SciPost Phys. 18, 173 (2025)

Higher spin swampland conjecture for massive AdS_3 gravity

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Abstract

In this paper, we show that a possible version of the swampland weak gravity conjecture for higher spin (HS) massive topological AdS_3 gravity can be expressed in terms of mass M_{hs}, charge Q_{hs} and coupling constant g_{hs} of 3D gravity coupled to higher spin fields as $\mathrm{M}_{hs} \leq \sqrt{2}\mathrm{Q}_{hs} g_{hs} M_{Pl}$. The higher spin charge is given by the $SO(1,2)$ quadratic Casimir $\mathrm{Q}^2_{hs} = s(s-1)$ and the HS coupling constant by $g^2_{hs} = 2/\left(M^2_{Pl} l^2_{AdS_3}\right)$ while the mass expressed like $\left(l_{AdS_3}\mathrm{M}_{hs}\right)^2$ is defined as $\left(1+\mu l_{AdS_3}\right)^2 s(s-1) + \left[1-\left(\mu l_{AdS_3}\right)^2 (s-1)\right]$.

Received 2024-07-26
Accepted 2025-05-15
Published 2025-06-03
Check for updates
doi:10.21468/SciPostPhys.18.6.173

Contents

1 Introduction

With the purpose of investigating and deriving possible swampland constraints [1–3] on topological gravity [4,5] coupled to higher spin massive fields, we consider the three dimensional AdS action with a negative cosmological constant in addition to a gravitational Chern-Simons (CS) term to build up the higher spin topological massive gravity (HSTMG) theory [6]- [9]. Particularly, we are interested in the higher spin Bañados-Teitelboim-Zanelli (BTZ) black hole solution [10,11] and its discharge.

This study is driven by several motives, mainly the non-supersymmetric AdS [12,13] conjecture and the weak gravity constraint [14–16]. In fact, it was stipulated that non supersymmetric AdS spaces—as well as locally lookalike AdS geometries— are at best metastable; they manifest a non perturbative instability and will ultimately decay [2,3]. And since the BTZ black hole is locally isometric to AdS_3, it should also a priori exhibit a similar instability [4].

Actually, black holes in AdS spaces are of two types [17]: we either have large black holes in equilibrium with their thermal bath, or small unstable black holes in need of discharging by radiating away their charge. This aligns with the weak gravity conjecture, which requires the emission of a super-extremal particle with a constraint on its mass to charge ratio. However, it was argued in [17] that this mild version of the WGC, demanding a single super-extremal state, is not sufficient and a lattice refinement of the constraint is more suitable. To insure the decay of BTZ black holes in AdS_3, one must guarantee that the emitted particles reach the AdS_3 boundary and don't bounce back to form a self-interacting particle condensate that could eventually become sub-extremal [17]. Indeed, the boundary conditions on the AdS_3 cylinder can box the discharged particles enabling them to interact in a sub-extremal cloud of emitted particles. Therefore, one must require instead a stronger version of the weak gravity conjecture, a lattice WGC, where in each charge sector, there should be a super-extremal state [17].

For unstable BTZ black holes in HSTMG, and in order to comply the super-extremality constraint of the WGC, there must be a set of emitted super-extremal particles that ought to be charged and massive higher spin particles. Now, is it possible to formulate such WGC constraint for higher spin topological massive gravity to regulate the discharge of unstable higher spin BTZ black holes?

To the best of our knowledge, this inquiry was never investigated in Literature. The WGC constraint was only established for a disjointed setting, where the gravitational and gauge sectors are separated by considering 3D gravity in addition to a U(1) gauge field [17,18] but never for massive AdS_3 gravity in CS formulation coupled to higher spin fields.

In this paper, we intend to fill in this gap by first reviewing known results on the D-dimensional black holes and their WGC constraints to formulate our hypothesis about the expected super-extremality bound for the HS-BTZ black hole (section 2). Then, we construct the mass and charge operators to build the higher spin states (section 3). Once we have all the tools needed, we derive the swampland constraint for higher spin BTZ black holes and compute the tower of super-extremal higher spin states (section 4). And before concluding, we discuss the relevance of our findings with regard to recent progress in the swampland program Literature (section 5).

2 Weak gravity conjecture in $D \geq 4$

This section aims to motivate a Swampland conjecture for higher spin massive AdS_3 gravity to regulate the discharge of higher spin BTZ black holes using commonly accepted arguments. To pave the way for this 3D Swampland constraint, we intend to align it with the weak gravity conjecture (WGC) governing the decays of black holes in space dimensions $D > 3$ [14]. This bridging between HS- AdS_3 models and effective gauge theories coupled to D-gravity (EFF_D) is sustained by several facts and features; in particular:

(**A**) the existence of a Chern-Simons (CS) gauge formulation of higher spin AdS_3 gravity [19,20]. In this formalism, one uses standard gauge fields valued in the Lie algebra of the CS gauge symmetry $\mathcal{G}_{\text{hs}} \times \tilde{\mathcal{G}}_{\text{hs}}$; this, as we will see, permits to replicate the construction of certain constraints regulating the decay of D-black holes for unstable HS-BTZ black holes. For the remainder of this investigation, we focus on higher spin BTZ black hole solutions in the CS formulation with rank 2 gauge symmetries namely: (**1**) the higher spin SL(3,ℝ) model [21] having two spins $s = 2, 3$ as a representative of the HS theory with SL(N,ℝ) family [22]. (**2**) The higher spin model with SO(2, 3) group having also two spins $s = 2, 4$ as a representative of the HS ortho-symplectic families with gauge symmetries given by the real split forms of B_N, C_N and D_N Lie groups [23]. And (**3**) the exceptional G_2 higher spin model [24] with spin spectrum given by $s = 2, 6$. As well, this G_2 can be viewed as a representative of the exceptional family of finite dimensional Lie algebras. Useful characteristic properties of these HS topological gauge models are as follows

symmetry $\mathcal{G}_{hs}$	spin set $J_{\mathcal{G}}$	generators	$\dim \mathcal{G}_{hs}$
SL(3,ℝ)	2, 3	$W^{(1)}_{m_1} \oplus W^{(2)}_{m_2}$	$3+5$
SO(2, 3)	2, 4	$W^{(1)}_{m_1} \oplus W^{(3)}_{m_3}$	$3+7$
G_2	2, 6	$W^{(1)}_{m_1} \oplus W^{(5)}_{m_5}$	$3+11$

(1)

with label m_j taking integral values as $-j \leq m_j \leq j$. The $W^{(j)}_{m_j}$ are the generators of the spin $s_j = j + 1$; they form an isospin j representation of the principal sl(2;ℝ) partitioning the $\mathcal{G}_{hs}$ generators as exhibited by the two last columns of the above table.

The second feature supporting the EFT_D-HS AdS_3 cross over is (**B**) the AdS_3/CFT_2 correspondence [28] allowing to relate topological aspects of HS- AdS_3 gravity such as Wilson lines with conformal highest weight representations and conformal observables; which will be fundamental for the computation of the HS Swampland constraint. And lastly, the possibility to (**C**) realise both the masses and the charges required by the WGC in terms of the quantum numbers of the CS gauge symmetry, as well as the coupling constants and the Planck mass M_{Pl}. The EFT_D-HS AdS_3 crossing is therefore based on matching the D- dimensional WGC ingredients with those of 3D higher spin gravity as follows:

$D > 3$	$D = 3$
D-gravity + U(1) charged matter	3D gravity coupled to higher spin fields
Effective field models	Higher spin Chern-Simons formulation
Electrically charged Black holes	Higher spin BTZ black holes
Electric charge q_e	Spin charge Q_{hs}
Mass $\boldsymbol{m}$	Mass of HS particles M_{hs}

(2)

We show throughout this paper that the emitted particle states $|s;\{\lambda\}\rangle$ of the 3D BTZ black hole carry, in addition to the higher spins s, masses M_{hs} and charges Q_{hs} which are functions of s. These states will be denoted below like

$$|s; \mathrm{M}_{\text{hs}}, \mathrm{Q}_{\text{hs}} > , \tag{3}$$

where the masses M_{hs}, and the charges Q_{hs} are eigenvalues of some function of commuting observables $\mathcal{O}_i$ of the gauge theory with symmetry group $\mathcal{G}_{\text{hs}} \times \tilde{\mathcal{G}}_{\text{hs}}$. Candidates for these $\mathcal{O}_i$s are given by the Cartan charge operators of $\mathcal{G}_{\text{hs}} \times \tilde{\mathcal{G}}_{\text{hs}}$ and their Casimirs.

