182188416	76233400110735695520
W[(8,7,6,6,2)]	29	-137	2913373653270128828358720	1313382488163858590583360
W[(8,7,7,1,1)]	24	-42	2760956499680	-35309984768
W[(8,7,7,2,0)]	24	-39	193411225936	-55127514240
W[(8,7,7,2,1)]	25	-62	3377194221012096	19460158766688
W[(8,7,7,2,2)]	26	-83	3047920567708923264	466497292363869360
W[(8,7,7,3,0)]	25	-56	51294957112992	-23657221999872
W[(8,7,7,3,1)]	26	-80	761479183438470384	9764259633947136

Continued on next page

Table 15 – continued

Web	deg(W)	h(W)	n_W	d_W
W[(8,7,7,3,2)]	27	-102	612363334253114849568	161547529688023582240
W[(8,7,7,3,3)]	28	-122	118167437729899245359856	53848312436074659256320
W[(8,7,7,4,0)]	26	-71	3881643757375656	-2421429008571216
W[(8,7,7,4,1)]	27	-96	56225444916409818816	952690031556127200
W[(8,7,7,4,2)]	28	-119	44276794310053780437840	15741021131132583500640
W[(8,7,7,4,3)]	29	-140	8644490285282936739547776	5310824139950279186413056
W[(8,7,7,5,0)]	27	-84	119602242975339008	-91914770340089280
W[(8,7,7,5,1)]	28	-110	1805398290592460769984	35900356719080933088
W[(8,7,7,5,2)]	29	-134	1469714544745529210370240	636966041838597182072352
W[(8,7,7,6,0)]	28	-95	1810611871504105272	-1616941505273075616
W[(8,7,7,6,1)]	29	-122	29632320884127789040512	655738370875420162368
W[(8,7,7,7,0)]	29	-104	15060587910821007264	-15002076051597343200
W[(8,8,3,3,2)]	24	-63	9867511282052976	1185430621063936
W[(8,8,3,3,3)]	25	-80	2006276928131711424	439198792163267520
W[(8,8,4,2,2)]	24	-60	3345447617629184	316457581635168
W[(8,8,4,3,1)]	24	-57	822222499846400	7826902863232
W[(8,8,4,3,2)]	25	-77	713471511849776160	122281608111146832
W[(8,8,4,3,3)]	26	-95	135135632721772486224	41111092665529983696
W[(8,8,4,4,0)]	24	-48	4136092740352	-1559208686912
W[(8,8,4,4,1)]	25	-71	62675569121448240	784225690196688
W[(8,8,4,4,2)]	26	-92	49318322079952346112	11721366876475799760
W[(8,8,4,4,3)]	27	-111	9090329485714422369312	3788700251223612606992
W[(8,8,4,4,4)]	28	-128	624198973773146079716784	353812311322644431070252
W[(8,8,5,2,1)]	24	-51	99885127862240	604016711936
