Resumming post-Minkowskian and post-Newtonian gravitational waveform expansions

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Abstract

We derive formulae that resum, at a given order in the soft limit, the infinite series of Post-Minkowkian (small gravitational coupling) or Post-Newtonian (small velocities) corrections to the gravitational waveform produced by particles moving along a general (open or closed) trajectory in Schwarzschild geometry in the probe limit. Specifying to the case of circular orbits, we compute the wave form and the energy flux to order 30PN, and compare it against the available results in the literature. The results are based on a novel hypergeometric representation of the solutions of the Heun equation (and its confluence), that leads to a simple mathematical proof of the Heun connection formula.

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1 Introduction

Recent advances in the sensitivity of experimental apparatus for detecting gravitational waves (GWs) necessitate increasingly precise theoretical computations. The next generation of space-based experiments, such as the Laser Interferometer Space Antenna (LISA), promises to revolutionize the GW detection by significantly enhancing sensitivity and exploring entirely new wavelength ranges compared to ground-based interferometers [1]. LISA will detect long-wavelength GWs originating from sources like supermassive black hole mergers, extreme mass-ratio inspirals (EMRIs), galactic binaries, and the Stochastic Gravitational Wave Background—sources largely inaccessible to ground-based observatories.

The gravitational signals from these sources are diverse. For instance, GWs from supermassive black hole mergers will produce strong signals with large signal-to-noise ratios, while those from EMRIs, though weaker, will carry exceptionally detailed information about the gravitational sources. Indeed, EMRIs waveforms, spanning thousands of cycles during their inspiral phase, provide a unique opportunity to probe fundamental questions about the nature of gravitational sources, including their compactness, the presence or absence of horizons, light rings, echoes [2], and the validity of Einstein's theory of gravity and its possible extensions [3,4].

Two main analytical tools have been developed for studying the problem we just described: the multipole expansion [5–7] and perturbation theory [8–19]. The latter, valid in the probe limit of the binary, the so called EMRI systems, is the focus of this paper. In this framework, GWs are computed by solving the Einstein equations to linear order in the perturbations induced by the motion of a lighter body orbiting around a very massive black hole (BH). The radial and angular propagations of the waves are governed by ordinary differential equations of the confluent Heun type. Analytic solutions of these equations are not known, but it is possible to write them as an expansion in the Newton constant (Post-Minkowskian, or PM, approximation) or in the orbital velocity (Post-Newtonian, or PN, approximation).

In the case of circular orbits, the PM and PN approximations are linked and characterised by a single expansion parameter v (the tangential velocity). The current state-of-the-art for the emitted energy in perturbation theory is the 22nd PN (order v^{44}) computation in [20]. The expansion is known to be converging slowly [9,21] due to the presence of logarithmic terms, leading to an infinite tower of tails and tails of tails non-local interactions [22–25]. In [26–35] novel techniques, capitalizing on recent advances in quantum field theories, have been exploited to derive an exact formula for the Heun connection matrix and alternative formulations of the waveforms were proposed in terms of the instanton partition function of a $SU(2)^2$ quiver gauge theory. Following this path the infinite tower of logarithmic terms have been resummed into exponentials.

More recently, multipole methods [36–38], scattering amplitude techniques [42–53] and black hole perturbation theory [54] have been applied to the study of gravitational scattering processes, fostering a fruitful interplay and mutual cross-checking among these complementary approaches. Within the quantum field theory framework, the PM series describes the expansion in loops (weak coupling) of the scattering amplitude keeping finite the relative velocity of the binary. Using perturbation theory, we will show that at any fixed order in the gravitational coupling, the entire series of PN corrections to the probe limit of the gravitational waveform can be resummed and expressed in terms of a single hypergeometric function. Analogous results hold for the PN expansion itself, where the exact dependence on the gravitational coupling corrections is determined at any given order in the small-velocity limit.

The PM and PN series provide novel hypergeometric representations of confluent Heun function closely in spirit to the well studied incoming and upgoing MST solutions [15]. The crucial difference between the two, is that while the MST Ansatz involves an infinite sum of hypergeometric functions, our solution is written in terms of a single hypergeometric function and its first derivative. The crucial advantage of this representation is that it provides a simple mathematical proof of the Heun connection formulae originally found in [26, 27, 29, 30]. The connection matrix coefficients are given as ratio of gamma functions with arguments depending on a single (but highly non-trivial) function codifying the holonomies of the confluent Heun function around the singular point. This function, known as the "renormalized angular momentum" in the MST context [10–19], can be related to the so called Seiberg-Witten period of a $\mathcal{N}=2$ supersymmetric gauge theory [31–35] or the "anomalous dimension" in the effective field theory description of the gravitational wave [55–58]. The Heun connection formula has been recently applied in a variety of physical contexts, including the study of QNMs for gravity backgrounds (BHs, fuzzballs, D-branes), correlators in thermal field theories, holography, cosmological perturbations and scattering amplitudes in gravity [27, 28, 32, 33, 35, 59–87].

To keep the material self-contained, in this paper we present a derivation of the Heun solutions (and its confluences) from first principles with no reference to supersymmetric gauge theory or localization, which sit in the background as inspiration to some of our Ansätze, that nonetheless could be also accepted as such. Finally, we remark, that even if the main motivation for resuming PM and PN series relies on the study of open orbits, our hypergeometric representations provide an algorithm which is very efficient when implemented in a Mathematica code to describe PN expansions of bounded orbits. Indeed, here we compute the gravitational waveform and the energy flux, in the case of circular orbits, up to the 30th PN order.[1] This result will be tested against the 22nd PN order computation available in literature [20], and results based on MST and numerical methods from the Black Hole Perturbation Toolkit (BHTK) [88]. The analytic expressions of all ingredients contributing to the 30PN expansion of the wave form and energy flux are collected in ancillary mathematical files. Numerical studies of unbounded systems are postponed to future investigations.

This is the plan of the paper: in Section 2 we derive the PM and PN expansion of the confluent Heun function. In Section 3 we apply the results of Section 2 to the study of the gravitational waveforms emitted by a particle moving in a Schwarzschild background. In Section 4 we test our results against those obtained via the MST method and numerical integration. In Appendix A we derive the PM and PN expansions of the non-confluent Heun function. Section 5 contains our conclusions.

Conventions

By confluent Heun equation we will refer to the ordinary differential equation (6) of second order, with two regular singularities at $z=1,\infty$ and an irregular one at $z=0$. The equation is specified by five complex parameters dubbed here as $\{m_1, m_2, m_3, u, x\}$. The expansion of the general solution of this equation in the "soft limit" $x \ll 1$ is derived in section 2. These results apply to any physical problem governed by an equation of confluent Heun type. The focus of this paper is on the study of gravitational wave forms in Schwarzschild geometry. The

[1] The ancillary Mathematica files coming with this paper should help the reader to check our computations.

Heun gravity dictionary in this case reads

$$
\begin{aligned}
x &= 4i\omega M\,, & y &= 2i\omega r\,, & z &= \frac{2M}{r}\,,\\
m_1 &= \tfrac{x}{2}\,, & m_2 &= m_3^* = -2+\tfrac{x}{2}\,, & u &= \left(\ell+\tfrac{1}{2}\right)^2+\tfrac{5x}{2}+\tfrac{x^2}{4}\,,
\end{aligned}
\tag{1}
$$

with $y = x/z$, M the mass of the BH, ω and ℓ the frequency and orbital number of the gravitational wave and r the radial distance. Along the paper we set $G = 1$ and use the mostly plus signature $(-+++)$.

2 The confluent Heun equation

The confluent Heun differential equation

$$
\mathrm{Hc}''(t)+\left(\frac{\beta_0}{t}+\frac{\beta_1}{t-1}+\lambda\right)\mathrm{Hc}'(t)+\frac{\alpha_1 t-\alpha_0}{t(t-1)}\mathrm{Hc}(t)=0\,, \tag{2}
$$

is an ordinary differential equation of second order with two regular singularities at $t = 0, 1$ and an irregular one at infinity. A basis of solutions is given by

$$
\begin{aligned}
Hc_+(t) &= \mathrm{HeunC}[\alpha_0,\alpha_1,\beta_0,\beta_1,\lambda,t]\,,\\
Hc_-(t) &= t^{1-\beta_0}\mathrm{HeunC}[\alpha_0+(1-\beta_0)(\lambda-\beta_1),\alpha_1+\lambda(1-\beta_0),2-\beta_0,\beta_1,\lambda,t]\,,
\end{aligned}
\tag{3}
$$

with HeunC the confluent Heun function.

2.1 Confluent Heun-Gauge dictionary

The confluent Heun equation can be viewed as the quantum version of the Seiberg-Witten (SW) curve governing the dynamics of a $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with three flavours of masses m_i, transforming in the fundamental representation of the gauge group [27, 28] living on a curved Nekrasov-Shatashvili background [89]. The quantum curve can be written in the form[2]

$$
G_1''(z)+G_1'(z)\left(\frac{1-m_1-m_2}{z-1}-\frac{2(m_3-1)}{z}+\frac{x}{z^2}\right)+G_1(z)\left(\frac{\frac{1}{4}-u+m_3(m_3-1)}{z^2}+\frac{u+\frac{3}{4}+\delta}{(z-1)z}\right)=0\,, \tag{6}
$$

with $\delta = m_1m_2+m_1m_3+m_2m_3-m_1-m_2-m_3$, x the gauge coupling and u the Coulomb branch parameter. (6) is related to the standard form (2) of the Heun equation by the Heun gauge dictionary

$$
\begin{aligned}
G_1(z) &= z^{-\frac{\alpha_1}{\lambda}}\mathrm{Hc}(\tfrac{1}{z})\,, & t &= z^{-1}\,, & \lambda &= -x\,,\\
\alpha_0 &= u-\tfrac{1}{4}+m_1(1-m_1)+(m_1+m_3-1)x\,, & & & \alpha_1 &= (m_1+m_3-1)x\,,\\
\beta_0 &= 1-m_1+m_2\,, & & & \beta_1 &= 1-m_1-m_2\,.
\end{aligned}
\tag{7}
$$

[2] Equation (6) can be written in the normal Schrödinger like form

$$
\Psi''(z)+Q_{\text{gauge}}(z)\Psi=0\,, \qquad G_1(z)=e^{\frac{x}{2z}}z^{m_3-1}(1-z)^{\frac{m_1+m_2-1}{2}}\Psi(z)\,, \tag{4}
$$

with

$$
Q_{\text{gauge}}=-\frac{x^2}{4z^4}+\frac{xm_3}{z^3}+\frac{1-(m_1-m_2)^2}{4(z-1)z}+\frac{1-(m_1+m_2)^2}{4(z-1)^2z}+\frac{u-\frac{1}{4}+\frac{x}{2}(m_1+m_2-1)}{(z-1)z^2}\,. \tag{5}
$$

Equivalently, (6) can be written as a differential equation in the variable $y = x/z$, taking the form

$$G_0''(y)+G_0'(y)\left(\frac{1-m_1-m_2}{y-x}+\frac{m_1+m_2+2m_3-1}{y}-1\right)+G_0(y)\left(\frac{u+\frac{3}{4}+\delta}{(x-y)y}+\frac{1+\delta+m_3(m_3-1)}{y^2}\right)=0\,, \tag{8}$$

with

$$G_1(\tfrac{x}{y}) = g(x)G_0(y)\,, \tag{9}$$

where $g(x)$ is a y-independent function. In these variables, the confluent Heun equations (CHEs) (6) and (8) have irregular singularities at $z = 0$ or $y = 0$ and their solutions transform non-trivially under rotations around these points. We denote by $G^0_\pm(y)$, $G^1_\pm(z)$, the eigenvectors of this action, i.e. solutions of the CHEs satisfying

$$G^0_\alpha\left(y\,e^{2\pi i}\right) = e^{-2\pi i \alpha a} G^0_\alpha(y)\,, \qquad G^1_\alpha\left(z\,e^{2\pi i}\right) = e^{2\pi i \alpha a} G^1_\alpha(z)\,, \tag{10}$$

with $\alpha = \pm$ and $a(u, x, m_i)$ is an unknown function of the Heun parameters. The quantity a will play a fundamental role in the following. One of the lessons of the Heun gravity correspondence is that the solutions of the CHE and the connection matrices look simpler if expressed in terms of a rather than in terms of u.