In this regard, we restrict to the principal SL(2,$\mathbb{R}$) symmetry observables within the gauge symmetry $\mathcal{G}_{hs}$; they are given by the Cartan charge L_0 and the quadratic Casimir $\mathcal{C}_2$ with the following commutation relations

$$[L_n, L_m] = (m-n)\, L_{n+m}\,, \tag{4}$$
$$\mathcal{C}_2 = L_0^2 - L_0 - L_+ L_-\,. \tag{5}$$

Particularly, we are interested in the mass $\hat{\mathrm{M}}_{\mathrm{hs}}$ and the charge $\hat{\mathrm{Q}}_{\mathrm{hs}}$ operators with spectrums as follows

$$\begin{aligned} \mathrm{M}_{\mathrm{hs}} &:= \mathrm{spect}\left(\hat{\mathrm{M}}_{\mathrm{hs}}\right)\,, \\ \mathrm{Q}_{\mathrm{hs}} &:= \mathrm{spect}\left(\hat{\mathrm{Q}}_{\mathrm{hs}}\right)\,. \end{aligned} \tag{6}$$

They can be expanded in terms of the commuting Cartan charge L_0 and the Casimir $\mathcal{C}_2$ of the principal SL(2,$\mathbb{R}$) symmetry like

$$\hat{\mathrm{M}}^2_{\mathrm{hs}} = \mathfrak{m}_0 L_0 + \mathfrak{m}_2 \mathcal{C}_2\,, \tag{7}$$
$$\hat{\mathrm{Q}}^2_{\mathrm{hs}} = \mathfrak{q}_0 L_0 + \mathfrak{q}_2 \mathcal{C}_2\,, \tag{8}$$

with some positive $\mathfrak{m}_i$ and $\mathfrak{q}_i$ to be determined later on. The mass $\hat{\mathrm{M}}_{\mathrm{hs}}$ and charge $\hat{\mathrm{Q}}_{\mathrm{hs}}$ observable operators act on the higher spin-s particle $|s;\mathrm{M}_{\mathrm{hs}},\mathrm{Q}_{\mathrm{hs}} >$ states as

$$\begin{aligned} \hat{\mathrm{M}}_{\mathrm{hs}}|s;\mathrm{M}_{\mathrm{hs}},\mathrm{Q}_{\mathrm{hs}}> &= \mathrm{M}_{\mathrm{hs}}|s;\mathrm{M}_{\mathrm{hs}},\mathrm{Q}_{\mathrm{hs}}>\,, \\ \hat{\mathrm{Q}}_{\mathrm{hs}}|s;\mathrm{M}_{\mathrm{hs}},\mathrm{Q}_{\mathrm{hs}}> &= \mathrm{Q}_{\mathrm{hs}}|s;\mathrm{M}_{\mathrm{hs}},\mathrm{Q}_{\mathrm{hs}}>\,. \end{aligned} \tag{9}$$

Before proceeding any further, we pause to carefully examine and comment on the structure of the operators (7-8) by leveraging well-known principles from the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [25, 26] and SL_2 isospin representation framework:

(1) for the $\hat{\mathrm{M}}^2_{\mathrm{hs}}$ expansion (7), the block term generated by L_0 can be attributed to the CFT_2 relationship $m \sim h + \bar{h}$ with the eigenvalue equations $L_0\,|h\rangle = h\,|h\rangle$ and $\bar{L}_0\,\left|\bar{h}\right\rangle = \bar{h}\,\left|\bar{h}\right\rangle$. Regarding the block term generated by C_2, it can be motivated by the Sugawara construction of the conformal energy momentum tensor from the affine SL_2 Kac-Moody current [27].

(2) As for the expansion of the operator $\hat{\mathrm{Q}}_{\mathrm{hs}}$ in (8), it can be restricted to the Casimir block $\hat{\mathrm{Q}}^2_{\mathrm{hs}} = q_2 C_2$ with some $q_2 > 0$. This is because the Casimir operator C_2 captures information on the SL_2 isospin Δ while the charge operator L_0 (thought of as J_z) captures data on the isospin projection Δ_z.

Taking all of the aforementioned into account, one might speculate that the mass $\hat{\mathrm{M}}_{\mathrm{hs}}$ and the charge $\hat{\mathrm{Q}}_{\mathrm{hs}}$ operators are indeed linked to each other like

$$\hat{\mathrm{M}}^2_{\mathrm{hs}} = \mathfrak{m}_0 L_0 + \frac{\mathfrak{m}_2}{\mathfrak{q}_2}\hat{\mathrm{Q}}^2_{\mathrm{hs}}\,. \tag{10}$$

Such property justifies the interest in the search for a Swampland conjecture for higher spin gravitational models. Moreover, by acting on the quantum states $|\Delta, N\rangle$ of (unitary) representations $\mathcal{R}^{\pm}_{\Delta}$ of the SL(2,$\mathbb{R}$) symmetry group with both sides of the above equation, we get the following mass relation

$$\begin{aligned} \mathcal{R}^{+}_{\Delta} : \mathrm{M}^2_{\Delta,N_+} &= +\mathfrak{m}_0\,(\Delta + N) + \frac{\mathfrak{m}_2}{\mathfrak{q}_2}\Delta\,(\Delta - 1)\,, \\ \mathcal{R}^{-}_{\Delta} : \mathrm{M}^2_{\Delta,N_-} &= -\mathfrak{m}_0\,(\Delta + N) + \frac{\mathfrak{m}_2}{\mathfrak{q}_2}\Delta\,(\Delta - 1)\,. \end{aligned} \tag{11}$$

The structure of the representations $\mathcal{R}_{\Delta}^{\pm}$ and the properties of the states $|\Delta_{\pm}, N_{\pm}\rangle$ will be thoroughly investigated in *subsection 3.1*. Meanwhile notice that the set of HS quantum states $|s;\mathrm{M_{hs}},\mathrm{Q_{hs}} >$ has a group theoretic basis; they can be perceived as the $|\Delta, N\rangle$ of the SL_2 representation group theory which will be proven to be accurate.

Returning to the HS Swampland conjecture issue, we seek to show that the decay of small HS- BTZ black holes in AdS_3 gravity is accompanied by the emission of super-extremal higher spin-s states $|s; \mathrm{M_{hs}}, \mathrm{Q_{hs}} >$ with spin dependent masses $\mathrm{M_{hs}}$ and charges $\mathrm{Q_{hs}}$ constrained as follows

$$\mathrm{M_{hs}} \leq \sqrt{2}\mathrm{Q_{hs}} g_{\mathrm{hs}} M_{\mathrm{Pl}} \,, \tag{12}$$

with g_{hs} standing for the higher spin coupling constant to be determined later. Below, we refer to (12) as the Swampland higher spin conjecture (**HSC**) for massive AdS_3 gravity. At first impression, one might wonder about the interpretation of such constraint and whether this inequality is a true swampland conjecture. However by way of construction, the swampland HSC will prove to be a version of the WGC that regulates the discharge of higher spin BTZ back hole solutions of HSTMG carrying charges beyond the usual U(1) of [17]. The HSC accounts for 3D black holes solutions with different backgrounds than the ones already considered in Literature [17, 18], and can be therefore perceived as a complement to the work conducted in AdS_3 framework regarding the derivation of the WGC.

Moreover, the condition (12) has interesting properties shared by unstable D-black holes. Particularly, the constraint (12) has a quite similar structure to the well known 4D weak gravity conjecture formulated by the following inequality [14]

$$\boldsymbol{m} \leq \sqrt{2} q g_{_{U(1)}} M_{\mathrm{Pl}} \,. \tag{13}$$

This well established constraint relation (13) will be used as a guiding principle for the derivation of the **HSC** (12). To avoid confusion between the 3D and 4D parameters, we use the following convention notations

black hole	mass	charge	coupling
4D charged BH	$\boldsymbol{m}$	q	$g_{_{U(1)}}$
3D BTZ	$\mathrm{M_{hs}}$	$\mathrm{Q_{hs}}$	g_{hs}

(14)

In dimensions $D \geq 4$, the weak gravity conjecture (WGC) requires the existence of at least one super extremal state $|\boldsymbol{m}, q\rangle$ in the particle spectrum of the effective U(1) gauge theory coupled to D- gravity with mass $\boldsymbol{m}$ and charge $Q_{_{U(1)}} = q g_{_{U(1)}}$ satisfying the condition [2, 14]

$$q^2 g_{_{U(1)}}^2 \geq \frac{D-3}{D-2} \boldsymbol{m}^2 M_{\mathrm{Pl}}^{2-D} \,, \tag{15}$$

where the gauge coupling constant $g_{_{U(1)}}$ scales like $\text{MASS}^{2-D/2}$ and M_{Pl} is the D- Planck mass. This constraint relation puts a condition on the allowed space time dimensions as it requires $D \geq 3$; although the $D = 3$ is a critical value. By putting D=3, the relation (15) leads to a trivial condition $q^2 g_{_{U(1)}}^2 \geq 0$ with no reference whatsoever to the value of the mass $\boldsymbol{m}^2$. Even with a reverse reasoning, if we consider instead $\boldsymbol{m}^2 \leq q^2 g_{_{U(1)}}^2 M_{\mathrm{Pl}}^{D-2}(D-2)/(D-3)$, all we learn is that $\boldsymbol{m}^2 \leq \infty$ lacking any information on the value of q^2.