W[(8,8,5,2,2)]	25	-71	100010833402440120	12238194838809408
W[(8,8,5,3,0)]	24	-45	1535514818112	-531223501536
W[(8,8,5,3,1)]	25	-68	24840263013168672	280061917894368
W[(8,8,5,3,2)]	26	-89	20610476713078747200	4443062729668252800
W[(8,8,5,3,3)]	27	-108	3937816074403396325984	1485322356047272900832
W[(8,8,5,4,0)]	25	-59	126532108859856	-62415555336480
W[(8,8,5,4,1)]	26	-83	1859138760210276768	27442949686074432
W[(8,8,5,4,2)]	27	-105	1458934381540619175680	429548694784566103072
W[(8,8,5,4,3)]	28	-125	276957561140740817672064	141959405252146760964096
W[(8,8,5,4,4)]	29	-143	20081002987269237834527544	13937339478667000119236400
W[(8,8,5,5,0)]	26	-71	3881643757375656	-2421429008571216
W[(8,8,5,5,1)]	27	-96	57539715512784775920	998655920868343392
W[(8,8,5,5,2)]	28	-119	45903620873883847141776	16604759404714367599872
W[(8,8,5,5,3)]	29	-140	9056138605664527122727872	5683140308380169857515456
W[(8,8,6,1,1)]	24	-39	927231904320	-9200679520
W[(8,8,6,2,0)]	24	-36	65215569408	-16642956928
W[(8,8,6,2,1)]	25	-59	1335301022489328	7932337626384
W[(8,8,6,2,2)]	26	-80	1329629977546611936	193232919723748536
W[(8,8,6,3,0)]	25	-53	20350993239840	-8738280013680
W[(8,8,6,3,1)]	26	-77	331824982853181696	4137353643774720
W[(8,8,6,3,2)]	27	-99	285713812377367900976	72150775131428807216
W[(8,8,6,3,3)]	28	-119	57992334650190556570488	25428538397714935697520
W[(8,8,6,4,0)]	26	-68	1692511359568896	-1000067627325696
W[(8,8,6,4,1)]	27	-93	26146648523244093888	430155649709142688
W[(8,8,6,4,2)]	28	-116	21685078061596939766784	7419452787535790690304
W[(8,8,6,4,3)]	29	-137	4407669387277685669560512	2614878610961089413438336
W[(8,8,6,5,0)]	27	-81	55456767284050560	-40752668020556480
W[(8,8,6,5,1)]	28	-107	878968309413423252864	17001413171806593504
W[(8,8,6,5,2)]	29	-131	746119098679547758175328	312334294647361791229536
W[(8,8,6,6,0)]	28	-92	875827020273329664	-752251254607854720
W[(8,8,6,6,1)]	29	-119	14913192160928967502416	321538657422443256528
W[(8,8,7,1,0)]	24	-21	140436672	-16170272
W[(8,8,7,1,1)]	25	-44	5601159429504	-83145266112