We will focus on the "soft limit" where x is taken small (large wave lengths), keeping finite either the variable z or y. We will refer to these two limits as PM and PN respectively. The two choices lead to equations of hypergeometric type. Indeed, sending $x \to 0$ in (6) and (8) lead to hypergeometric equations with solutions

$$\begin{aligned} \text{PM}: &\qquad G^0_\alpha(y) \underset{\substack{x\to 0\\ y\ \text{finite}}}{\approx} H^0_\alpha(y)\,, \\ \text{PN}: &\qquad G^1_\alpha(z) \underset{\substack{x\to 0\\ z\ \text{finite}}}{\approx} H^1_\alpha(z)\,, \end{aligned} \tag{11}$$

where

$$\begin{aligned} H^0_\alpha(y) &= y^{\frac{1}{2}-\alpha a-m_3}\,{}_1F_1\left(\tfrac{1}{2}-\alpha a-m_3; 1-2\alpha a; y\right)\,, \\ H^1_\alpha(z) &= z^{-\frac{1}{2}+\alpha a+m_3}\,{}_2F_1\left(\frac{1}{2}+\alpha a-m_1, \frac{1}{2}+\alpha a-m_2; 1+2\alpha a; z\right)\,. \end{aligned} \tag{12}$$

At this order, $a \underset{x\to 0}{\approx} \sqrt{u}$. For $x \neq 0$ but small one finds

$$\begin{aligned} \text{PM}_k: &\qquad G^0_\alpha(y) = y^{\frac{1}{2}-\alpha a-m_3}\sum_{i=0}^{k}\sum_{j=0}^{\infty} c^{PM}_{ij}(\alpha a) x^i y^{j-i} + O(x^{k+1})\,, \\ \text{PN}_k: &\qquad G^1_\alpha(z) = z^{-\frac{1}{2}+\alpha a+m_3}\sum_{j=0}^{k}\sum_{i=0}^{\infty} c^{PN}_{ij}(\alpha a) z^{i-j} x^j + O(x^{k+1})\,, \end{aligned} \tag{13}$$

and

$$u = a^2 + \sum_{i=1}^{k} u_i(a) x^i\,, \tag{14}$$

with the coefficients $c^{PM}_{ij}(a)$, $c^{PN}_{ij}(a)$, $u_i(a)$ determined by solving the CHE equations order by order in x. Inverting (14) one finds the dependence of the monodromy eigenvalue $a(u)$ on the parameter u. Moreover, since $G^1_\alpha(z)$ and $G^0_\alpha(\frac{x}{z})$ are solutions of the same equation and have the same monodromy around $z = 0$ they should be proportional to each other. We conclude therefore that in the overlap region, the so called near zone, where $x \ll z \ll 1$ we find

$$\text{Near zone}: \quad G^1_\alpha(z) = g_\alpha(x) G^0_\alpha\left(\tfrac{x}{z}\right)\,. \tag{15}$$

The main result of this Section is that, the series PM_k and PN_k in (13) can be exactly resummed and written in terms of a single hypergeometric function. This will provide us with a first principle proof of the connection formulae for the confluent Heun functions, where the whole non-triviality of the Heun equation is codified into a single function $a(u)$.

For concreteness, we will focus on the domain $z \in [0,1]$. The general solution $G_1(z) = \sum_\alpha c_\alpha G^1_\alpha(z)$ behaves near the boundaries as

$$\begin{aligned} G_1(z) &\underset{z\to 0}{\approx} B_-(1+\ldots) + B_+ z^{2m_3} e^{\frac{x}{z}}(1+\ldots)\,, \\ G_1(z) &\underset{z\to 1}{\approx} D_-(1+\ldots) + D_+(1-z)^{m_1+m_2}(1+\ldots)\,, \end{aligned} \tag{16}$$

where the dots stand for subleading terms in these limits. Specifying boundary conditions corresponds to fix two linear combinations made out of the four coefficients $B_\pm$, $D_\pm$. We will derive connection formulae relating the coefficients $B_\pm$ to $D_\pm$ and viceversa.

2.2 PM approximation: Exterior region

In this subsection we consider the limit $x \ll 1$ keeping y finite. This corresponds to the PM approximation and we refer to this region as *the exterior region*: The reason of it will be clear in the next Section from the dictionary with the gravity side (see (45)).

We denote the basic solution as $G^0_\alpha(\frac{x}{z})$ and at leading order in x we take

$$G^0_\alpha(y) = H^0_\alpha(y)\,. \tag{17}$$

For higher orders in x, proceeding by trial and error, we come to the following Ansätze

$$G^0_\alpha(y) = P_0(y)H^0_\alpha(y) + \widehat{P}_0(y)\, y H^{0\prime}_\alpha(y)\,, \tag{18}$$

where the prime stands for derivation and

$$\begin{aligned} P_0(y) &= 1 + \sum_{i=1}^{k}\sum_{j=0}^{i-1} c_{ij}\, x^i y^{j-i}\,, \\ \widehat{P}_0(y) &= \sum_{i=1}^{k}\sum_{j=0}^{i-1} \widehat{c}_{ij}\, x^i y^{j-i}\,, \end{aligned} \tag{19}$$

are polynomials of order k in x.[3] We plug this Ansätze into (6) and use the differential equation to express higher derivatives of H^0_α in terms of H^0_α and its first derivative. Therefore one finds a system of algebraic linear equations that determines the coefficients c_{ij}, $\widehat{c}_{ij}$ and u_i (whose complete forms are given in (71)). Setting $c_{00} = 1$, one finds, for the first term in the recursion

$$\begin{aligned} c_{10} = u_1 &= \frac{1}{2}(1 - m_1 - m_2 - m_3) - \frac{2m_1 m_2 m_3}{4a^2 - 1}\,, \\ \widehat{c}_{10} = \partial_{m_3} u_1 &= -\frac{1}{2} - \frac{2m_1 m_2}{4a^2 - 1}\,. \end{aligned} \tag{20}$$

It is easy to check that c_{ij}, $\widehat{c}_{ij}$ are invariant under $a \to -a$. This is true at any PM order, so all the dependence on α of the solution is codified in H^0_α. Explicit computations show that the coefficients $c_{i,i-1}$, $\widehat{c}_{i,i-1}$ can be always written in terms of derivatives of u_i. Indeed, one finds

$$\begin{aligned} c_{i,i-1} &= u_i\,, \\ \gamma_m &= 1 + \sum_{i=1}^{\infty} \widehat{c}_{i,i-1} x^i = e^{-\partial_{m_3}\mathcal{F}_{\text{inst}}}\,, \end{aligned} \tag{21}$$

[3] For $k = 1$ M. Billò, M. L. Frau and A. Lerda first obtained (18) resumming the contributions of the Young tableaux entering the partition function of the supersymmetric gauge theory.

with

$$\mathcal{F}_{\text{inst}} \equiv -\sum_{i=1}^{\infty} \frac{u_i}{i} x^i \,. \tag{22}$$

The subscript "inst" reminds the reader that this quantity is interpreted as the instanton prepotential in the gauge theory description of the Heun equation. Indeed in the limit $z \to 0$, where one of the two gauge couplings is turned off, the quiver theory reduces to a $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory with three massive fundamental flavours with prepotential $\mathcal{F}_{\text{inst}}$.

We notice that (18) is exact in y at the k-th PM order, so we can extrapolate it to the region of large y using the standard hypergeometric transformation rule

$$H^0_\alpha(y) = \sum_{\alpha'=\pm} B_{\alpha\alpha'} \widetilde{H}^0_{\alpha'}(y) \,, \tag{23}$$

with

$$\widetilde{H}^0_\alpha(y) = y^{-m_3(1+\alpha)} e^{\frac{y(1+\alpha)}{2}} {}_2F_0\left(\frac{1}{2} - a + \alpha m_3, \frac{1}{2} + a + \alpha m_3, \frac{\alpha}{y}\right), \tag{24}$$

and

$$B_{\alpha\alpha'} = e^{\frac{i\pi}{2}(1-\alpha')(\frac{1}{2}-\alpha a - m_3)} \frac{\Gamma(1-2\alpha a)}{\Gamma\left(\frac{1}{2} - \alpha a - \alpha' m_3\right)} \,. \tag{25}$$

We can therefore introduce an equivalent basis of solutions, naturally defined in the *far region*.

$$\widetilde{G}^0_\alpha(y) = P_0(y) \widetilde{H}^0_\alpha(y) + \widehat{P}_0(y) y \widetilde{H}^{0\prime}_\alpha(y) \,, \tag{26}$$

related to the previous one by the connection formulae

$$G^0_\alpha(y) = \sum_{\alpha'=\pm} B_{\alpha\alpha'} \widetilde{G}^0_{\alpha'}(y) \,. \tag{27}$$

2.3 PN approximation: Interior region

In this subsection we consider the limit $x \ll 1$ keeping z finite. This corresponds to the PN approximation and we refer to this region as *the interior region*. The analysis proceeds *mutatis mutandis* following the same steps of the previous subsection, exchanging the roles of y and z.