However, to retrieve additional insights, we concentrate on the interesting four dimensional theory ($D = 4$). The condition (15) reads like

$$\boldsymbol{m}^2 \leq 2 q^2 g_{_{U(1)}}^2 M_{\mathrm{Pl}}^2 \,, \tag{16}$$

with

$$q = \int_{\mathbb{S}^2} \mathbf{E}_{u(1)} . d\boldsymbol{\sigma} \,, \tag{17}$$

and where $\mathbf{E}_{u(1)} = -\nabla V - \partial_t \mathbf{A}$ is the usual electric field. For this U(1) abelian gauge theory, gauge group elements $\mathcal{U}$ are given by $e^{i\mathbf{Q}_{u(1)}}$ with generator

$$\mathbf{Q}_{u(1)} = g_{u(1)} \mathfrak{Q}\,, \tag{18}$$

acting on charged quantum states like

$$\mathbf{Q}_{u(1)} |\boldsymbol{m}, q\rangle = q g_{u(1)} |\boldsymbol{m}, q\rangle \ , \qquad \hat{\boldsymbol{M}}^2 |\boldsymbol{m}, q\rangle = \boldsymbol{m}^2 |\boldsymbol{m}, q\rangle \ , \tag{19}$$

where $\hat{\boldsymbol{M}}$ is the mass operator. In the upcoming section, we construct the higher spin homologue of (19) for HS-AdS$_3$ gravity.

3 Higher spin particle states

An essential key component to the derivation of the relation (12), is the set of emitted super extremal particle states $|s; \mathrm{M}_{\mathrm{hs}}, \mathrm{Q}_{\mathrm{hs}} >$. It is therefore crucial, before all else, to define these states. We identify these particles as eigenstates of some higher spin charge operator defined like $\boldsymbol{Q}_{\mathrm{hs}} = g_{\mathrm{hs}} Q$ analogously to the 4D charge operator $\mathbf{Q}_{u(1)} = g_{u(1)} \mathfrak{Q}$ given by eq(18). It acts as follows

$$\boldsymbol{Q}_{\mathrm{hs}} |s; \mathrm{M}_{\mathrm{hs}}, \mathrm{Q}_{\mathrm{hs}}\rangle = g_{\mathrm{hs}} \mathrm{Q}_{\mathrm{hs}} |s; \mathrm{M}_{\mathrm{hs}}, \mathrm{Q}_{\mathrm{hs}}\rangle \ , \tag{20}$$

where g_{hs} is the higher spin coupling constant of the higher spin gauge theory, it will be computed later on [see eq(62)].

To manoeuvre the set of these states, we use the principal SL(2,$\mathbb{R}$) representations since all the rank 2 gauge symmetries $\mathcal{G}_{\mathrm{hs}} \times \tilde{\mathcal{G}}_{\mathrm{hs}}$ we are considering can be obtained via the principal embedding of SL(2,$\mathbb{R}$). Therefore, we deem it necessary to briefly recall results on the principal SL(2,$\mathbb{R}$) subgroup of the gauge symmetry $\mathcal{G}_{\mathrm{hs}}$ and its unitary representations.

3.1 Unitary representations of SL(2,$\mathbb{R}$)

SL(2,$\mathbb{R}$) is a non compact group homomorphic to the Lorentz SO(1,2) and generated by L_0, $L_\pm$ with commutation relations $[L_n, L_m] = (m-n)L_{n+m}$ labelled by $n, m = 0, \pm$. It has several families of irreducible representations that can classified into two sets [30, 31], non unitary and unitary. The latter will be the focus of the upcoming discussion.

Unitary irreducible representations (UIR) are infinite dimensional, they are obtained by requiring the hermiticity condition $L_n^\dagger = L_{-n}$ and the positivity of the quantum states norms; i.e: $\| |\psi >\| > 0$. An interesting type of these UIRs is given by the discrete series denoted like $\mathcal{R}_\Delta^\pm$ [29–31]:

(1) Discrete series $\mathcal{R}_\Delta^+$ are generated by the quantum states $|\Delta, N\rangle$ as follows

$$\begin{aligned} L_+ |\Delta, N\rangle &= \sqrt{(N+1)(N+2\Delta)}\, |\Delta, N+1\rangle \ , \\ L_- |\Delta, N\rangle &= \sqrt{N(N+2\Delta-1)}\, |\Delta, N-1\rangle \ , \\ L_0 |\Delta, N\rangle &= (N+\Delta)\, |\Delta, N\rangle \ , \\ \mathcal{C}_2 |\Delta, N\rangle &= \Delta(\Delta-1)\, |\Delta, N\rangle \ , \end{aligned} \tag{21}$$

where $\mathcal{C}_2$ is the SL(2,$\mathbb{R}$) quadratic Casimir $L_0^2 - L_0 - L_+ L_-$. From these relations, one can compute useful quantities to draw several properties; in particular:

(i) the norm $< \Delta, N | L_+ L_- | \Delta, N >$ which is equal to $N(N+2\Delta-1)$. And its homologue $< \Delta, N | L_- L_+ | \Delta, N >$ given by $(N+1)(N+2\Delta)$.

(**ii**) The representation $\mathcal{R}_\Delta^+$ is bounded from below indicating that $L_- |\Delta, N\rangle = 0$ and requiring therefore $N(N + 2\Delta - 1) = 0$.

This latter constraint can be solved for $N = 0$, and the state $|\Delta, 0\rangle$ with positive definite Δ is thus a lowest weight state obeying the following lowest weight relations

$$\begin{aligned} L_- |\Delta, 0\rangle &= 0\,, \\ L_0 |\Delta, 0\rangle &= \Delta |\Delta, 0\rangle\,, \\ \mathcal{C}_2 |\Delta, 0\rangle &= \Delta(\Delta - 1) |\Delta, 0\rangle\,. \end{aligned} \tag{22}$$

With L_+ acting on $|\Delta, 0\rangle$ as $L_+ |\Delta, 0\rangle = \sqrt{2\Delta} |\Delta, 1\rangle$.

(2) Discrete series $\mathcal{R}_\Delta^-$ are also generated by the states $|\Delta, N\rangle$ and can be constructed as follows

$$\begin{aligned} L_- |\Delta, N\rangle &= -\sqrt{(N+1)(N+2\Delta)} |\Delta, N+1\rangle\,, \\ L_+ |\Delta, N\rangle &= -\sqrt{N(N+2\Delta-1)} |\Delta, N-1\rangle\,, \\ L_0 |\Delta, N\rangle &= -(N+\Delta) |\Delta, N\rangle\,, \\ \mathcal{C}_2 |\Delta, N\rangle &= \Delta(\Delta - 1) |\Delta, N\rangle\,, \end{aligned} \tag{23}$$

from which we can compute:

(**i**) The norm $< \Delta, N | L_+ L_- | \Delta, N >$ giving $(N+1)(N+2\Delta)$; and the homologue $< \Delta, N | L_- L_+ | \Delta, N >$ given by $N(N + 2\Delta - 1)$.

(**ii**) Conversely to the $\mathcal{R}_\Delta^+$ representation, $\mathcal{R}_\Delta^-$ is bounded from above with the constraint $L_+ |\Delta, N\rangle = 0$ requiring $N(N + 2\Delta - 1) = 0$.

Analogously, if we impose $N = 0$, the highest weight state $|\Delta, 0\rangle$ annihilated by L_+ satisfies the relations

$$\begin{aligned} L_+ |\Delta, 0\rangle &= 0\,, \\ L_0 |\Delta, 0\rangle &= -\Delta |\Delta, 0\rangle\,, \\ \mathcal{C}_2 |\Delta, 0\rangle &= \Delta(\Delta - 1) |\Delta, 0\rangle\,, \end{aligned} \tag{24}$$

with L_- action given by $L_- |\Delta, 0\rangle = -\sqrt{2\Delta} |\Delta, -1\rangle$.

Notice that the two discrete representations $\mathcal{R}_\Delta^+$ and $\mathcal{R}_\Delta^-$ are isomorphic; the isomorphism $\iota : \mathcal{R}_\Delta^+ \to \mathcal{R}_\Delta^-$ is given by the 1:1 correspondence $\iota(L_n) = -L_{-n}$ as manifestly exhibited by the relations (21) and (23).

3.2 Higher spin AdS_3 gravity

Focussing on HS- BTZ black holes with rank 2 symmetries of eq(1), the gauge theory is described by the 3D HS gravity action $\mathcal{S}_0^{\text{GRAV}}$ given in terms of two copies of Chern-Simons (CS) fields A and $\tilde{A}$ as follows [19, 20]

$$\mathcal{S}_0^{\text{GRAV}} = \frac{\mathrm{k}}{4\pi} \int tr\left(AdA + \frac{2}{3}A^3\right) - \frac{\tilde{\mathrm{k}}}{4\pi} \int tr\left(\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3\right)\,, \tag{25}$$

with CS level $\tilde{\mathrm{k}} = \mathrm{k}$. This positive integer number is related to the AdS_3 radius and the 3D Newton coupling constant like $\mathrm{k} = l_{\text{AdS}_3}/(4G_N)$.Being a discrete relation, this quantity can be imagined as a quantization relation of the 3D Newton constant expressed like $G_N^{[\mathrm{k}]} = l_{\text{AdS}_3}/(4\mathrm{k})$, showing in turns that $G_N^{[1]} = l_{\text{AdS}_3}/4$.