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Table 15 – continued

Web	deg(W)	h(W)	n_W	d_W
W[(8,8,7,2,0)]	25	-41	391409808576	-119442727776
W[(8,8,7,2,1)]	26	-65	8359186921934400	46040552189760
W[(8,8,7,2,2)]	27	-87	8999069221638485760	1469239641997479152
W[(8,8,7,3,0)]	26	-59	126532108859856	-62415555336480
W[(8,8,7,3,1)]	27	-84	2251004857405139040	29808201189352224
W[(8,8,7,3,2)]	28	-107	2125395706871678176416	599695012064780172192
W[(8,8,7,3,3)]	29	-128	475983733946283470184480	232464084388228505851296
W[(8,8,7,4,0)]	27	-75	11461248223336400	-7656245352109120
W[(8,8,7,4,1)]	28	-101	196113949182715052160	3469894328671355136
W[(8,8,7,4,2)]	29	-125	179029959278599331183904	68216655150306510669456
W[(8,8,7,5,0)]	28	-89	419093788958668992	-345708325307878560
W[(8,8,7,5,1)]	29	-116	7382355219180836928000	153930935927106810240
W[(8,8,7,6,0)]	29	-101	7505020393627384992	-7217177104026255792
W[(8,8,8,0,0)]	24	0	80	4
W[(8,8,8,1,0)]	25	-23	347078520	-46049040
W[(8,8,8,1,1)]	26	-47	15746747463456	-285485916336
W[(8,8,8,2,0)]	26	-44	1096632086784	-368134868160
W[(8,8,8,2,1)]	27	-69	27132798580132512	137150695673232
W[(8,8,8,2,2)]	28	-92	33641794610965862400	5923901626220768640
W[(8,8,8,3,0)]	27	-63	408865565088240	-219322647849280
W[(8,8,8,3,1)]	28	-89	8424735061429730304	115480640308379520
W[(8,8,8,3,2)]	29	-113	9106787701513392933312	2772036316581728759040
W[(8,8,8,4,0)]	28	-80	42801528135993600	-30974226462442944
W[(8,8,8,4,1)]	29	-107	844343421475958442384	15670595020056748992
W[(8,8,8,5,0)]	29	-95	1810611871504105272	-1616941505273075616
W[(9,5,5,4,4)]	27	-120	96959783616963943030208	50915057095808531822560
W[(9,5,5,5,3)]	27	-117	42482450279496628079616	20127314721021911676928
W[(9,5,5,5,4)]	28	-134	2754129399126265116578688	1771748988189311827248096
W[(9,5,5,5,5)]	29	-149	85113274584443674676815872	66936916351993885652216832
W[(9,6,4,4,4)]	27	-117	45328721421594382226880	22390120482647854754272
W[(9,6,5,4,3)]	27	-114	19781742107681508906368	8820756832744533115968
W[(9,6,5,4,4)]	28	-131	1321756163879149610446896	801414379844189789048640
W[(9,6,5,5,2)]	27	-108	3194412266903770307840	1003282043598184664064
W[(9,6,5,5,3)]	28	-128	588017185467849919723392	322017472455505977946944
W[(9,6,5,5,4)]	29	-146	41753519869402757126310048	30986533923874230605211360
W[(9,6,6,3,3)]	27	-108	3965695431311102575520	1505336844006318854880
W[(9,6,6,4,2)]	27	-105	1468942204326701835456	434985047971986769440
W[(9,6,6,4,3)]	28	-125	280190865205345515202656	144746838738025811348976
W[(9,6,6,4,4)]	29	-143	20420265705357860466725736	14305111033066829808964272
W[(9,6,6,5,1)]	27	-96	57866098737278532384	1011424106940266496
W[(9,6,6,5,2)]	28	-119	46407141063611686053552	16899949831877607148848
W[(9,6,6,5,3)]	29	-140	9205492921536158089178400	5827961507523161595876576
W[(9,6,6,6,0)]	27	-81	55456767284050560	-40752668020556480
W[(9,6,6,6,1)]	28	-107	886526887354737645072	17294122346233404288
W[(9,6,6,6,2)]	29	-131	757338998385448742259408	319351886711731639092048
W[(9,7,4,4,3)]	27	-108	4302653199079323937920	1721332405243795563776
W[(9,7,4,4,4)]	28	-125	308970407090480886905664	168965147904886287828768
W[(9,7,5,3,3)]	27	-105	1852914051293922601984	670374726927803695104
W[(9,7,5,4,2)]	27	-102	684339544738107833984	193221962896538278144
W[(9,7,5,4,3)]	28	-122	136517982401519160854256	67481808424198752531456
W[(9,7,5,4,4)]	29	-140	10288587378544353628959936	6913304732636292162876864
W[(9,7,5,5,1)]	27	-93	26729153888914424832	450524230700139520
W[(9,7,5,5,2)]	28	-116	22480410038381810846784	7838457970390311440448
W[(9,7,5,5,3)]	29	-137	4625435952934481006642688	2809517795490346324727808
W[(9,7,6,3,2)]	27	-96	131895336423525168320	31875712216477298048
W[(9,7,6,3,3)]	28	-116	28271565861485256037152	11940345167441877911232
W[(9,7,6,4,1)]	27	-90	12023512502480326944	192137049043395264