We denote by $G^1_\alpha(z)$ the basic solutions and at leading order in x we take

$$G^1_\alpha(z) = H^1_\alpha(z) \,. \tag{28}$$

Higher orders follow from the Ansätze

$$G^1_\alpha(z) = P_1(z) H^1_\alpha(z) + \widehat{P}_1(z) z H^{1\prime}_\alpha(z) \,, \tag{29}$$

where the prime stands for derivation and

$$\begin{aligned} P_1(z) &= 1 + \sum_{j=1}^{k} \sum_{i=0}^{j-1} d_{ij} z^{i-j} x^j \,, \\ \widehat{P}_1(z) &= \sum_{j=1}^{k} \sum_{i=0}^{j} \widehat{d}_{ij} z^{i-j} x^j \,. \end{aligned} \tag{30}$$

Setting $d_{00} = 1$, one finds that the first non-trivial coefficients are

$$d_{01} = \frac{1}{2} + \frac{2m_3(1-m_3)}{4a^2-1} \,, \qquad \widehat{d}_{01} = -\widehat{d}_{11} = \frac{2m_3}{4a^2-1} \,. \tag{31}$$

We notice that the results are invariant under $a \to -a$. This is true at any PN order, so all the dependence on α of the solution is codified in H^1_α.

Proceeding as before, we can extrapolate the PN representation to the region $z \approx 1$ using the hypergeometric transformation rule

$$H^1_\alpha(z) = \sum_{\alpha'=\pm} F_{\alpha\alpha'} \widetilde{H}^1_{\alpha'}(z)\,, \tag{32}$$

with

$$\widetilde{H}^1_\alpha(z) = z^{a+m_3-\frac{1}{2}}(1-z)^{\frac{1}{2}(m_1+m_2)(1+\alpha)}\, {}_2F_1\left(\frac{1}{2}+a+\alpha m_1, \frac{1}{2}+a+\alpha m_2; 1+\alpha(m_1+m_2); 1-z\right), \tag{33}$$

and

$$F_{\alpha\alpha'} = \frac{\Gamma(1+2\alpha a)\Gamma\left(-\alpha'(m_1+m_2)\right)}{\Gamma\left(\frac{1}{2}+\alpha a-\alpha' m_1\right)\Gamma\left(\frac{1}{2}+\alpha a-\alpha' m_2\right)}\,. \tag{34}$$

An equivalent basis of solutions defined in the *near horizon region* is therefore given by

$$\widetilde{G}^1_\alpha(z) = P_1(z)\widetilde{H}^1_\alpha(z) + \widehat{P}_1(z)\, z\, \widetilde{H}^{1\prime}_\alpha(z)\,, \tag{35}$$

satisfying the connection formulae

$$G^1_\alpha(z) = \sum_{\alpha'=\pm} F_{\alpha\alpha'} \widetilde{G}^1_{\alpha'}(z)\,. \tag{36}$$

2.4 Boundary conditions and connection formulae

The functions $G^0_\alpha(\frac{x}{z})$ and $G^1_\alpha(z)$ are solutions of the same differential equation and have the same monodromy around $z=0$, so they should be proportional to each other. We denote by $g_\alpha(x)$ their ratio

$$g_\alpha(x) = \frac{G^1_\alpha(z)}{G^0_\alpha(\frac{x}{z})} = x^{\alpha a+m_3-\frac{1}{2}}\left[1+x\left(\frac{1}{4}-\frac{m_3(m_1+m_2+2m_3-2)-m_1m_2}{4a^2-1}+\frac{8\,\alpha\, a\, m_1m_2m_3}{(4a^2-1)^2}\right)+\ldots\right]. \tag{37}$$

Expanding for small z and y one can check explicitly that this ratio depends only on $x = zy$. Furthermore one finds

$$\gamma(x) = \frac{g_+(x)}{g_-(x)} = x^{2a} e^{-\partial_a \mathcal{F}_{\text{inst}}(a,x)}\,, \tag{38}$$

with $\mathcal{F}_{\text{inst}}(a,x)$ defined in (22). Collecting all the different pieces together, we have at our disposal four different basis of solutions $\{G^0_\alpha, \widetilde{G}^0_\alpha, G^1_\alpha, \widetilde{G}^1_\alpha\}$ related to each other by the connection matrices $B_{\alpha\alpha'}$, $F_{\alpha\alpha'}$. More precisely, the general solution can be written as

$$G_1(z) = \sum_\alpha c_\alpha g_\alpha(x) G^0_\alpha(\tfrac{x}{z}) = \sum_{\alpha,\alpha'} c_\alpha g_\alpha(x) B_{\alpha\alpha'} \widetilde{G}^0_{\alpha'}(\tfrac{x}{z}) = \sum_\alpha c_\alpha G^1_\alpha(z) = \sum_{\alpha,\alpha'} c_\alpha F_{\alpha\alpha'} \widetilde{G}^1_{\alpha'}(z)\,. \tag{39}$$

The first line corresponds to the PM approximation where x is taken small keeping y finite. The second equality in this line is obtained using the connection formulae (27). Similarly, the second line corresponds to the PN approximation where x is taken small keeping z finite and the right hand side follows from (36). We stress that these connection formulae are exact. As in the case of the hypergeometric connection formulae, they can be written as ratios of gamma functions with the *caveat* that the arguments of these functions depend on $a(u)$. The whole non-triviality of the confluent Heun equation is codified in a single function $a(u)$. Expanding at the leading order the two right hand sides in (39) and using (21), one finds the asymptotics

$$\begin{aligned} G_1(z) &\underset{z\to 0}{\approx} \sum_{\alpha,\alpha'} c_\alpha g_\alpha(x) B_{\alpha\alpha'} \left(\tfrac{x}{z}\right)^{-m_3(1+\alpha')} e^{\frac{x(1+\alpha')}{2z}} e^{-\frac{1+\alpha'}{2}\partial_{m_3}\mathcal{F}_{\text{inst}}}\,, \\ G_1(z) &\underset{z\to 1}{\approx} \sum_{\alpha,\alpha'} c_\alpha F_{\alpha\alpha'} (1-z)^{\frac{1+\alpha'}{2}(m_1+m_2)} h_{\alpha'}\,, \end{aligned} \tag{40}$$

with

$$h_{\alpha'} = P_1(1) + \frac{1+\alpha'}{2}(m_1+m_2)\widehat{P}_1{}'(1)\,. \tag{41}$$

Once again the ratio of these coefficients takes a simple form

$$\frac{h_+}{h_-} = e^{\partial_{m_1+m_2}\mathcal{F}_{\text{inst}}}\,. \tag{42}$$

Comparing against (16) one finds the asymptotic coefficients

$$\begin{aligned} B_{\alpha'} &= \sum_\alpha c_\alpha g_\alpha(x) B_{\alpha\alpha'} x^{-m_3(1+\alpha')} e^{-\frac{1+\alpha'}{2}\partial_{m_3}\mathcal{F}_{\text{inst}}}\,, \\ D_{\alpha'} &= \sum_\alpha c_\alpha F_{\alpha\alpha'} h_{\alpha'}\,. \end{aligned} \tag{43}$$

3 Black hole perturbation

In this Section we apply the results obtained in the previous one to the study of gravitational waves in the Schwarzschild geometry.

3.1 Heun gravity correspondence

BHs perturbations are typically described by equations of the confluent Heun type. For example, the Teukolsky equation for spin s perturbations in the Schwarzschild geometry is given by

$$\frac{1}{\Delta(r)^s}\frac{d}{dr}\left[\Delta(r)^{s+1}R'(r)\right] + \left(\frac{\omega^2 r^4 - 2is(r-M)\omega r^2}{\Delta(r)} + 4is\omega r - (\ell-s)(\ell+s+1)\right)R(r) = 0\,, \tag{44}$$

with $\Delta = r(r-2M)$. We are interested in the ψ_4 fundamental mode [8,90] corresponding to $s=-2$. (44) can be put into the CHE form (6) after the following identifications

$$\begin{aligned} R(z) &= e^{-\frac{x}{2z}}(1-z)^{2-\frac{x}{2}}\left(\tfrac{x}{z}\right)^{-1-\frac{x}{2}} G_1(z)\,, \qquad & z &= \frac{2M}{r}\,, \qquad x = 4iM\omega\,, \\ m_1 &= \tfrac{x}{2}\,, \qquad m_2 = m_3^* = -2 + \tfrac{x}{2}\,, & u &= \left(\ell + \tfrac{1}{2}\right)^2 + \tfrac{5x}{2} + \tfrac{x^2}{4}\,. \end{aligned} \tag{45}$$

We denote by $R_{\text{in}}(z)$ a solution satisfying incoming boundary conditions at the horizon and by $R_{\text{up}}(z)$ that one satisfying upgoing boundary conditions at infinity, i.e.

$$\begin{aligned} R_{\text{in}}(z) &\approx \begin{cases} B_-^{\text{in}} e^{-\frac{x}{2z}} z^{-1-m_3} + B_+^{\text{in}} e^{\frac{x}{2z}} z^{m_3-1}\,, & z\to 0\,, \\ D_-^{\text{in}} (1-z)^{1-\frac{m_1+m_2}{2}}\,, & z\to 1\,, \end{cases} \\ R_{\text{up}}(z) &\approx \begin{cases} B_+^{\text{up}} e^{\frac{x}{2z}} z^{m_3-1}\,, & z\to 0\,, \\ D_-^{\text{up}} (1-z)^{1-\frac{m_1+m_2}{2}} + D_+^{\text{up}} (1-z)^{1+\frac{m_1+m_2}{2}}\,, & z\to 1\,, \end{cases} \end{aligned} \tag{46}$$

where we have set to zero the coefficients $D_+^{\rm in}$ and $B_-^{\rm up}$. In radial coordinates $R_{\rm in,up}(r) \equiv R_{\rm in,up}[z(r)]$. In terms of these solutions the Green function reads

$$G(r,r') = \frac{1}{W} \begin{cases} R_{\rm in}(r')R_{\rm up}(r)\,, & r' < r\,, \\ R_{\rm in}(r)R_{\rm up}(r')\,, & r < r'\,, \end{cases} \tag{47}$$

with

$$W = \frac{R_{\rm in}(r)R'_{\rm up}(r) - R'_{\rm in}(r)R_{\rm up}(r)}{\Delta(r)} = \frac{\mathrm{i}\omega}{2M^2} B_-^{\rm in} B_+^{\rm up}\,, \tag{48}$$

a constant built out of the Wronskian which is conveniently computed for $r \to \infty$, since its first derivative is zero as it follows using (44). In the following we use as independent variables the two dimensionless combinations

$$y = 2\mathrm{i}\omega r\,, \qquad x = 4\mathrm{i}\omega M\,. \tag{49}$$

The incoming and upgoing solutions can be written as linear combinations

$$R_{\rm in,up}(y) = \sum_\alpha c_\alpha^{\rm in,up} g_\alpha R_\alpha(y)\,, \tag{50}$$

of the two independent solutions of the Teukolsky equation, let us say in the PM representation, that according to (45) are given by

$$R_\alpha(y) = e^{-\frac{y}{2}} \left(1 - \frac{x}{y}\right)^{2-\frac{x}{2}} y^{-1-\frac{x}{2}} G_\alpha^0(y)\,. \tag{51}$$

$c_\alpha^{\rm in,up}$ are some coefficients, chosen such that the incoming boundary conditions at the horizon or upgoing at infinity are satisfied, i.e.