The conversion to the metric formulation is quite straightforward and mainly based on expressing both the dreibein E_μ and the spin connection Ω_μ in terms of the two CS gauge potentials A_μ and $\tilde{A}_\mu$ as follows

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2} Tr\left(E_\mu E_\nu\right), \\ \Phi_{\mu_1 \ldots \mu_s} &= Tr\left(E_{(\mu_1} \ldots E_{\mu_s)}\right), \\ E_\mu &= A_\mu - \tilde{A}_\mu, \\ \Omega_\mu &= A_\mu + \tilde{A}_\mu. \end{aligned} \tag{26}$$

Notice also that here the 1-form gauge connections A and $\tilde{A}$ as well as the E and Ω are non abelian 3D fields; they are valued in the Lie algebra of the gauge symmetry $\mathcal{G}_{hs} \times \tilde{\mathcal{G}}_{hs}$ and satisfy the Grumiller-Riegler (GR) boundary conditions [32] for a more general set-up. The field equations of motion of (25) are given by $\mathcal{F}_{\mu\nu} = 0$ and $\tilde{\mathcal{F}}_{\mu\nu} = 0$ where the $\mathcal{F}_{\mu\nu}$ and $\tilde{\mathcal{F}}_{\mu\nu}$ are the gauge fields strengths reading as $\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $\partial_{[\mu} \tilde{A}_{\nu]} + [\tilde{A}_\mu, \tilde{A}_\nu]$.

Because of the vanishing value of the gauge field strengths, gauge invariants similar to the 4D electric field $\mathbf{E}_{u(1)}$ of (25) and the associated electric charge $q = \int_{\mathbb{S}^2} \mathbf{E}_{u(1)} . d\boldsymbol{\sigma}$ are unavailable in the higher spin AdS_3 gravity. Instead, there are alternative gauge invariants given by (**i**) the Wilson loops

$$\mathcal{W}_{\mathcal{R}}[\gamma] = Tr_{\mathcal{R}}\left[P \exp\left(\int_\gamma \boldsymbol{A}\right) P \exp\left(\int_\gamma \tilde{\boldsymbol{A}}\right)\right], \tag{27}$$

with $\mathcal{R}$ being a representation of the gauge symmetry, γ a loop in AdS_3, $\boldsymbol{A}$ as well as $\tilde{\boldsymbol{A}}$ some gauge connections expanding as $A_\mu dx^\mu$ and $\tilde{A}_\mu dx^\mu$. And (**ii**) topological defects given by line operators constructed as [33, 34]

$$\mathcal{W}_{\mathcal{R}}\left[y_i, y_f\right] = \langle U_i | \, Tr_{\mathcal{R}}\left[P \exp\left(\int_{\Upsilon_{if}} \boldsymbol{A}\right) P \exp\left(\int_{\Upsilon_{if}} \tilde{\boldsymbol{A}}\right)\right] \left| U_f \right\rangle, \tag{28}$$

where $\left(y_i, y_f\right)$ are the end points of the curve Υ_{if} parameterised by y and where $U(y)$ is a probe field on Υ_{if} with boundary condition $U(y_i) = U\left(y_f\right) = I_{id}$. As illustrations, we give the expansion of the potential A_μ for the SL(3,$\mathbb{R}$) and G_2 models in the higher spin basis. For SL(3,$\mathbb{R}$), we have the following splitting

$$\begin{aligned} SL(3,\mathbb{R}): \quad A_\mu &= \sum_{m_1=-1}^{1} \mathcal{A}_\mu^{m_1} W_{m_1}^{(1)} + \sum_{m_2=-2}^{2} \mathcal{W}_\mu^{m_2} W_{m_2}^{(2)} \\ &:= \sum_{m=-1}^{1} \mathcal{A}_\mu^m L_m + \sum_{n=-2}^{2} \mathcal{W}_\mu^n W_n, \end{aligned} \tag{29}$$

with the commutation relations [36]

$$\begin{aligned} \left[L_i, L_j\right] &= (j-i) L_{i+j}, \\ \left[L_i, W_m\right] &= (m-2i) W_{i+m}, \\ \left[W_n, W_m\right] &= \frac{1}{3}(n-m)\left(2m^2 + 2n^2 - mn - 8\right) L_{n+m}. \end{aligned} \tag{30}$$

Similarly for G_2, we can write

$$\begin{aligned} G_2: \qquad A_\mu &= \sum_{m_1=-1}^{1} \mathcal{A}_\mu^{m_1} W_{m_1}^{(1)} + \sum_{m_5=-5}^{5} \mathcal{W}_\mu^{m_5} W_{m_5}^{(5)} \\ &:= \sum_{m=-1}^{1} \mathcal{A}_\mu^m L_m + \sum_{n=-5}^{5} \mathcal{W}_\mu^n W_n, \end{aligned} \tag{31}$$

with

$$
\begin{aligned}
\left[L_i, L_j\right] &= (j-i)\, L_{i+j}\,, \\
\left[L_i, W_m\right] &= (m-5i)\, W_{i+m}\,, \\
\left[W_n, W_m\right] &= f_{m,n|2}^{(1,5)} L_{n+m}\,,
\end{aligned}
\tag{32}
$$

where $f_{m,n|2}^{(1,5)}$ are constant structures obtained by solving the Jacobi identities. Notice that in the HS- basis, the 8 generators of SL(3,$\mathbb{R}$) are split into two blocks 3+5 given by: (**i**) the three $W_{m_1}^{(1)}$ with label $m_1 = 0, \pm 1$; they are just the usual generators L_m of the principal SL(2,$\mathbb{R}$). (**ii**) The five $W_{m_2}^{(2)} \equiv W_n$ with index $m_2 = n = 0, \pm 1, \pm 2$; they generate the coset space SL(3,$\mathbb{R}$)/SL(2,$\mathbb{R}$). Quite similar relations can be written for the 14 generators of G_2 that split as $3+11$.

As far as these types of HS- expansions are concerned, notice the following features depicted for the case of SL(3,$\mathbb{R}$) model: (**i**) The commutation relations of sl(3,$\mathbb{R}$) in the HS basis can be presented in a condensed form as follows

$$
\left[W_{m_j}^{(j)}, W_{n_k}^{(k)}\right] = \sum_{r_1=-1}^{1} \mathfrak{f}_{n_k,m_j|1}^{(j,k)} \delta_{m_j+n_k}^{r_1} W_{r_1}^{(1)} + \sum_{r_2=-2}^{2} \mathfrak{f}_{n_k,m_j|2}^{(j,k)} \delta_{m_j+n_k}^{r_2} W_{r_2}^{(2)}\,,
\tag{33}
$$

with the constant structures $\mathfrak{f}_{m_j,n_k|s}^{(j,k)}$ given by

$$
\mathfrak{f}_{n_1,m_1|1}^{(1,1)} = m_1 - n_1\,,
\tag{34}
$$

$$
\mathfrak{f}_{n_2,m_1|2}^{(1,2)} = m_1 - 2n_2\,,
\tag{35}
$$

$$
\mathfrak{f}_{n_2,m_2|2}^{(2,2)} = \frac{1}{3}(n_2 - m_2)\left(2m_2^2 + 2n_2^2 - m_2 n_2 - 8\right).
\tag{36}
$$

In general, we can express these commutations in a shorter form like

$$
\left[W_{m_\tau}^{(\tau)}, W_{n_\sigma}^{(\sigma)}\right] = \sum_{\upsilon} \sum_{r_\upsilon} \mathfrak{f}_{n_\sigma,m_\tau|\upsilon}^{(\tau,\sigma)} \delta_{m_\tau+n_\sigma}^{r_\upsilon} W_{r_\upsilon}^{(\upsilon)}\,.
\tag{37}
$$

(**ii**) Higher spin theories are characterised by the spins- s of the principal SL(2,$\mathbb{R}$) within SL(3,$\mathbb{R}$); it is defined by the usual commutation relations (4) where we have set $L_m = W_{m_1}^{(1)}$. As such, it is interesting to use the formal decomposition

$$
SL(3,\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \frac{SL(3,\mathbb{R})}{SL(2,\mathbb{R})}\,,
\tag{38}
$$

to split the gauge potentials A_μ and $\tilde{A}_\mu$ as follows

$$
A_\mu = A_\mu^{sl_2} + A_\mu^{sl_{3/2}}\,,
\tag{39}
$$

$$
\tilde{A}_\mu = \tilde{A}_\mu^{sl_2} + \tilde{A}_\mu^{sl_{3/2}}\,.
\tag{40}
$$

4 Derivation of the HS Swampland conjecture

In this section, we target the derivation of the HS Swampland conjecture in AdS_3 (12) which can be articulated as in the following statement:

Higher spin Swampland conjecture in AdS$_3$

A higher spin BTZ black hole solution of 3D topologically massive gravity with a negative cosmological constant $\Lambda < 0$ should be able to discharge by emitting super-extremal higher spin particles with mass $\mathrm{M_{hs}}$ and charge $\mathrm{Q_{hs}}$ such that

$$\mathrm{M_{hs}} \leq \alpha_3 \mathrm{Q_{hs}} g_{\mathrm{hs}} M_{\mathrm{Pl}}\,, \tag{41}$$

where g_{hs} is the higher spin gauge coupling and α_3 is some constant that we set as $\alpha_3 = \sqrt{2}$.