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Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(9,7,6,4,2)]	28	-113	105472162871721858662400	34757136133378838296320
W[(9,7,6,4,3)]	29	-134	2240131001946976373134464	1285699665017929239033504
W[(9,7,6,5,0)]	27	-78	25383601647232320	-17802901998922240
W[(9,7,6,5,1)]	28	-104	424552799703766464912	80001138987631987200
W[(9,7,6,5,2)]	29	-128	37733747329936434989212 8	1527856126915690687518 72
W[(9,7,6,6,0)]	28	-89	419093788958668992	-345708325307878560
W[(9,7,6,6,1)]	29	-116	7465381920406413655872	1571740834551085559 04
W[(9,7,7,2,2)]	27	-84	4005072998835530048	623145599186074560
W[(9,7,7,3,1)]	27	-81	1000944581364139008	12940864211175424
W[(9,7,7,3,2)]	28	-104	1010995377658903433280	274043841200077763232
W[(9,7,7,3,3)]	29	-125	238119849054264183598080	112332804021170119948800
W[(9,7,7,4,0)]	27	-72	5101035246140064	-3238317072282240
W[(9,7,7,4,1)]	28	-98	930228493784610019 68	160368903949480608 0
W[(9,7,7,4,2)]	29	-122	894018817344730772802 72	329061483277284528098 88
W[(9,7,7,5,0)]	28	-86	198280729061595552	-156843054827785632
W[(9,7,7,5,1)]	29	-113	366701946017537470924 8	746489112754555484 16
W[(9,7,7,6,0)]	29	-98	3704581973944705776	-3435204329200397376
W[(9,8,4,3,3)]	27	-96	17604874903788298179 2	54398198798074388800
W[(9,8,4,4,2)]	27	-93	6434205857899096243 2	1553432095854360243 2
W[(9,8,4,4,3)]	28	-113	14880953729756521482240	6366888695682697842144
W[(9,8,4,4,4)]	29	-131	1250148843664649970662352	733085308674475362988800
W[(9,8,5,3,2)]	27	-90	26964980823333578944	5907389929476004224
W[(9,8,5,3,3)]	28	-110	6470374366380782830464	2506621507185094907424
W[(9,8,5,4,1)]	27	-84	2436667741602374528	36360838742410016
W[(9,8,5,4,2)]	28	-107	240195348206440929758 4	726440453016521093136
W[(9,8,5,4,3)]	29	-128	55691017048079367048240 0	29544395708272623471091 2
W[(9,8,5,5,0)]	27	-72	5101035246140064	-3238317072282240
W[(9,8,5,5,1)]	28	-98	95308266595738550640	16850253695412844 80
W[(9,8,5,5,2)]	29	-122	9287386758125630597184 0	3479052097621389355046 4
W[(9,8,6,2,2)]	27	-81	1755969430220426496	259636655223086848
W[(9,8,6,3,1)]	27	-78	438395891787545984	5516943176462464
W[(9,8,6,3,2)]	28	-101	475191508986206197632	123424186729923646080
W[(9,8,6,3,3)]	29	-122	117846752322559629255984	53548008570062637417696
W[(9,8,6,4,0)]	27	-69	2235977596096128	-1345692401785920
W[(9,8,6,4,1)]	28	-95	43589536992230477208	7303245101468637 12
W[(9,8,6,4,2)]	29	-119	44162182140827544636144	15659653975155880831536
W[(9,8,6,5,0)]	28	-83	92700939550359360	-70199768003592720
W[(9,8,6,5,1)]	29	-110	1801675431551746250880	357054554862821233 92
W[(9,8,6,6,0)]	29	-95	1810611871504105272	-1616941505273075616
W[(9,8,7,2,1)]	27	-66	11256418126498304	60842853741632
W[(9,8,7,2,2)]	28	-89	1531819852977699206 4	2579059445631070176
W[(9,8,7,3,0)]	27	-60	170193515484672	-85781801925696
W[(9,8,7,3,1)]	28	-86	3833600055583272480	514945231181784 00
W[(9,8,7,3,2)]	29	-110	442111988226108274924 8	129606910315661695276 8
W[(9,8,7,4,0)]	28	-77	19503820669876800	-13461969999093600
W[(9,8,7,4,1)]	29	-104	408994002687679224816	7411795886398986624
W[(9,8,7,5,0)]	29	-92	875827020316064256	-752251254614400960
W[(9,8,8,1,1)]	27	-48	22072657897776	-425570551040
W[(9,8,8,2,0)]	27	-45	1535514818112	-531223501536
W[(9,8,8,2,1)]	28	-71	48279338403693048	230627260020528
W[(9,8,8,2,2)]	29	-95	7293989525630949768 0	1340420997576916939 2
W[(9,8,8,3,0)]	28	-65	725912434085952	-405156007308576
W[(9,8,8,3,1)]	29	-92	182758150765721388 48	254920513927767648
W[(9,8,8,4,0)]	29	-83	92700939550359360	-70199768003592720
W[(9,9,3,3,3)]	27	-81	2670197361402514944	595724740895599616
W[(9,9,4,3,2)]	27	-78	951716992566363648	166286435635301824
W[(9,9,4,3,3)]	28	-98	29752032637462684689 6	94786453063647970848