$$D_+^{\rm in} = B_-^{\rm up} = 0\,, \tag{52}$$

with asymptotic coefficients given by (16), (45)

$$\begin{aligned} B_{\alpha'}^{\rm in,up} &= \sum_\alpha c_\alpha^{\rm in,up} g_\alpha(x) B_{\alpha\alpha'} x^{1-\alpha' m_3} e^{-\frac{1+\alpha'}{2}\partial_{m_3}\mathcal{F}_{\rm inst}}\,, \\ D_{\alpha'}^{\rm in,up} &= \sum_\alpha c_\alpha^{\rm in,up} F_{\alpha\alpha'} h_{\alpha'} e^{-\frac{x}{2}}\, x^{1+m_3}\,. \end{aligned} \tag{53}$$

We are interested in the PN expansion of the solutions, since we want to describe binary systems in the limit where the distance between the two objects is large and velocities small. For example, for circular orbits with tangential velocity v, this corresponds to the limit where v is small, $y \sim v$ and $x \sim v^3$. Therefore the PN expansion can be written as

$$R_\alpha(y) = y^{\frac{3}{2}-\alpha a} \sum_{n=0}^\infty R_n(\alpha a, y)\,, \tag{54}$$

with

$$\begin{aligned} R_0(a,y) &= 1\,, \\ R_1(a,y) &= \frac{2y}{1-2a}\,, \\ R_2(a,y) &= \frac{(17-2a)y^2}{16(2a-1)(a-1)} + \frac{(2a-3)x}{4y}\,, \\ R_3(a,y) &= \frac{(7-2a)y^3}{8(2a-1)(2a-3)(a-1)} + \frac{(11-6a)x}{4(2a-1)}\,, \\ R_4(a,y) &= \frac{\left(4a^2-72a+163\right)y^4}{512(a-2)(a-1)(2a-3)(2a-1)} + \frac{\left(-4a^2+48a-119\right)yx}{64(2a-1)(a-1)} + \frac{\left(8a^3-4a^2-18a+9\right)x^2}{64y^2(a+1)}\,, \end{aligned} \tag{55}$$

and

$$a(\ell)=\ell+\frac{1}{2}+\frac{\left(15\ell^4+30\ell^3+28\ell^2+13\ell+24\right)x^2}{8\ell(\ell+1)(2\ell+1)\left(4\ell^2+4\ell-3\right)}+\dots \tag{56}$$

3.2 The incoming solution

The incoming solution is obtained by requiring no outgoing waves at the horizon, i.e. by setting $D_+^{\rm in}=0$. According to (53), such condition leads to

$$\sum_{\alpha=\pm} c_\alpha^{\rm in} F_{\alpha+}=0\,. \tag{57}$$

This determines the ratio of the two coefficients to be

$$\frac{c_+^{\rm in}}{c_-^{\rm in}}=-\frac{F_{-+}}{F_{++}}=\frac{\Gamma(-2a)\Gamma\left(\frac{1}{2}-m_1+a\right)\Gamma\left(\frac{1}{2}-m_2+a\right)}{\Gamma(2a)\Gamma\left(\frac{1}{2}-m_1-a\right)\Gamma\left(\frac{1}{2}-m_2-a\right)}\,. \tag{58}$$

On the other hand, using the first of (40), one finds the asymptotic behaviour at infinity

$$R_{\rm in}(y)\underset{y\to\infty}{\approx}\sum_{\alpha,\alpha'} c_\alpha^{\rm in} g_\alpha B_{\alpha\alpha'} e^{\frac{\alpha' y}{2}} y^{1-\alpha' m_3} e^{-\frac{1+\alpha'}{2}\partial_{m_3}\mathcal{F}_{\rm inst}}\,. \tag{59}$$

Comparing against the first line of $R_{\rm in}$ in (46), one finds

$$B_{\alpha'}^{\rm in}=\sum_\alpha c_\alpha^{\rm in} g_\alpha B_{\alpha\alpha'} x^{1-\alpha' m_3} e^{-\frac{1+\alpha'}{2}\partial_{m_3}\mathcal{F}_{\rm inst}}\,. \tag{60}$$

We notice that the incoming boundary condition determines only the ratio $c_+^{\rm in}/c_-^{\rm in}$, leaving undetermined the overall normalization $c_-^{\rm in}$. This dependence however cancels out in the Green function after dividing by the factor W (48). It is convenient therefore to define a ratio which does not depend on overall normalizations

$$\mathfrak{R}_{\rm in}(y)=\frac{R_{\rm in}(y)}{xB_-^{\rm in}}=C_{\rm in}\left[R_-(y)-\gamma\tfrac{F_{-+}}{F_{++}}R_+(y)\right], \tag{61}$$

with

$$C_{\rm in}=\frac{x^{\frac{x}{2}}}{B_{--}}\left(1-\gamma\tfrac{F_{-+}B_{+-}}{F_{++}B_{--}}\right)^{-1}\underset{x\to 0}{\approx}(-1)^{\ell+1}\frac{\Gamma(\ell-1)}{\Gamma(2\ell+2)}+\dots \tag{62}$$

We notice that in the PN limit $R_\alpha(y)\sim y^{\frac{3}{2}-\alpha a}$ and $\gamma\sim x^{2a}$, so the γR_+-contribution in (61) is suppressed by an extra factor $(x/y)^{2a}$ with respect to R_-. The incoming solution is therefore dominated by the R_- component. Using (56) one finds the PN expansion

$$\begin{aligned} R_-(y)=y^{a+\frac{3}{2}}\Bigg[&1+\frac{y}{1+\ell}+\left(\frac{(9+\ell)y^2}{8(1+\ell)(3+2\ell)}-\frac{(\ell+2)x}{2y}\right)+\left(\frac{(\ell+4)y^3}{8(\ell+1)(\ell+2)(2\ell+3)}-\frac{(7+3\ell)x}{4(\ell+1)}\right) \\ &+\left(\frac{\left(50+19\ell+\ell^2\right)y^4}{128(\ell+1)(\ell+2)(2\ell+3)(2\ell+5)}+\frac{(\ell^2-1)(\ell+2)x^2}{4(2\ell-1)y^2}-\frac{(\ell+4)(\ell+9)xy}{16(\ell+1)(2\ell+3)}\right)+\dots\Bigg]. \end{aligned} \tag{63}$$

It is important to observe that at higher orders both $R_\pm(y)$ exhibit poles in the limit where $\ell\to\mathbb{Z}$, but one can check that they cancel against each other in the combination (61). For example, for $\ell=2$ the poles in the two components first appear at order v^{16}.

3.3 The upgoing solution

Upgoing boundary conditions correspond to setting $B_-^{\rm up}=0$, that according to (53) leads to

$$\frac{c_-^{\rm up}}{c_+^{\rm up}}=-\gamma\frac{B_{+-}}{B_{--}}\,. \tag{64}$$

Following the same passages of the incoming solution, we can write the asymptotic behaviour of $R_{\rm up}(y)$ at infinity, which is the same as (59) but with $c_\alpha^{\rm up}$ in place of $c_\alpha^{\rm in}$. This implies that also the coefficients $B_\alpha^{\rm up}$ take the same form of (60) with the same substitution $c_\alpha^{\rm in}\to c_\alpha^{\rm up}$. Again we introduce the ratio

$$\mathfrak{R}_{\rm up}(y)=(2M)^3\frac{R_{\rm up}(y)}{B_+^{\rm up}}=C_{\rm up}\left[R_+(y)-\tfrac{B_{+-}}{B_{--}}R_-(y)\right], \tag{65}$$

with

$$C_{\rm up}=\frac{(2M)^3e^{\partial_{m_3}\mathcal{F}_{\rm inst}}}{x^{3+\frac{x}{2}}B_{++}}\left(1-\tfrac{B_{-+}B_{+-}}{B_{++}B_{--}}\right)^{-1}=\frac{(2M)^3}{x^{3+\frac{x}{2}}}\frac{\Gamma(2-\ell)}{\Gamma(-2\ell)}+\dots \tag{66}$$

We notice that the leading term on the right hand side is finite in the limit where ℓ becomes an integer with $\ell\geq 2$. The same result is obtained by setting $\ell\to\mathbb{N}$ from the very beginning and sending then $x\to 0$. In the PN limit, the solution is dominated by the $R_+(y)$ component with PN expansion

$$\begin{aligned}R_+(y)=y^{\frac{3}{2}-a}\Bigg[1-\frac{y}{\ell}+\left(\frac{(\ell-1)x}{2y}-\frac{(\ell-8)y^2}{8l(2\ell-1)}\right)+\left(-\frac{(3\ell-4)x}{4\ell}+\frac{(\ell-3)y^3}{8(\ell-1)\ell(2\ell-1)}\right)\\
+\left(\frac{(\ell-1)\ell(\ell+2)x^2}{4(2\ell+3)y^2}-\frac{(\ell-8)(\ell-3)xy}{16\ell(2\ell-1)}+\frac{\left(32-17\ell+\ell^2\right)y^4}{128(\ell-1)\ell(2\ell-3)(2\ell-1)}\right)\Bigg].\end{aligned} \tag{67}$$

It is important to observe that again the poles in $R_\pm(y)$ in the limit where $\ell\to\mathbb{Z}$ cancel against each other in the combination (65). For $\ell=2$ the poles in the two components first appear at order v^7.

4 Wave form and energy flux at 30PN

In this Section we use perturbation theory to compute the probe limit of the wave form and the energy flux produced by a circular EMRI binary system at order 30PN, i.e. v^{60}, with v the tangential velocity of the light particle. The results will be compared against the 22PN results available in the literature [20] and the numerical computations using the package *Teukolsky* of the Black Hole Perturbation Toolkit (BHTK) [88] based on the MST method [10–19] and on a numerical integrations of the differential equation. We stress the fact that by 30PN, we mean that the polynomials coefficients P_0, $\widehat{P}_0$ and the exponent a, defining the solution are truncated at order 30PN, but the hypergeometric functions and the $c_\alpha^{\rm in}(a)$ coefficients specifying the solution are kept exact, so an infinite number of terms beyond 30PN are taken into account. In particular, the infinite tower of tail contributions (log divergences, π's and transcendental numbers) are resummed into exponentials and Gamma functions. The observables we consider are the (ℓ,m) harmonic modes of the incoming solution $\mathfrak{R}_{\rm in}$ and the energy flux radiated to infinity by the system.

Since for circular orbits $v^2=M/r$, and r should be taken bigger than the innermost stable circular orbit (ISCO) radius $r_{\rm ISCO}=6M$, the domain of variability of the tangential velocity is $v\in[0,\frac{1}{\sqrt{6}}]$. We find that the first 12 digits of 30PN results are stable for any choice v in this domain, so we will always display this number of digits. The numerical computations and those with the MST method are performed setting in the BHTK Mathematica files: AccuracyGoal = 1000, PrecisionGoal = 70, WorkingPrecision = 80, SetPrecision=80. With this choice of parameters both numerical and MST results agree to a 12 digits precision. In our tables these results will be denoted as BHTK.