The constraint (41) bears a mighty resemblance to the inequality (16) regulating the decay of charged 4D black holes,

$$\begin{array}{ccccccc} & \text{3D HS-BTZ} & & \longleftrightarrow & & \text{4D charged BH,} & \\ \mathrm{M_{hs}} & \leq & \sqrt{2}\mathrm{Q_{hs}} g_{\mathrm{hs}} M_{\mathrm{Pl}} & \longleftrightarrow & \boldsymbol{m} & \leq & \sqrt{2} q g_{_{U(1)}} M_{\mathrm{Pl}}\,, \end{array} \tag{42}$$

but instead of the abelian U(1) parameters, we must determine the higher spin $\mathrm{M_{hs}}$, $\mathrm{Q_{hs}}$ and g_{hs} quantities for the $\mathcal{G}_{\mathrm{hs}}$ symmetry. For this purpose, we first promote the 3D gravity theory described by the field action $\mathcal{S}_0^{\mathrm{GRAV}}$ to a higher spin topologically massive AdS$_3$ gravity [6, 8] in order for our, as of yet, massless higher spin states to acquire mass.

The pure 3D gravity is known to be topological due to the absence of local degrees of freedom, offering simple settings that allow for tractable studies of gravitational theories, including those coupled to higher spin fields. This pure theory can be extended by incorporating massive degrees of freedom, by deforming the AdS$_3$ action with a gravitational CS term [6–9]:

$$\mathcal{S}_1^{\mathrm{GRAV}} = \frac{M_{\mathrm{Pl}}}{2\mu} \int_{\mathcal{M}_{3D}} Tr\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right), \tag{43}$$

where Γ is the Christoffel symbol and where μ is a massive parameter. The modified equations of motion are as follows:

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0\,, \tag{44}$$

where the Einstein tensor is given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{l^2_{AdS_3}} g_{\mu\nu}\,, \tag{45}$$

and the Cotton tensor by:

$$C_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu}^{\ \alpha\beta} \nabla_\alpha R_{\beta\nu} + (\mu \longleftrightarrow \nu)\,. \tag{46}$$

The theory develops a diffeomorphism anomaly given by the difference between the right c_+ and the left c_- central charges as we will discuss more thoroughly in the next section.

Now, regarding the mass of the new massive mode of topologically massive spin 2 gravity has been computed in [37, 38] and is given by:

$$m^2_{(2)} = \left(\mu + \frac{2}{l_{\mathrm{AdS}_3}}\right)^2 - \frac{1}{l^2_{\mathrm{AdS}_3}} = \frac{1}{l^2_{\mathrm{AdS}_3}} \left(\mu l_{\mathrm{AdS}_3} + 3\right)\left(\mu l_{\mathrm{AdS}_3} + 1\right). \tag{47}$$

It can also be written as:

$$m^2_{(2)} = \frac{(2-1)}{l^2_{\mathrm{AdS}_3}} \left((2-1)\mu l_{\mathrm{AdS}_3} + (2+1)\right)\left(\mu l_{\mathrm{AdS}_3} + 1\right). \tag{48}$$

A possible generalisation of the s=2 formula for higher spin super extremal states M_{hs} is therefore a function of the parameter μ and the conformal spin s such that $\mathrm{M}_{\mathrm{hs}} = \mathrm{M}(s,\mu)$. Following the conjecture of [4, 37, 38], M_{hs} can be formulated as

$$\mathrm{M}_{\mathrm{hs}}^2 = \frac{1+\mu l_{\mathrm{AdS}_3}}{l_{\mathrm{AdS}_3}^2}(s-1)\left[(s-1)\mu l_{\mathrm{AdS}_3} + (s+1)\right]. \tag{49}$$

This is a remarkable relation that can be put into a covariant form using observables of the principal SL(2,ℝ) symmetry of the higher spin theory. In fact, by putting $M_{\mathrm{AdS}_3} = 1/l_{\mathrm{AdS}_3}$ into the above $\mathrm{M}_{\mathrm{hs}}^2$ relation and after rearranging the terms, we end up with the distinguishable expression

$$\mathrm{M}_{\mathrm{hs}}^2 = \left(M_{\mathrm{AdS}_3} + \mu\right)^2 s(s-1) + \left(M_{\mathrm{AdS}_3}^2 - \mu^2\right)(s-1), \tag{50}$$

more thoroughly investigated below. The relationship between this conjectured mass formula and those $\mathrm{M}_{\Delta,N_+}^2$ and $\mathrm{M}_{\Delta,N_-}^2$ given by (11), associated with the two unitary representations $\mathcal{R}_\Delta^+$ and $\mathcal{R}_\Delta^-$, will be commented in subsection 4.2. Before that, let us see how the swampland constraint relation can be derived from (50).

4.1 From eq(50) towards eq(41)

Using the relation $s = 1 + j$, linking the values of the conformal spins- s of the higher spin AdS_3 gravity to the isospin j representation weights of SL(2,ℝ), the above conjectured mass relation $\mathrm{M}_{\mathrm{hs}}^2$ becomes

$$\begin{aligned}\mathrm{M}_{\mathrm{hs}}^2 &= \left(M_{\mathrm{AdS}_3} + \mu\right)^2 j(j+1) + \left(M_{\mathrm{AdS}_3}^2 - \mu^2\right) j \\ &= \left(\frac{1+\mu l_{\mathrm{AdS}_3}}{l_{\mathrm{AdS}_3}}\right)^2 j(j+1) + \left(\frac{1-\mu^2 l_{\mathrm{AdS}_3}^2}{l_{\mathrm{AdS}_3}^2}\right) j,\end{aligned} \tag{51}$$

exhibiting two well known quantum numbers of SL(2,ℝ) representations namely the second Casimir $j(j+1)$ and the Cartan charge j of highest weight (HW) state. By framing (49) in the form (51), the dependence on the second Casimir comes as no surprise especially in holographic contexts like ours. In AdS/CFT, the mass spectrum of states is encoded in the eigenvalues of the Casimir operator. In fact, the action of the quadratic Casimir is related to the mass of the quantum states as follows [39–44]:

$$m^2 l_{AdS}^2 = C_2, \tag{52}$$

exactly as given by the first half of (51) namely:

$$\hat{\mathrm{M}}_{\mathrm{hs}}^2 = \mathfrak{m}_2 \mathcal{C}_2. \tag{53}$$

However, a state in the spectrum is uniquely identified once considering both the Casimir and its Cartan charge $(j(j+1), j)$. For example, states in the same HW representation share the same Casimir value, so to distinguish between individual states within that representation, we must take into account, in addition to the Casimir, the Cartan charge operator L_0 as given in (7):

$$\hat{\mathrm{M}}_{\mathrm{hs}}^2 = \mathfrak{m}_0 L_0 + \mathfrak{m}_2 \mathcal{C}_2. \tag{54}$$

Then, we use the conjecture to identify and established a formula of the coefficients $\mathfrak{m}_0$ and $\mathfrak{m}_2$ in terms of the parameters of the HSTMG theory at hand, we have:

$$\mathfrak{m}_2 = \left(\frac{1+\mu l_{\mathrm{AdS}_3}}{l_{\mathrm{AdS}_3}}\right)^2 , \tag{55}$$

$$\mathfrak{m}_0 = \left(\frac{1-\mu^2 l^2_{\mathrm{AdS}_3}}{l^2_{\mathrm{AdS}_3}}\right) , \tag{56}$$

giving thus (51).

Moreover, since $j \geq 1$ due to the condition $s \geq 2$, we have the property $j(j+1) > j$ implying that the dominant term in the $\mathrm{M}^2_{\mathrm{hs}}$ formula is given by the block term $\left(M_{\mathrm{AdS}_3}+\mu\right)^2 j(j+1)$. Furthermore, we can note two additional valuable features:

(**i**) In the region of the parameter space of the higher spin theory where $M^2_{\mathrm{AdS}_3} - \mu^2$ is negative definite (i.e: $1-\mu^2 l^2_{\mathrm{AdS}_3} < 0$); we have

$$\mu^2 > M^2_{\mathrm{AdS}_3} \quad \Leftrightarrow \quad \mu^2 > \frac{1}{l^2_{\mathrm{AdS}_3}} , \tag{57}$$

and then the mass formula (51) induces the following inequality

$$\mathrm{M}^2_{\mathrm{hs}} < \left(M_{\mathrm{AdS}_3}+\mu\right)^2 j(j+1) \quad \Leftrightarrow \quad \mathrm{M}^2_{\mathrm{hs}} < \left(\frac{1+\mu l_{\mathrm{AdS}_3}}{l_{\mathrm{AdS}_3}}\right)^2 j(j+1) , \tag{58}$$

which corresponds precisely to (41); thus offering a natural candidate for the swampland conjecture regarding HS topological AdS_3 massive gravity.

(**ii**) For the critical value $\mu^2 = \mu_c^2 = M^2_{\mathrm{AdS}_3}$, the block term $\left(M^2_{\mathrm{AdS}_3} - \mu_c^2\right) j$ in (51) vanishes; and the mass formula $\mathrm{M}^2_{\mathrm{hs}}$ in (51) is equal to $(\mathrm{M}^2_{\mathrm{hs}})_c = 4M^2_{\mathrm{AdS}_3} j(j+1)$.