Continued on next page

Table 15 – continued

Web	deg(W)	$h(W)$	n_W	d_W
W[(9,9,4,4,1)]	27	-72	83845604993916000	1063377379790976
W[(9,9,4,4,2)]	28	-95	1090364393046919486608	27149794915803480576
W[(9,9,4,4,3)]	29	-116	309071917403310096222722	1374036371622675075360
W[(9,9,5,2,2)]	27	-72	134329574207275104	16773356868339520
W[(9,9,5,3,1)]	27	-69	33386278590988800	381144946134016
W[(9,9,5,3,2)]	28	-92	45945931585469072928	1038893765633275353 6
W[(9,9,5,3,3)]	29	-113	13510524682548173641728	544206362250647134617 6
W[(9,9,5,4,0)]	27	-60	170193515484672	-85781801925696
W[(9,9,5,4,1)]	28	-86	4165834565805729216	63502596678612960
W[(9,9,5,4,2)]	29	-110	502924861806198031075 2	1581805356183712228896
W[(9,9,5,5,0)]	28	-74	8765016259161504	-5758034709276000
W[(9,9,5,5,1)]	29	-101	201260631020776796160	36553847542558417 92
W[(9,9,6,2,1)]	27	-60	1823776449179136	10747142102848
W[(9,9,6,2,2)]	28	-83	3046779784226901072	465966091929471216
W[(9,9,6,3,0)]	27	-54	27765085214112	-12215408263200
W[(9,9,6,3,1)]	28	-80	761207182511922096	9751729607816928
W[(9,9,6,3,2)]	29	-104	1009523697408684687360	27317925301442744400 0
W[(9,9,6,4,0)]	28	-71	3881643757375656	-2421429008571216
W[(9,9,6,4,1)]	29	-98	92904071198229763248	1598269886739062400
W[(9,9,6,5,0)]	29	-86	198280729061595552	-156843054827785632
W[(9,9,7,1,1)]	27	-45	7933211814912	-126220459008
W[(9,9,7,2,0)]	27	-42	553728279360	-174588053440
W[(9,9,7,2,1)]	28	-68	20278800720533664	105019516952544
W[(9,9,7,2,2)]	29	-92	33639461017770972672	592179744190913606 4
W[(9,9,7,3,0)]	28	-62	305922925426848	-160791639748800
W[(9,9,7,3,1)]	29	-89	8424230964009345024	115413671534048256
W[(9,9,7,4,0)]	29	-80	42801528146793216	-30974226462689184
W[(9,9,8,1,0)]	27	-24	539120544	-76342880
W[(9,9,8,1,1)]	28	-50	42951164308896	-929847901440
W[(9,9,8,2,0)]	28	-47	2981800050480	-1093125957120
W[(9,9,8,2,1)]	29	-74	112932475851555408	486039827347776
W[(9,9,8,3,0)]	29	-68	1692511362069504	-1000067627051904
W[(9,9,9,0,0)]	27	0	112	0
W[(9,9,9,1,0)]	28	-26	1272585120	-203310240
W[(9,9,9,1,1)]	29	-53	113958894140160	-2891432289792
W[(9,9,9,2,0)]	29	-50	7888589144400	-3138370134624

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