4.1 The incoming and upgoing solutions

In this subsection we collect all the ingredients needed in the computations of the incoming, upgoing waveforms at order k=30 in the hypergeometric PM representation. We write[4]

$$\begin{aligned}\mathfrak{R}_{\text{in}}(y) &= C_{\text{in}} R_{-}(y) + (a \to -a)\,,\\ \mathfrak{R}_{\text{up}}(y) &= C_{\text{up}} R_{+}(y) + (a \to -a)\,,\end{aligned} \tag{68}$$

with[5]

$$\begin{aligned}
R_\alpha(y) &= e^{-\frac{y}{2}}\left(1-\tfrac{x}{y}\right)^{2-\frac{x}{2}} y^{-1-\frac{x}{2}}\left[P_0(y)\,H^0_\alpha(y) + \widehat{P}_0(y)\,y\,H^{0\prime}_\alpha(y)\right],\\
H^0_\alpha(y) &= y^{\frac{5}{2}+\frac{x}{2}-\alpha a}\,{}_1F_1\left(\tfrac{5}{2}+\tfrac{x}{2}-\alpha a; 1-2\alpha a; y\right),\\
C_{\text{in}} &= x^{\frac{x}{2}} e^{-\frac{i\pi}{2}(5+2a+x)} \frac{\Gamma(-\frac{3}{2}+a-\frac{x}{2})}{\Gamma(1+2a)} (1-\gamma_{\text{tot}})^{-1}\,,\\
C_{\text{up}} &= \frac{(2M)^3}{x^{3+\frac{x}{2}}\,\gamma_m} \frac{\Gamma(\frac{5}{2}-a+\frac{x}{2})}{\Gamma(1-2a)} \left(1 - e^{-2\pi i a}\,\frac{\cos\frac{\pi}{2}(x+2a)}{\cos\frac{\pi}{2}(x-2a)}\right)^{-1}\,,\\
\gamma_m &= 1 + \sum_{i=1}^{\infty} \widehat{c}_{i,i-1} x^i\,, \qquad \gamma = x^{2a} e^{-\partial_a \mathcal{F}_{\text{inst}}}\,,\\
\gamma_{\text{tot}} &= \gamma\, e^{-2\pi i a}\,\frac{\Gamma(-2a)^2\Gamma\left(-\frac{3}{2}+a-\frac{x}{2}\right)\Gamma\left(\frac{1}{2}+a-\frac{x}{2}\right)\Gamma\left(\frac{5}{2}+a-\frac{x}{2}\right)}{\Gamma(2a)^2\Gamma\left(-\frac{3}{2}-a-\frac{x}{2}\right)\Gamma\left(\frac{1}{2}-a-\frac{x}{2}\right)\Gamma\left(\frac{5}{2}-a-\frac{x}{2}\right)}\,.
\end{aligned} \tag{69}$$

The coefficients $c_{ij}, \widehat{c}_{ij}$ in the polynomials

$$P_0(y) = 1 + \sum_{i=1}^{k}\sum_{j=0}^{i-1} c_{ij}\, x^i y^{j-i}\,, \qquad \widehat{P}_0(y) = \sum_{i=1}^{k}\sum_{j=0}^{i-1} \widehat{c}_{ij}\, x^i y^{j-i}\,, \tag{70}$$

are determined by the recursion relations

$$\begin{aligned}
c_{ij} &= \Big[A_1\, c_{i,j-1} + A_2\, c_{i-1,j} + A_3\, c_{i-1,j-1} + \widehat{A}_1 \widehat{c}_{i,j-1} + \widehat{A}_2 \widehat{c}_{i-1,j} + \widehat{A}_3 \widehat{c}_{i-1,j-1}\\
&\quad + \sum_{s=1}^{k} c_{s,s-1}(C_1\, c_{i-s,j-s} + \widehat{C}_1\, \widehat{c}_{i-s,j-s})\Big]\left[8\Delta_{ij}(\Delta_{ij}^2 - 4a^2)\right]^{-1}\,,\\
\widehat{c}_{ij} &= \Big[B_1\, c_{i,j-1} + B_2\, c_{i-1,j} + B_3\, c_{i-1,j-1} + \widehat{B}_1 \widehat{c}_{i,j-1} + \widehat{B}_2 \widehat{c}_{i-1,j} + \widehat{B}_3 \widehat{c}_{i-1,j-1}\\
&\quad + \sum_{s=1}^{k} c_{s,s-1}(C_2\, c_{i-s,j-s} + \widehat{C}_2\, \widehat{c}_{i-s,j-s})\Big]\left[8\Delta_{ij}(\Delta_{ij}^2 - 4a^2)\right]^{-1}\,,\\
u_i &= c_{i,i-1}\,,
\end{aligned} \tag{71}$$

[4]Here we rewrite the second terms in the right hand sides of (61), (65), as the images under $a \to -a$ of the first terms, using the transformation rules $R_- \to R_+$, $\gamma \to \gamma^{-1}$, $F_{-\alpha'} \leftrightarrow F_{+\alpha'}$ and $B_{-\alpha'} \leftrightarrow B_{+\alpha'}$.

[5]We notice that $\gamma_{\text{tot}} = \gamma \frac{F_{-+}B_{+-}}{F_{++}B_{--}} = e^{-2\pi i a_D} = e^{-\partial_a(\mathcal{F}_{\text{tree}}+\mathcal{F}_{\text{one-loop}}+\mathcal{F}_{\text{inst}})}$ collects the tree level, one-loop and instanton contributions to the dual SW period a_D.

starting from $c_{0i} = \delta_{i0}$, $\hat{c}_{0i} = 0$, with

$$
\begin{aligned}
&A_1 = -8\left(\Delta_{ij}+1\right)\left(\Delta_{ij}-x-5\right), && \hat{A}_1 = 4\left(4a^2-(x+5)^2\right)\Delta_{i,j},\\
&A_3 = 8\Delta_{ij}\left(\Delta_{ij}-x-5\right), && \hat{A}_3 = 4\left(4a^2-(x+5)^2\right)\left(-\Delta_{i,j}+x-2\right),\\
&A_2 = 2\left(3x^2+6x-12a^2-13\right)\Delta_{ij}+8\Delta_{ij}^3-8(x-1)\Delta_{ij}^2+8a^2(x-7)-2(x^3+3x^2-13x-15), &&\\
&\hat{A}_2 = \left(4a^2-(x+5)^2\right)\left(-2(x-1)\Delta_{ij}+4a^2+x^2-2x-3\right), &&\\
&B_1 = -16\left(\Delta_{ij}+1\right), && \hat{B}_1 = 8\Delta_{ij}\left(\Delta_{ij}+x+5\right),\\
&\hat{B}_3 = 8\left(-\Delta_{i,j}^2-7\Delta_{ij}+x^2+3x-10\right), && B_3 = 16\Delta_{ij},\\
&B_2 = 4\left(-2(x-1)\Delta_{i,j}+4a^2+x^2-2x-3\right), &&\\
&\hat{B}_2 = 2\left(-\Delta_{ij}+x+1\right)\left(-4\Delta_{ij}^2-8\Delta_{ij}+12a^2+x^2+2x-15\right), &&\\
&C_1 = 8\left(\Delta_{ij}-x-5\right), && \hat{C}_1 = 16a^2-4(x+5)^2,\\
&C_2 = 16, && \hat{C}_2 = 8\left(\Delta_{ij}+x+5\right),
\end{aligned}
\tag{72}
$$

and $\Delta_{ij} = i - j$.

Finally one has to compute $a(\ell)$. Given $u_i(a)$ by (71), one can invert the series (13) to compute $a(u)$ and then set (from (45))

$$
u = (\ell+\tfrac{1}{2})^2 + \tfrac{5x}{2} + \tfrac{x^2}{4}. \tag{73}
$$

Inverting the series at high PM order can be time consuming for a Mathematica code. There is an alternative, more direct way to find the function $a(u)$ based on the connection between this quantity and the quantum SW period of a SU(2) theory with three fundamental flavours. The quantum SW period a is known to satisfy the continuous fraction equation [91]

$$
\frac{xM(a+1)}{P(a+1)-\frac{xM(a+2)}{P(a+2)-\ldots}} + \frac{xM(a)}{P(a-1)-\frac{xM(a-1)}{P(a-2)-\ldots}} - P(a) = 0, \tag{74}
$$

with

$$
\begin{aligned}
P(a) &= a^2+ax+2x-\frac{3x^2}{4}x-(\ell+\tfrac{1}{2})^2,\\
M(a) &= (a-\tfrac{x}{2}-\tfrac{1}{2})(a-\tfrac{x}{2}+\tfrac{3}{2})(a+\tfrac{x}{2}+\tfrac{3}{2}).
\end{aligned}
\tag{75}
$$

Equation (74) can be solved for $a(\ell)$ order by order in x. For example for $\ell = 2,3$ one finds

$$
\begin{aligned}
a(2) &= \tfrac{5}{2} + \tfrac{107x^2}{840} - \tfrac{1695233x^4}{148176000} + \tfrac{76720109901233x^6}{30764716012800000} - \tfrac{716388065858865707261481x^8}{996443210846050007040000000} + \ldots\\
a(3) &= \tfrac{7}{2} + \tfrac{13x^2}{168} - \tfrac{10921x^4}{4346496} + \tfrac{95353832269x^6}{493385539046400} - \tfrac{23627105510827613x^8}{1148838359958761472000} + \ldots
\end{aligned}
\tag{76}
$$

4.2 The luminosity

In this subsection, we collect the formulae needed to evaluate the luminosity at infinity for the case of a particle moving along a circular orbit in a Schwarzschild geometry.