So, using $\mu^2 \simeq M^2_{\mathrm{AdS}_3} + \delta\mu^2$ with positive $\delta\mu^2$, eq(51) becomes

$$\mathrm{M}^2_{\mathrm{hs}} = 4M^2_{\mathrm{AdS}_3} j(j+1) - \left(\delta\mu^2\right) j , \tag{59}$$

thus leading to the inequality

$$\mathrm{M}^2_{\mathrm{hs}} \leq 4M^2_{\mathrm{AdS}_3} j(j+1) \quad \Leftrightarrow \quad \mathrm{M}^2_{\mathrm{hs}} \leq \frac{4}{l^2_{\mathrm{AdS}_3}} j(j+1) . \tag{60}$$

In comparison with (41) stipulating $\mathrm{M}^2_{\mathrm{hs}} \leq 2\mathrm{Q}^2_{\mathrm{hs}} g^2_{\mathrm{hs}} M^2_{\mathrm{Pl}}$, one can deduce the expressions of both the charge Q_{hs} and the coupling constant g_{hs}; they are given by

$$\mathrm{Q}^2_{\mathrm{hs}} = j(j+1) = s(s-1) , \tag{61}$$

$$g^2_{\mathrm{hs}} = \frac{2M^2_{\mathrm{AdS}_3}}{M^2_{\mathrm{Pl}}} = \frac{2}{M^2_{\mathrm{Pl}} l^2_{\mathrm{AdS}_3}} , \tag{62}$$

with higher spin $s = 1+j$. The expression of the coupling constant (62) can be presented otherwise by using the CS level relation $\mathrm{k} = l_{\mathrm{AdS}_3}/(4G_N)$, which gives

$$g^2_{\mathrm{hs}} = \frac{1}{8\mathrm{k}^2 G_N^2 M^2_{\mathrm{Pl}}} , \tag{63}$$

where the dependence on the Chern-Simons coupling k, the Newton constant G_N as well as Planck mass M_{Pl} is exhibited. As these constant are interconnected, we can further unclutter the expression by using the relation $M_{\rm Pl}G_N = 1/(8\pi)$ to showcase that $g_{\rm hs}$ is merely the inverse of the Chern-Simons k:

$$g_{\rm hs}^2 = \left(8\pi^2\right)/{\rm k}^2\,. \tag{64}$$

To justify interpreting $j(j+1) = s(s-1)$ as a higher spin charge, we draw connections to results from the Literature on charged black holes in higher dimensions, including comparisons with: **(i)** the extremality constraints for Kerr-Newman black hole in D-dimensions, and **(ii)** the extremality constraint on HS-BTZ entropy.

We begin by recalling that the mass spectrum of charged states in D-dimensional effective field theories coupled to gravity (D>3) can be categorised into two distinct regimes based on their mass to charge ratios, as stipulated by the weak gravity conjecture [45]. These regimes are referred to as sub-extremal and super-extremal, as described below:

(a)- *sub-extremal regime*:
This phase consists of massive charged states $|M_{BH}, Q_{BH}\rangle$ whose mass is bounded from below like $M_{BH}^2 \geq \frac{g^2}{\sqrt{G}} Q_{BH}^2$ where g is the gauge coupling of the charge symmetry and G is the "Newton" constant. These states correspond to black holes solutions and are to referred to as sub-extremal. In the particular instance where $\left(M_{BH}^2\right)_{ext}$ is equal to $\frac{g^2}{\sqrt{G}}\left(Q_{BH}^2\right)_{ext}$, the charged black hole is said to be extremal; its mass is given by the charge, up to a multiplicative constant.

(b)- *super-extremal regime*:
This phase is composed of quantum particle states $|m, Q\rangle$ with mass bounded from above like $m^2 \leq \frac{g^2}{\sqrt{G}} Q^2$. In this regime, the masses of the states satisfy $m < M_{BH}$; they describe super-extremal particles emitted during the discharge of the black hole. The kinematical properties of these states are governed by two main conservation laws: the total energy-momentum conservation inducing the mass inequality $\sum_i m_i \leq M_{BH}$ and the charge conservation given by the equation $Q_{BH} = \sum_i Q_i$. Following [46], the presence of super-extremal particles ensures the decay of the black hole by maintaining $\frac{Q}{m} \geq (\frac{Q_{BH}}{M_{BH}})_{ext} \sim O(1)$. Notice that the existence of super-extremal states is necessary for the consistency of the quantum gravity theory. It is also worth noting that particle states with the particular mass $m^2 = \frac{g^2}{\sqrt{G}} Q^2$ can be viewed as BPS-like states.

In our setting, we similarly identify two distinct regimes **(a)** and **(b)**. First, the sub-extremal higher spin BTZ black hole corresponding to a state $|M_{BH}, Q_{BH}\rangle$ in regime-**(a)**, characterised by the inequality $M_{BH}^2 \geq \frac{g^2}{\sqrt{G}} Q_{BH}^2$. The second regime-**(b)** concerns the super-extremal higher spin particles $|{\rm M}_{\rm hs}, {\rm Q}_{\rm hs} >$ emitted by the unstable higher spin BTZ black holes. These particle states have masses constrained by ${\rm M}_{\rm hs}^2 \leq M_{BH}^2$; see eqs (3.34-3.35) and (3.75) in [46]. Additionally, they verify the WGC condition:

$${\rm M}_{\rm hs}^2 \leq \left(\frac{1}{l_{{\rm AdS}_3}} + \mu\right)^2 j(j+1) \leq M_{BH}^2\,.$$

Comparing our conjecture (51) with the general super-extremality constraint for emitted particles, namely $m^2 \leq \left(g^2/\sqrt{G}\right) Q^2$ [45], we identify a correspondence between $\sqrt{j(j+1)}$ and the quantised HS charge ${\rm Q}_{\rm hs}$. Using this correspondence, we obtain ${\rm Q}_{\rm hs} = \sqrt{s(s-1)}$ where $j = s-1$ as shown in eq(4.10). Moreover, for sufficiently large enough values of spin s, the charge ${\rm Q}_{\rm hs}$ mirrors the spin in agreement with HS theory labeled by quantum numbers of sl(2,$\mathbb{R}$) and aligning with the CFT's conserved currents at the boundary of AdS. Furthermore, this spin-dependent charge ${\rm Q}_{\rm hs}$ is not novel in other physical contexts. For instance, in spintronics, spin accumulation at the boundaries of a material creates a spin-dependent electric field that links spin to the distribution of charges. This enables the conversion of a charge current into a spin current through a process known as the Spin Hall effect [56] or conversely through the inverse spin hall effect [57].

(i)- Comparison with extremality conditions of Kerr-Newman type of Black holes
First, recall that the extremality constraint for Kerr-Newman black hole in D-dimensions (i.e. **(a)**-regime black holes) with metric, [47]

$$ds^2_{KN} = -\frac{\delta}{\rho^2}\left(dt - \mathrm{a}_{KN}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\mathrm{a}_{KN}dt - \left(r^2 + \mathrm{a}^2_{KN}\right)d\phi\right]^2 .$$

where $\delta = r^2 - 2M_{KN}r + \mathrm{a}^2_{KN} + Q^2_{KN}$ and $\rho = r^2 + \mathrm{a}^2_{KN}\cos^2\theta$, relates the black hole's mass M_{KN}, its electric charge Q_{KN} and angular momentum a_{KN} by the quadratic relation $M^2_{KN} = Q^2_{KN} + \mathrm{a}^2_{KN}$ [47]. This triangular identity highlights the interchangeable roles of the electric charge Q^2_{KN} and angular momentum a^2_{KN} in the *extremality* condition ($\delta = 0$), as the permutation $Q_{KN} \longleftrightarrow \mathrm{a}_{KN}$ leaves the *extremal* mass M^2_{KN} invariant. Since the WGC bounds in the **(b)**-regime reflects contributions from the same physical parameters as in the **(a)**-regime, we expect an analogous triangular structure to emerge for the **(b)**-regime states, as conjectured for the emitted higher spin particles M_{hs}. In our case, the HS-BTZ black hole possesses a HS charge and momentum, both of which are similarly encoded in M_{hs}.
In this regard, we reference a WGC bound that has been proposed in [48] for Kerr-Newman AdS black holes. It remarkably incorporates the extremal black hole's parameters along with the charge Q of particles in the vicinity of the black hole horizon. It is of the form:

$$\frac{Q a_{KN}\,(Q_{KN})_{ext}}{(M_{KN})_{ext} + 4a^3_{KN}/l^2} \geq 1 .$$

For particles with charge $Q = \frac{1}{a_{KN}}$ and $\frac{a_{KN}}{l} << 1$, the bound leads to the extremality constraint $(Q_{KN}/M_{KN})_{ext} \sim O(1)$. This suggests that imposing conditions on **(b)**-regimes particles can, a priori, imply constraints on **(a)**-regime black holes. However, can a similar analysis be extended to HS-BTZ black holes in AdS_3?

(ii)- Comparing (51) with an extremality constraint on HS-BTZ entropy:
We begin by recalling that for higher spin BTZ black holes in AdS_3, an extremality constraint was proposed for higher spin sl(3) gravity in [49], see also [50, 51]. It is based on deriving entropy using holonomies of flat connections in Chern-Simons theory of the higher spin gravity, and then imposing the reality condition, which requires the inequalities:

$$|\mathcal{W}_+| \leq |\eta\,(\mathcal{L}_+)^{3/2}|\,, \quad |\mathcal{W}_-| \leq |\eta\,(\mathcal{L}_-)^{3/2}|\,, \tag{65}$$

where $(\mathcal{L}_+, \mathcal{W}_+)$ as well as $(\mathcal{L}_-, \mathcal{W}_-)$ define the left and the right conserved currents at the AdS_3 boundary respectively. To precisely explore the connection between this extremality condition on the (**a**)-regime black holes and our (**b**)-regime constraint on higher spin massive particles within the framework of higher spin topologically *massive* gravity (HSTMG), we need:
(1)- A constraint relation analogous to the reality condition of entropy (65). Unfortunately, to the best of our knowledge, extremality constraints for black holes in 3D higher spin massive gravity have yet to be established.
(2)- To formulate an extremality bound for HS BTZ black holes in HSTMG. First, one must consider theories beyond the standard massive gravity [54] used in our investigation. For instance, minimal models [52], or chiral theories [53] could provide a much simplified framework for entropy calculations. Then, extend these models to incorporate higher spin fields to compute the entropy in terms of the CS flat connections, impose the reality condition, deduce the extremality bounds and subsequently investigate the implications for the weak gravity conjecture.

Given these challenges, we plan to pursue this as a direction for future research.

4.2 Refining eqs(61-62) and the super-extremal tower

The emergence of quantum numbers of the SL(2,$\mathbb{R}$) representations in the conjectured mass formula (51) makes one ponder about other hidden facets of $\mathrm{M}_{\mathrm{hs}}^2$. Below, we give two interesting features allowing to refine the eqs(61-62):

The first feature concerns the algebraic interpretation of the expression $\mathrm{M}_{\mathrm{hs}}^2$ (51); in fact, $\mathrm{M}_{\mathrm{hs}}^2$ can be perceived as the eigenvalue of a mass operator $\hat{\mathrm{M}}^2$ acting on the quantum particle states $|\Delta, N\rangle$ emitted by the HS-BTZ black hole as follows

$$\hat{\mathrm{M}}^2 |\Delta, N\rangle = \mathrm{M}_{\mathrm{hs}}^2 |\Delta, N\rangle \,, \tag{66}$$

with positive inetegers Δ and N. Acting on these quantum states $|\Delta, N\rangle$ by the mass operator

$$\hat{\mathrm{M}}^2 = \left(M_{\mathrm{AdS}_3} + \mu\right)^2 \mathcal{C}_2 + \left(M_{\mathrm{AdS}_3}^2 - \mu^2\right) L_0 \,, \tag{67}$$

we get its eigenvalues in terms of the quantum numbers Δ and N; they read as follows

$$\mathrm{M}_{\Delta,N}^2 = \left(M_{\mathrm{AdS}_3} + \mu\right)^2 \Delta(\Delta - 1) + \left(M_{\mathrm{AdS}_3}^2 - \mu^2\right)(\Delta + N) \,. \tag{68}$$

Because $N \in \mathbb{N}$, the quantum states $|\Delta, N\rangle$ define an infinite tower of states candidates for the emitted particles of the HS-BTZ black hole. In this regard, recall that in AdS_3 one must upgrade the mild WGC to stronger forms like the lattice WGC of [17]. The refined HSC is given by the tower WGC [58] occupied by super extremal higher spin states (68) fulfilling the mass to charge constraint (41). In fact, the self-interacting particle condensate stills forms for HSTMG because of the AdS_3 boundary conditions, which can act as a box that reflects back the emitted particles, enabling them to self-interact in a sub-extremal cloud. Our setting and the physics therein, with the additional mass deformation, is still governed by the choice of the boundary conditions, similar to the standard theory. This suggests that the same arguments presented in the standard case [17] for the WGC refinement beyond the mild version are still applicable.

However, there is another way to justify the need for a refined version for the class of theories we are considering. Under diagonal boundary conditions, one can identify the higher spin symmetry $SL(N) \times SL(N)$, with the affine $U(1)^{2(N-1)}$ asymptotically. This shows the presence of multiple U(1) gauge fields at the boundary which further motivates the need for a WGC formulation beyond the mild version involving a single U(1) [59–61].

Now which version of the refined WGC should we apply? Usually, in the presence of multiple U(1)s, one can impose the convex hall condition [62]. However, it only requires the emission of a super-extremal vector (a multi-particle state) which would be problematic for the condensate in this case. Our proposed refinement is more natural as it is based on the sl(2,$\mathbb{R}$) representation. Since the emitted super-extremal states with masses $(\mathrm{M}_-^2)_{\Delta,N}$ are closely related to the unitary SL_2 representation $\mathcal{R}_\Delta^-$, the tower of states fulfilling HS Swampland conjecture was then given by the quantum states of $\mathcal{R}_\Delta^-$. We therefore identified the refinement of the WGC as the tower WGC.

The second feature regards the HS Swampland conjecture (42) namely $\mathrm{M}_{\mathrm{hs}} \leq \sqrt{2}\mathrm{Q}_{\mathrm{hs}} g_{\mathrm{hs}} M_{\mathrm{Pl}}$. This inequality puts a constraint on the appropriate unitary representation of SL_2 where the tower of super extremal particle states $|\Delta, N\rangle$ emitted by the HS-BTZ black hole resides. Because $M_{\mathrm{AdS}_3}^2 - \mu^2$ has an indefinite sign, we can distinguish three types of mass operators according to the value of μ^2 compared to $M_{\mathrm{AdS}_3}^2$. We have

$$\begin{aligned} (a): &\quad \mu^2 > M_{\mathrm{AdS}_3}^2 \,, \\ (b): &\quad \mu^2 = M_{\mathrm{AdS}_3}^2 \,, \\ (c): &\quad \mu^2 < M_{\mathrm{AdS}_3}^2 \,, \end{aligned} \tag{69}$$

these three phases are very common in the study of TMG theories [6, 7, 63, 64]. In fact by considering the additional gravitational CS term (43), the massive HS gravity theory develops a diffeomorphism anomaly given by the difference between the right c_+ and the left c_- central charges

$$c_\pm = \frac{3l_{AdS_3}}{G_N}\left(1 \pm \frac{1}{\mu l_{AdS_3}}\right), \tag{70}$$

leading to

$$\frac{1}{\mu} = \frac{G_N}{6}(c_+ - c_-)\,. \tag{71}$$

The value of μ is therefore a measure of the violation of parity in TMG. Additionally, one must note that the central charges are positive definite when $\frac{1}{\mu l_{AdS_3}} \leq 1$. As for the critical value $\mu l_{AdS_3} = 1$, it implies the vanishing of the central charges c_- and the resulting TMG theory was shown to be dual to a logarithmic CFT [37].

For all three phases (69), the mass operator takes the following forms

$$\begin{aligned}
(a):\quad & \hat{\mathrm{M}}_-^2 = \left(M_{AdS_3} + \mu\right)^2 \mathcal{C}_2 - \left|M_{AdS_3}^2 - \mu^2\right| L_0\,,\\
(b):\quad & \hat{\mathrm{M}}_0^2 = 4M_{AdS_3}^2 \mathcal{C}_2\,,\\
(c):\quad & \hat{\mathrm{M}}_+^2 = \left(M_{AdS_3} + \mu\right)^2 \mathcal{C}_2 + \left|M_{AdS_3}^2 - \mu^2\right| L_0\,.
\end{aligned} \tag{72}$$

Acting by these operators on the particle states $|\Delta, N\rangle$, we obtain the eigenvalues

$$\begin{aligned}
(\mathrm{M}_-^2)_{\Delta,N} &= \left(M_{AdS_3} + \mu\right)^2 \Delta(\Delta - 1) - \left|M_{AdS_3}^2 - \mu^2\right| (\Delta + N)\,,\\
(\mathrm{M}_0^2)_{\Delta,N} &= 4M_{AdS_3}^2 \Delta(\Delta - 1)\,,\\
(\mathrm{M}_+^2)_{\Delta,N} &= \left(M_{AdS_3} + \mu\right)^2 \Delta(\Delta - 1) + \left|M_{AdS_3}^2 - \mu^2\right| (\Delta + N)\,,
\end{aligned} \tag{73}$$

which for $\Delta > 1$, they obey the inequalities

$$\begin{aligned}
(\mathrm{M}_-^2)_{\Delta,N} &< \left(M_{AdS_3} + \mu\right)^2 \Delta(\Delta - 1)\,,\\
(\mathrm{M}_0^2)_{\Delta,N} &= 4M_{AdS_3}^2 \Delta(\Delta - 1)\,,\\
(\mathrm{M}_+^2)_{\Delta,N} &> \left(M_{AdS_3} + \mu\right)^2 \Delta(\Delta - 1)\,,
\end{aligned} \tag{74}$$

showing that $(\mathrm{M}_0^2)_{\Delta,N}$ is a critical mass. This feature allows to think about the $(\mathrm{M}_-^2)_{\Delta,N}$ inequality as follows

$$(\mathrm{M}_-^2)_{\Delta,N} \leq 4M_{AdS_3}^2 \Delta(\Delta - 1) \quad \Leftrightarrow \quad \mathrm{M}_{hs} \leq \sqrt{2}\mathrm{Q}_{hs} g_{hs} M_{Pl}\,, \tag{75}$$

from which we deduce the HS charge Q_{hs} and the HS coupling constant g_{hs} supported by the representation theory,

$$\mathrm{Q}_{hs} = \sqrt{\Delta(\Delta - 1)} \quad \Leftrightarrow \quad g_{hs} = \sqrt{2}\frac{M_{AdS_3}}{M_{Pl}}\,. \tag{76}$$

Notice finally that expressing eqs(73) as

$$(\mathrm{M}_\pm^2)_{\Delta,N} = \left(M_{AdS_3} + \mu\right)^2 \Delta(\Delta - 1) \pm \left|M_{AdS_3}^2 - \mu^2\right| (\Delta + N)\,, \tag{77}$$

we see that these masses $(\mathrm{M}_\pm^2)_{\Delta,N}$ are intimately related to the unitary SL_2 representations $\mathcal{R}_\Delta^\pm$. The tower of states fulfilling HS Swampland conjecture (75) is then given by the quantum states of $\mathcal{R}_\Delta^-$.