After expanding in harmonics, the corresponding waveform measured by a far away observer sitting at a point with coordinates (T,R,Θ,Φ) can be written as

$$
h = h_+ - \mathrm{i}h_\times \underset{R\to\infty}{\approx} -\frac{4G}{R}\int\frac{d\omega}{2\pi}\sum_{\ell,m} e^{-\mathrm{i}\omega(T-R_*)}\frac{Z_{\ell m}(\omega)}{2(\mathrm{i}\omega)^2}\,{}_{-2}Y^{\ell m}(\Theta,\Phi), \tag{77}
$$

with R_* the tortoise coordinate. The harmonic coefficients $Z_{\ell m}$ can be written in terms of derivatives of the incoming solution

$$\begin{aligned} Z_{\ell m} = {} & \mathfrak{R}_{\text{in}}(y)\left(\frac{Ex^2 b_{0\ell m}}{2M^2 y(x-y)} + \frac{\mathrm{i}Jx^3\left(y^2+4y-4x\right)b_{1\ell m}}{8M^3y^3(x-y)} + \frac{J^2x^4\left(2x-4y-y^2\right)b_{2\ell m}}{64EM^4y^3(x-y)}\right) \\ & + y\partial_y\mathfrak{R}_{\text{in}}(y)\left(\frac{\mathrm{i}Jx^3 b_{1\ell m}}{4M^3y^3} + \frac{J^2x^4\left(2x-2y-2y^2\right)b_{2\ell m}}{16EM^4y^5}\right) \\ & - y^2\partial_y^2\mathfrak{R}_{\text{in}}(y)\frac{J^2x^4(x-y)b_{2\ell m}}{16EM^4y^5}\,, \end{aligned} \tag{78}$$

with

$$\begin{aligned} b^0_{\ell m} &= \frac{\pi}{2}\sqrt{(\ell^2-1)\ell(\ell+2)}\,Y_0^{\ell m}(\tfrac{\pi}{2},0)\,, \\ b^1_{\ell m} &= \pi\sqrt{(\ell-1)(\ell+2)}Y_{-1}^{\ell m}(\tfrac{\pi}{2},0)\,, \\ b^2_{\ell m} &= \pi\, Y_{-2}^{\ell m}(\tfrac{\pi}{2},0)\,, \end{aligned} \tag{79}$$

and

$${}_sY_{\ell m}(\theta,\phi) = e^{im\phi}\sin^{2\ell}\left(\frac{\theta}{2}\right)\sqrt{\frac{(2\ell+1)(\ell-m)!(\ell+m)!}{4\pi(\ell-s)!(\ell+s)!}}\sum_{r=0}^{\ell-s}(-)^{\ell+m-r-s}\binom{\ell-s}{s}\binom{\ell+s}{r+s-m}\cot(\tfrac{\theta}{2})^{2r+s-m}\,, \tag{80}$$

the spin-weighted spherical harmonics and

$$E = \frac{\mu(1-2v^2)}{\sqrt{1-3v^2}}\,, \qquad \omega J = \frac{\mu m v^2}{\sqrt{1-3v^2}}\,, \tag{81}$$

the energy and angular momentum of the light particle. Finally, the luminosity, i.e. the amount of energy per unit time emitted by the system towards infinity, is computed by the formula (see [16] and references therein)

$$\frac{d\mathcal{E}}{dt} = \mu^2\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\frac{|Z_{\ell m}|^2}{2\pi\omega^2} = \left(\frac{d\mathcal{E}}{dt}\right)_N\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\eta_{\ell m}(v)\,, \tag{82}$$

where μ is the mass of the orbiting body. Rather than the luminosity, we will display its ratio against the luminosity computed in the Newton approximation

$$\left(\frac{d\mathcal{E}}{dt}\right)_N = \frac{32\mu^2 v^{10}}{5M^2}\,. \tag{83}$$

4.3 Numerical tests

In this subsection we use our formulae computed at order 30PM and set $M = 1$. In Figure 1, we plot the real and imaginary parts of the incoming and outgoing solutions obtained with this method for $\ell = 2$, $\omega = 0.1$.

The results are compared in Table 1 against those obtained by the MST method and numerical integration using the BHTK [88]. In all the subsequent tables, the displayed digits are those ones that are insensible to the contribution of terms of order higher than v^{60} in $P_0(y)$ and $\widehat{P}_0(y)$. The agreement is up to 10 and 14 digits even for small values of r, where the PM approximation is less reliable.

In Figure 2 we display the v-dependence of the luminosity for the $(\ell, m) = (2,2)$, $(3,3)$, $(2,1)$ harmonic modes computed from (61) at order $k = 30$ in the PN approximation.

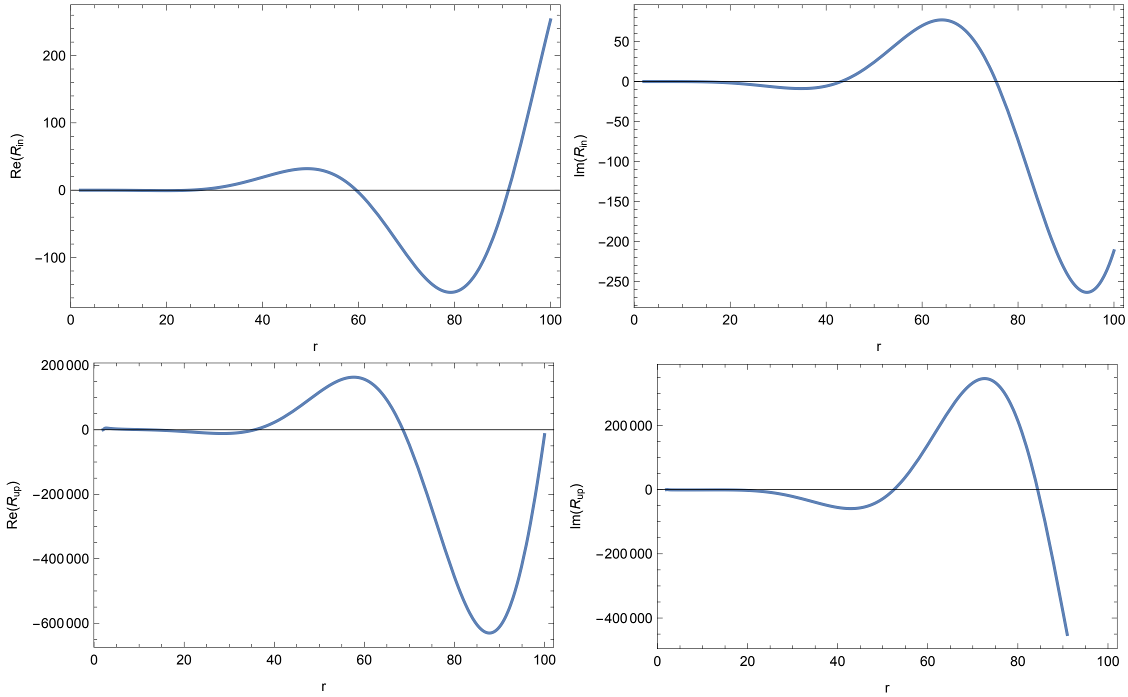

Figure 1: Plots of incoming and outgoing solutions for $\omega = 0.1, \ell = 2$. 1a: Re($\mathfrak{R}_{\text{in}}$), 1b: Im($\mathfrak{R}_{\text{in}}$), 2a: Re($\mathfrak{R}_{\text{up}}$), 2b: Im($\mathfrak{R}_{\text{up}}$).

Table 1: Comparison between the 30PM representation of confluent Heun solutions and MST/Numerical results for $\omega = 0.1, \ell = 2$.

$\mathfrak{R}_{\text{in}}$	$r = 5$	$r = 100$
HYP	-0.004086908784 -i 0.000826769654	252.949132499007 -i 211.392731187846
BHTK	-0.004086908784 -i 0.000826769654	252.949132499009 -i 211.392731187848
$\mathfrak{R}_{\text{up}}$	$r = 5$	$r = 100$
HYP	2632.048432 -i 840.387207	-15205.329855167 -i 989985.766975648
BHTK	2632.048432 -i 840.387207	-15205.329855168 -i 989985.766975651

The convergence speed of the results for some (ℓ, m) modes is showed in Tables 2 and 3. The digits in bold style underline the agreement between computations based on the PM hypergeometric representations and the MST/numerical results. As expected, the speed of convergence varies with v: in the case of the $(2,2)$ mode, for $v = 0.1$ an agreement at 12 digits is already reached at 10PM order, while for $v = 0.4$, the contributions at order 30PM are needed to reach a precision at all the considered digits. A similar behaviour is also found for higher (ℓ, m) modes. The last line of each sub-table shows the results coming from the 22PN computation in [20]. We observe that, opposite to what happens if one uses the 22PN expansion, our results agree against those obtained by the MST/Numerical methods for the first 10-12 digits independently of the ℓ mode. The difference is evident in the case of large ℓ, see for example the results for the $(10, 1)$ mode in Table 3.

Finally Table 4 we compare our 30PN results for the energy flux against those obtained by MST/Numerical methods and by the 22PN computation in [20].

We notice that in the whole range of v the results obtained with the 30PN hypergeometric representations for the total luminosity $d\mathcal{E}/dt$ agree with those obtained by the numerical integration with the MST method up to 11-13 digits.

Table 2: Convergence of the PM_k results for $\eta_{\ell,m}$, with $\ell = 2,3, 1 \leq m \leq \ell$, normalized with the Newtonian luminosity. The results obtained using the MST/ NUM methods and the 22PN computation in [20] are displayed in the last two lines of each sub-table.

k	$\eta_{2,2}(0.1)$	$\eta_{2,2}(0.3)$	$\eta_{2,2}(0.4)$
5	**0.9614837055**13	**0.841**181503167	**0.880**796957664
10	**0.961483705502**	**0.8411761**78745	**0.880**713162076
15	**0.961483705502**	**0.84117617132**8	**0.88070**5481610
20	**0.961483705502**	**0.841176171325**	**0.8807054**12127
25	**0.961483705502**	**0.841176171325**	**0.8807054116**30
30	**0.961483705502**	**0.841176171325**	**0.880705411627**
BHTK	**0.961483705502**	**0.841176171325**	**0.880705411627**
22PN	**0.961483705502**	**0.84117617132**6	**0.880705**608981

k	$\eta_{2,1}(0.1)$	$\eta_{2,1}(0.3)$	$\eta_{2,1}(0.4)$
5	**0.00027613091885**6	**0.0026641**1244976	**0.00572**740466492
10	**0.000276130918862**	**0.00266414**679182	**0.005730**06836479
15	**0.000276130918862**	**0.00266414686**348	**0.00573016**433854
20	**0.000276130918862**	**0.00266414686353**	**0.00573016551**089
25	**0.000276130918862**	**0.00266414686353**	**0.005730165520**70
30	**0.000276130918862**	**0.00266414686353**	**0.00573016552076**
BHTK	**0.000276130918862**	**0.00266414686353**	**0.00573016552076**
22PN	**0.000276130918862**	**0.00266414686353**	**0.00573016504468**

k	$\eta_{3,3}(0.1)$	$\eta_{3,3}(0.3)$	$\eta_{3,3}(0.4)$
5	**0.0127391414224**	**0.091120**2006109	**0.16**6992708659
10	**0.0127391414224**	**0.0911203195**830	**0.167029**390871
15	**0.0127391414224**	**0.091120319524**5	**0.167029**260059
20	**0.0127391414224**	**0.0911203195244**	**0.16702925823**9
25	**0.0127391414224**	**0.0911203195244**	**0.167029258222**
30	**0.0127391414224**	**0.0911203195244**	**0.167029258222**
BHTK	**0.0127391414224**	**0.0911203195244**	**0.167029258222**
22PN	**0.0127391414224**	**0.0911203195**262	**0.167029**722596

k	$10^6 \times \eta_{3,2}(0.1)$	$10^4 \times \eta_{3,2}(0.3)$	$10^3 \times \eta_{3,2}(0.4)$
5	**7.695632933**	**5.99470**001058	**2.24151**427554
10	**7.695632933**	**5.994701 1**1376	**2.24157**466375
15	**7.695632933**	**5.99470112089**	**2.2415769**1351
20	**7.695632933**	**5.99470112089**	**2.24157695**298
25	**7.695632933**	**5.99470112089**	**2.24157695337**
30	**7.695632933**	**5.99470112089**	**2.24157695337**
BHTK	**7.695632933**	**5.99470112089**	**2.24157695337**
22PN	**7.695632933**	**5.9947011208**8	**2.241576**33248

k	$10^6 \times \eta_{3,1}(0.1)$	$10^6 \times \eta_{3,1}(0.3)$	$10^5 \times \eta_{3,1}(0.4)$
5	**1.1829878889**	**8.212183**6172	**1.328**87186107
10	**1.1829878889**	**8.21218325**69	**1.32886**658816
15	**1.1829878889**	**8.2121832518**	**1.3288658**4761
20	**1.1829878889**	**8.2121832518**	**1.328865835**99
25	**1.1829878889**	**8.2121832518**	**1.32886583589**
30	**1.1829878889**	**8.2121832518**	**1.32886583589**
BHTK	**1.1829878889**	**8.2121832518**	**1.32886583589**
22PN	**1.1829878889**	**8.212183251**7	**1.32886**577283