5 Piecing HSC in the WGC framework

The weak gravity conjecture is one of the seminal ideas in the swampland program, and may very well be the most properly argued swampland criteria. It has been studied in numerous settings with various parametrisations and configurations, giving many formulations that differ both in their assumptions as well as in their regime of applicability, for an extensive review refer to [45]. Pertaining to our concern, we will briefly look over some of its statements and implications for AdS theories.

5.1 WGC in AdS background

A prerequisite of any potential WGC formulation in a curved AdS_d space is the possibility to recover the usual bound of the flat space once the curvature $l_{AdS_\mathrm{d}} \to \infty$ [45]. Unfortunately, a general AdS formulation of the WGC is still a pending issue. However, there are many proposals like the one in [65]:

$$\frac{\delta^2}{l^2_{AdS_\mathrm{d}}} \leq \frac{d-2}{d-3}\frac{e^2q^2}{G_N^2}\,, \tag{78}$$

where δ is the conformal scaling dimension related to the mass $\mathbf{m}$ via

$$\delta = \frac{d-1}{2} + \sqrt{\frac{(d-1)^2}{4} + l^2_{AdS_\mathrm{d}}\mathbf{m}^2}\,. \tag{79}$$

In addition to the bound (78) having the d=3 singularity, it is not satisfied for all CFTs and it is unclear why this particular condition is most likely to hold universally [45]. Another Anti de Sitter WGC reformulation is given by the charge convexity conjecture [66], it imposes bounds in terms of binding energy using the lowest dimension operator of the associated CFT. Although the convex charge constraint is believed to be more general than the WGC, we disregard it as it differs from the usual statements motivated by black holes decay or long range forces.

To overcome the triviality of the constraint (78) for d=3, there is an alternative method that exploits tools of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. In [17] and more generally in [18], the weak gravity conjecture was indeed derived using a conformal approach by demanding the partition function of the boundary CFT_2 to be modular invariant. In a disjointed setting [18], where the gravitational and gauge sectors are distinct by considering 3D gravity in addition to a U(1) gauge field, it is possible to establish a constraint on the conformal dimension of the lightest charged state as follows [18],

$$\delta - \delta_{\mathrm{VAC}} \simeq \frac{c}{6} + \frac{3}{2\pi} + \mathcal{O}\left(\frac{1}{c}\right). \tag{80}$$

This bound is not optimal, and can be enhanced via additional symmetries. In fact, for 2D supersymmetric CFT with $\mathcal{N} = (1,1)$ supercharges, the constraint (80) on the conformal weight improves to $\delta \simeq 1 + O(1/c)$.

However, the constraint (80) isn't suitable for HS-TMG as it doesn't consider charged higher spin fields and only concerns U(1) charges.

Formulations	AdS background	D=3	BH solution	Massive HS fields	HS charge
WGC in AdS [65]	x	-	x	-	-
Convex Charge Constraint [66]	x	x	-	-	-
WGC in AdS_3 [17]	x	x	x	-	-

(81)

5.2 Beyond electric U(1) charges

There are other formulations of the WGC that experimented with parameters beyond the typical electric U(1) charges. For instance, the so called spinning weak gravity conjecture [67] where quantum or higher derivative corrections lead to perturbed (BTZ) black holes obeying a rotating version of the WGC that follows from the holographic c-theorem. Another interesting case is the causality bounds on higher spin particles coupled to stringy gravity in 4D [68]. In fact, in order for a 4D gravitational theory coupled to a tower of higher spin states to be causal, a WGC-like constraint must be imposed on the lightest HS particle. The 4D causality bound is reminiscent of the spin-2 conjecture requiring a cutoff on gravitational theories with massive higher spin fields [69].

Formulations	AdS background	D=3	BH solution	Massive HS fields	HS charge
WGC in AdS [65]	x	-	x	-	-
Convex Charge Constraint [66]	x	x	-	-	-
WGC in AdS_3 [17]	x	x	x	-	-
A spinning WGC [67]	x	x	x	-	-
HS causality [68]	-	-	x	x	-
Our HSC proposal	x	x	x	x	x

(82)

As evidenced, the HSC addresses a setting with a particular configuration to investigate the WGC. We derive a WGC-like constraint for black hole solutions of higher spin topological massive gravity carrying higher spin charges. The HSC stems from the core SL(2) algebraic representations and provides a constraint on the HS fields masses and charges to regulate the discharge of the HS BTZ solutions. Before further discussing the difference between the HSTMG and the more standard setup of the WGC with local U(1) degrees of freedom, let us review some of the main similarities.

For a higher spin gravity theory with $SL(N)\times SL(N)$ symmetry, imposing diagonal boundary conditions generates asymptotic symmetries governed by the $U(1)^{(N-1)}\times U(1)^{(N-1)}$ affine algebra [59–61]. The higher spin particles, higher spin versions of the graviton, emerge through composites of the $U(1)$ photons via a twisted Sugawara construction at the boundary. The BTZ black hole solution in this $SL(N)$ higher spin gravity theory is therefore analogous to a charged BTZ black hole solution in AdS_3 Einstein gravity with SL(2)×SL(2) coupled to $U(1)^{(N-2)}\times U(1)^{(N-2)}$ gauge fields. In this case, we introduced massless higher spin degrees of freedom, endowing the BTZ black hole with higher spin charges which correspond to $U(1)$ charges in the diagonal representation.

However, with the inclusion of the gravitational Chern-Simons term, we induce a mass deformation in the theory's geometry. This is evident from the modified equations of motion $G_{\mu\nu}+\frac{1}{\mu}C_{\mu\nu}=0$ having non vanishing Cotton tensor $C_{\mu\nu}\neq 0$ due to the presence of the CS gravitational term. Therefore, unlike the additional gauge charges, the CS gravitational term invokes a mass deformation, yielding massive higher spin degrees of freedom that effect the geometry of the spacetimes and the associated metrics. It becomes necessary to adapt the WGC to these new degrees of freedom that affect the black hole's stability and dynamics.

Exploring swampland conjectures from the lens of holographic theories has been of great interest recently. While we mainly focused on the WGC, there is a substantial body of work relating the swampland distance conjecture to higher spin theories as in [70] and the ensuing [71,72]. For instance in [70], it has been proposed that at infinite distances all theories possess an emergent HS symmetry in such a manner that certain proprieties of the conformal manifolds can be written as a function of the HS spectrum.

6 Conclusion

In this paper, we investigated a well motivated inquiry regarding the discharge of higher spin BTZ black holes in a higher spin topological massive gravity setting with Chern-Simons formulation based on rank-2 higher spin gauge symmetries. We proposed a higher spin Swampland conjecture to regulate the emission of super-extremal higher spin particles given by an upper bound on their mass to charge ratio.

En route to derive the higher spin swampland conjecture, we first established a correspondence between the massive higher spin AdS_3 models and effective gauge theories coupled to D-gravity (EFF_D) to hypothesize a formulation of the swampland constraint for higher spin BTZ black holes. Exploiting the principal SL(2,$\mathbb{R}$) of the higher spin gauge symmetry, we constructed the charge (20) and the mass (67) operators as well as their eigenvalues (73, 61). We also computed the higher spin gauge coupling constant (62) and showcased its relation to the inverse of the Chern-Simons level k (64).

Furthermore by using the infinite dimensional unitary representations, particularly the discrete series $\mathcal{R}^-_\Delta$, we built a tower of higher spin states (77) occupied by the emitted higher spin particles in accordance with the lattice refinement required for the AdS_3 space. We must note that the mass operator leading to the tower of higher spin states ensues from the phase $\mu^2 > M^2_{AdS_3}$ assuring the positivity of the central charges (70) as well as the unitarity of the CFT.

On a final note, we discussed the various WGC formulations especially for AdS backgrounds in different settings to place the higher spin swampland conjecture within the WGC framework as a way to emphasize the pertinence of our work regarding recent advancements in the swampland program. Overall, the antecedent results may imply several interpretations:

(i) The inclusivity of topological massive gravity within the general Landscape of consistent quantum gravitational theories.

(ii) Particularly, the established link between the higher spin conjecture and the WGC constraint conveys the validity of the later for topological massive higher spin gravitational models.

(iii) The existence of the tower of higher spin states is strongly supported by algebraic properties of the core SL(2,$\mathbb{R}$) of the HS gravity namely the discrete infinite unitary representation $\mathcal{R}^-_\Delta$.

Acknowledgments

Funding Information The work of R. Sammani is funded by the National Center for Scientific and Technical Research (CNRST) under the PhD-ASsociate Scholarship- PASS.

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