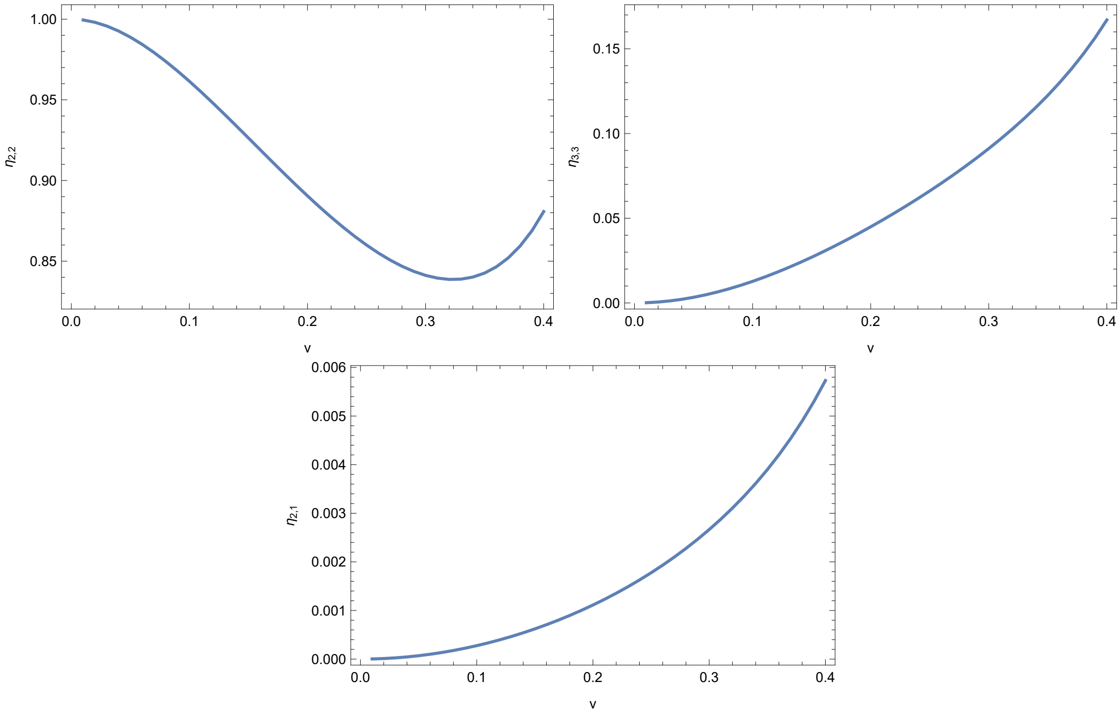

Figure 2: Luminosity vs velocity. We display the v-dependence of the ratio $\eta_{\ell m}/\eta^0_{22}$ for the three dominating modes. a: (2,2), b: (3,3), c: (2,1).

Table 3: Convergence of the PM$_k$ results for $\eta_{10,1}$ normalized by the Newtonian luminosity. The results obtained using the MST/ NUM methods and the 22PN computation in [20] are displayed in the last two lines.

k	$10^{39} \times \eta_{10,1}(0.1)$	$10^{31} \times \eta_{10,1}(0.3)$	$10^{29} \times \eta_{10,1}(0.4)$
5	**9.351820404**3	**8.6461**6460005	**4.13**218882098
10	**9.3518204044**	**8.64619838996**	**4.131878425**00
15	**9.3518204044**	**8.64619838996**	**4.13187842498**
20	**9.3518204044**	**8.64619838996**	**4.13187842498**
25	**9.3518204044**	**8.64619838996**	**4.13187842498**
30	**9.3518204044**	**8.64619838996**	**4.13187842498**
BHTK	**9.3518204044**	**8.64619838996**	**4.13187842498**
22PN	9.35182040436	8.64619786463	4.12989060363

Mathematica codes

Two Mathematica notebooks are given as ancillary files:

- **RinRupdRdt.nb** With this file it is possible to compute the $\eta_{\ell m}$ modes importing the data files P30.m, Pt30.m, gg30.m, gk30.m and a30.m, containing the polynomials $P_0(y)$, $\widehat{P}_0(y)$, the quantities γ, γ_κ and the a cycle. The results thus obtained will show a slower convergence with respect to the data of Tables 1, 2-3 and 4 which were obtained keeping higher orders than 30PN for $c_{ij}, \widehat{c}_{ij}$. The files with these data are quite large and can be obtained by writing to any of the authors.

Table 4: Total luminosity $d\mathcal{E}/dt$.

	$\nu = 0.1$	$\nu = 0.3$	$\nu = 0.4$
HYP	0.974719282291879	0.950003411484245	1.10991197121
BHTK	0.974719282291999	0.950003411484254	1.10991197120
22PN	0.974719282291877	0.950003411477562	1.10990619336

- **PNexpansions.nb** computes the PN expansions of $R_+(y)$, γ, γ_κ at any PM order lower than 10. This code imports the data files PNRpp10.m, PNgg10.m, PNgk10.m and PNaa10.m, containing the expansions of the quantities $R_+(y)$, γ, γ_κ and a at order 10PM. If these files are used to reproduce our results above the convergence will be a little slower.

5 Summary and conclusions

In this paper, using black hole perturbation theory, we derive exact formulae for the PM$_k$ and PN$_k$ expansions (at order k in the soft-limit $\omega M \ll 1$) of the gravitational wave form emitted by a particle moving on an arbitrary trajectory in Schwarzschild geometry at leading order in the probe limit. The results rely on a novel hypergeometric representation of the solutions of the Heun (or confluent Heun) equation (see Appendix A for the discussion of the non-confluence case). We build four equivalent basis of solutions, $\widetilde{G}^1_\alpha$, G^1_α, G^0_α and $\widetilde{G}^0_\alpha$ that describe with good accuracy different regions of the space time, and are connected to each other by Heun connection matrices. The final picture is summarized in the following scheme

$$\begin{array}{lll} \text{Near horizon} & \text{Near zone} & \text{Wave zone} \\ z \approx 1\,, & x \ll z \ll 1\,, & z \ll x \ll 1\,, \\ F_{\alpha\alpha'}\widetilde{G}^1_{\alpha'}\,, & G^1_\alpha = g_\alpha G^0_\alpha\,, & g_\alpha B_{\alpha\alpha'}\widetilde{G}^0_{\alpha'}\,, \end{array} \tag{84}$$

where $B_{\alpha\alpha'}$ and $F_{\alpha\alpha'}$ are the standard hypergeometric connection matrices relating z to $1/z$ or $1-z$ respectively. The whole non-triviality of the Heun connection formula is codified in the dependence of these matrices on the characteristic function $a(\ell)$. This simple result is a direct consequence of our Ansatz for the solution of the differential equation in terms of a single hypergeometric function and its derivative.

Specifying to circular orbits, we derive the probe limit of the wave form and energy flux at order 30PN. In contrast with the PN expansions that one can find in literature [7, 20], these formulae resum the infinite tower of logarithmically divergent terms (tail and tail of tails), responsible for the slow convergence of the PN series, into exponentials. Given that the results are too large to appear in print we collect them in ancillary Mathematica files. The results for the waveform have been compared against those obtained using the numerical package BHToolkit, finding an agreement from 9 to 12 digits in the whole range of parameters allowed for bounded orbits.

We remark that our results provide a hypergeometric representations of solutions of Heun equations of general type, so they apply to any physical problem governed by equations of Heun type. In particular, they can be easily adapted to the study of collisions of spinning BHs (Kerr geometry) or orbital motions in the backgrounds of topological stars, D-branes and cosmological perturbations [27, 28, 77–80]. It would also be very interesting to apply our results to the generation of templates of waveforms for GWs data analysis.[6] We hope to also come back to this issue in a future publication.

[6]We thank A. Nagar and M. Panzeri for useful discussions on this issue.

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A The Heun equation

The generalization of the previous results to the case of the non-confluent Heun equation is straightforward. The Heun differential equation is defined as

$$G_1''(z)+\left(\frac{\beta_0}{z}+\frac{\beta_1}{z-1}+\frac{\beta_2}{z-\lambda}\right)G_1'(z)+\frac{z\alpha_0\alpha_1-\sigma}{z(z-1)(z-\lambda)}G_1(z)=0\,, \tag{A.1}$$

with $\beta_2=1+\alpha_0+\alpha_1-\beta_0-\beta_1$. The solutions of the Heun equation can be written as linear combinations of

$$\begin{aligned}&\mathrm{Heun}_-(z)=\mathrm{Heun}[\lambda,\sigma,\alpha_0,\alpha_1,\beta_0,\beta_1,z]\,,\\&\mathrm{Heun}_+(z)=z^{1-\beta_0}\mathrm{Heun}[\lambda,\sigma+(1-\beta_0)(\beta_2+\lambda\beta_1),1+\alpha_0-\beta_0,1+\alpha_1-\beta_0,2-\beta_0,\beta_1,z]\,,\end{aligned} \tag{A.2}$$

where $\mathrm{Heun}[\lambda,\sigma,\alpha_0,\alpha_1,\beta_0,\beta_1,z]$ is the Heun function which is normalized to one at $z=0$, so that

$$\mathrm{Heun}_-(z)=1+\ldots\,,\qquad \mathrm{Heun}_+(z)=z^{1-\beta_0}(1+\ldots)\,. \tag{A.3}$$

The dots stand for corrections in z. The expansion coefficients of $\mathrm{Heun}_\alpha(z)$ can be obtained recursively, solving the equation order by order in z. The Heun differential equation can be related to the quantum SW curve of a $\mathcal{N}=2$ supersymmetric gauge theory with four fundamental hypermultiplets. To this aim, we write the six Heun parameters in terms of 4 masses m_i, a gauge coupling x and a Coulomb branch parameter u, defined via the identifications[7]

$$\begin{aligned}&\alpha_0=1-m_1-m_3\,,\qquad \alpha_1=1-m_2-m_3\,,\qquad \lambda=x\,,\\&\beta_0=1-m_3+m_4\,,\qquad \beta_1=1-m_1-m_2\,,\qquad \beta_2=1-m_3-m_4\,,\\&\sigma=u(x-1)+\left(\tfrac{1}{2}-m_3\right)^2+x\left(\tfrac{3}{4}-m_1-m_2-m_3+m_1m_2+m_1m_3+m_2m_3\right).\end{aligned} \tag{A.6}$$

The mathematical problem we are interested in is looking for a solution in an interval, let us say $z\in[0,1]$, satisfying some specific boundary conditions. The general solution

$$G_1(z)=\sum_\alpha B_\alpha\,\mathrm{Heun}_\alpha(z)\,, \tag{A.7}$$

[7]Equation (A.1) can be alternatively written in the normal form (4) with

$$G_1(z)=(1-z)^{-\frac{\beta_1}{2}}z^{-\frac{\beta_0}{2}}(z-\lambda)^{-\frac{\beta_2}{2}}\,\Psi(z)\,, \tag{A.4}$$

and

$$\begin{aligned}Q_{\rm gauge}=&\frac{\frac14-\left(\frac{m_3+m_4}{2}\right)^2}{(z-x)^2}+\frac{\frac14-\left(\frac{m_3-m_4}{2}\right)^2}{z^2}+\frac{\frac14-\left(\frac{m_1+m_2}{2}\right)^2}{(z-1)^2}+\frac{m_3^2+m_4^2+2m_1m_2-1}{2(z-1)z}\\&+\frac{(1-x)\left(u+\frac14-\frac{m_3^2}{2}-\frac{m_4^2}{2}\right)-\frac{x}{2}(1-m_1-m_2)(1-m_3-m_4)}{(z-1)z(z-x)}\,.\end{aligned} \tag{A.5}$$

behaves, near the boundaries, as

$$
\begin{aligned}
G_1(z) &\underset{z\to 1}{\approx} D_-(1+\ldots)+D_+(1-z)^{m_1+m_2}(1+\ldots), \\
G_1(z) &\underset{z\to 0}{\approx} B_-(1+\ldots)+B_+z^{m_3-m_4}(1+\ldots).
\end{aligned}
\tag{A.8}
$$

Specifying boundary conditions corresponds to fix two linear combinations made out of the four coefficients $B_\pm$, $D_\pm$.

A.1 The hypergeometric representation

As before we start from the PM/PN Ansätze

$$
G^0_\alpha(\tfrac{x}{z})g_\alpha(x) = G^1_\alpha(z)+\ldots, \tag{A.9}
$$

with

$$
\begin{aligned}
G^0_\alpha(y) &= P_0(y)H^0_\alpha(y)+\widehat{P}_0(y)\,yH^{0\prime}_\alpha(y), \\
G^1_\alpha(z) &= P_1(z)H^1_\alpha(z)+\widehat{P}_1(z)zH^{1\prime}_\alpha(z),
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
H^0_\alpha(y) &= y^{\frac{1}{2}-\alpha a-m_3}\,{}_2F_1\left(\tfrac{1}{2}-\alpha a-m_3,\tfrac{1}{2}-\alpha a-m_4;1-2\alpha a;y\right), \\
H^1_\alpha(z) &= z^{-\frac{1}{2}+\alpha a+m_3}\,{}_2F_1\left(\tfrac{1}{2}+\alpha a-m_1,\tfrac{1}{2}+\alpha a-m_2;1+2\alpha a;z\right),
\end{aligned}
\tag{A.11}
$$

and

$$
\begin{aligned}
P_0(y) &= 1+\sum_{i=1}^{k}\sum_{j=0}^{i-1}c_{ij}\,x^i y^{j-i}, \qquad & \widehat{P}_0(y) &= \sum_{i=1}^{k}\sum_{j=0}^{i}\widehat{c}_{ij}\,x^i y^{j-i}, \\
P_1(z) &= 1+\sum_{j=1}^{k}\sum_{i=0}^{j-1}d_{ij}\,z^{i-j}x^j, \qquad & \widehat{P}_1(z) &= \sum_{j=1}^{k}\sum_{i=0}^{j}\widehat{d}_{ij}\,z^{i-j}x^j.
\end{aligned}
\tag{A.12}
$$

We notice that now only the non-confluent hypergeometric functions are involved. As before, plugging the Ansätze (A.9) into (A.1) and setting to zero the coefficients of H^p_α $(p=0,1)$ and its derivatives, one finds a linear system of algebraic equations for the coefficients c_{ij}, $\widehat{c}_{ij}$, d_{ij}, $\widehat{d}_{ij}$. For example, for $k=1$, writing $u=a^2+u_1x+\ldots$, one finds

$$
\begin{aligned}
c_{10} &= \frac{1}{2}(1-m_1-m_2-m_3)-\frac{2m_1m_2m_3}{4a^2-1}, \qquad & \widehat{c}_{11} &= -\widehat{c}_{10} = \frac{1}{2}+\frac{2m_1m_2}{4a^2-1}, \\
d_{01} &= \frac{2(m_3-1)m_3m_4}{4a^2-1}-\frac{m_4}{2}, \qquad & \widehat{d}_{11} &= -\widehat{d}_{01} = \frac{2m_3m_4}{4a^2-1}+\frac{1}{2}, \\
u_1 &= \frac{2m_1m_2m_3m_4}{4a^2-1}+\frac{1}{8}\left(4a^2-1\right)+\frac{1}{2}\left(1-\sum_{i=1}^{4}m_i+\sum_{i<j}^{4}m_im_j\right).
\end{aligned}
\tag{A.13}
$$

As in the confluent case, we denote by

$$
g_\alpha(x) = \frac{G^1_\alpha(z)}{G^0_\alpha(\frac{x}{z})}, \tag{A.14}
$$

the ratio of the two solutions. Explicit computations show that, as in the confluent case, the ratio of the two components can be written as

$$
\gamma(x) = \frac{g_+(x)}{g_-(x)} = x^{2a}e^{-\partial_a\mathcal{F}_{\mathrm{inst}}(a,x)} = x^{2a}\left(1+x\left(a-\frac{16am_1m_2m_3m_4}{(1-4a^2)^2}\right)+\ldots\right), \tag{A.15}
$$

with $\mathcal{F}_{\mathrm{inst}}$ still is given by (22). The function $\mathcal{F}_{\mathrm{inst}}$ corresponds now to the prepotential of a $\mathcal{N}=2$ supersymmetric $SU(2)$ gauge theory with four fundamental flavours.

A.2 Connection formulae

As in the confluent case, connection formulae for the Heun function can be easily derived from the standard hypergeometric transformation rules of H^p_α. We introduce the two extra sets of hypergeometric functions

$$\begin{aligned}\widetilde{H}^0_\alpha(y) &= (e^{-i\pi}y)^{-\frac{1+\alpha}{2}m_{34}}\,{}_2F_1\left(\tfrac{1}{2}-a-m_3+\tfrac{1+\alpha}{2}m_{34}, \tfrac{1}{2}+a-m_3+\tfrac{1+\alpha}{2}m_{34}; 1+\alpha m_{34}; \tfrac{1}{y}\right),\\ \widetilde{H}^1_\alpha(z) &= z^{a+m_3-\frac{1}{2}}(1-z)^{\frac{1+\alpha}{2}(m_1+m_2)}\,{}_2F_1\left(\tfrac{1}{2}+a+\alpha m_1, \tfrac{1}{2}+a+\alpha m_2; 1+\alpha(m_1+m_2); 1-z\right),\end{aligned} \tag{A.16}$$

with $m_{34} = m_3 - m_4$. We notice that both $\widetilde{H}^0_\alpha(y)$ and $\widetilde{H}^1_\alpha(z)$ are invariant under the sign flipping $a \to -a$, due to hypergeometric identities. The functions (A.16) are related to those introduced in (A.11) by the standard hypergeometric relations

$$H^0_\alpha(y) = \sum_{\alpha'=\pm} B_{\alpha\alpha'}\widetilde{H}^0_{\alpha'}(y), \qquad H^1_\alpha(z) = \sum_{\alpha'=\pm} F_{\alpha\alpha'}\widetilde{H}^1_{\alpha'}(z), \tag{A.17}$$

with

$$\begin{aligned}B_{\alpha\alpha'} &= e^{-i\pi\left(\alpha a+m_3-\frac{1}{2}\right)}\frac{\Gamma(1-2\alpha a)\Gamma\left(\alpha'\left(m_4-m_3\right)\right)}{\Gamma\left(\frac{1}{2}-\alpha a-\alpha' m_3\right)\Gamma\left(\frac{1}{2}-\alpha a+\alpha' m_4\right)},\\ F_{\alpha\alpha'} &= \frac{\Gamma(1+2\alpha a)\Gamma\left(-\alpha'(m_1+m_2)\right)}{\Gamma\left(\frac{1}{2}+\alpha a-\alpha' m_1\right)\Gamma\left(\frac{1}{2}+\alpha a-\alpha' m_2\right)}.\end{aligned} \tag{A.18}$$

Plugging (A.17) into (A.10) one finds the connection formulae

$$\begin{aligned}G^0_\alpha(y) &= \sum_{\alpha'=\pm} B_{\alpha\alpha'}\widetilde{G}^0_{\alpha'}(y),\\ G^1_\alpha(z) &= \sum_{\alpha'=\pm} F_{\alpha\alpha'}\widetilde{G}^1_{\alpha'}(z),\end{aligned} \tag{A.19}$$

with

$$\begin{aligned}\widetilde{G}^0_\alpha(y) &= P_0(y)\widetilde{H}^0_\alpha(y)+\widehat{P}_0(\tfrac{x}{z})y\widetilde{H}^{0\prime}_\alpha(y),\\ \widetilde{G}^1_\alpha(z) &= P_1(z)\widetilde{H}^1_\alpha(z)+\widehat{P}_1(z)z\widetilde{H}^{1\prime}_\alpha(z),\end{aligned} \tag{A.20}$$

describing the solution in the *far region* and *near horizon zone*. The pairs $G^0_\alpha(z)$, $G^1_\alpha(z)$, $\widetilde{G}^0_\alpha(z)$, $\widetilde{G}^0_\alpha(z)$ provide four different bases of solutions of the Heun equation. Each basis covers a different patch of the interval $z \in [0,1]$. They are related to each other by the connection formulae (A.19). Remarkably the Heun connection formulae (A.19) take the familiar hypergeometric form, with the important difference that the argument of the gamma functions depends on the non-trivial function $a(u)$. Finally, we notice that the Heun function corresponds to the minus component of the far region basis

$$\text{Heun}_-(z) = \widetilde{G}^0_-(z) \underset{z\to 0}{\approx} 1+\ldots \tag{A.21}$$

Indeed in this limit $P_0 \to 1$, $\widetilde{H}^0_- \to 1$ and (A.21) follows from (A.20).

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