Entanglement Hamiltonian and effective temperature of non-Hermitian quantum spin ladders

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Abstract

Quantum entanglement plays a crucial role not only in understanding Hermitian many-body systems but also in offering valuable insights into non-Hermitian quantum systems. In this paper, we analytically investigate the entanglement Hamiltonian and entanglement energy spectrum of a non-Hermitian spin ladder using perturbation theory in the biorthogonal basis. Specifically, we examine the entanglement properties between coupled non-Hermitian quantum spin chains. In the strong coupling limit ($J_{\text{rung}} \gg 1$), first-order perturbation theory reveals that the entanglement Hamiltonian closely resembles the single-chain Hamiltonian with renormalized coupling strengths, allowing for the definition of an *ad hoc* temperature. Our findings provide new insights into quantum entanglement in non-Hermitian systems and offer a foundation for developing novel approaches for studying finite temperature properties in non-Hermitian quantum many-body systems.

1   Introduction

Quantum entanglement is a foundational concept that significantly enhances our understanding of many-body physics by elucidating quantum correlations between subsystems. The entanglement reveals concealed connections beyond classical physics. [1, 2] Suppose the total Hamiltonian $H = H_A + H_B + H_{AB}$ is written as summation of the subsystem Hamiltonians H_A and H_B, and their interaction H_{AB}. To further analyze the ground-state entanglement properties, the reduced density matrix $\rho_A = \text{Tr}_B |\psi_0\rangle\langle\psi_0|$ usually becomes an essential tool, where $|\psi_0\rangle$ is the normalized ground-state of the total Hamiltonian H, and the partial trace is performed on tracing the degrees of freedom of subsystem B. One common measure for quantifying entanglement between the subsystem A and B is the von Neumann entanglement entropy, defined by $S_{\text{von}} = -\sum_i \omega_i \ln \omega_i$, where ω_i is the ith eigenvalue of ρ_A. For example, in gapless systems where the low-energy theory is described by conformal field theory, the ground-state entanglement entropy exhibits a logarithmic scaling behavior with respect to subsystem size. This scaling allows for extraction of the central charge, a key parameter in the conformal field theory, which serves as an indicator of the phase transition's universality class. [3, 4]. For gapped systems, the ground-state entanglement entropy follows an area law, meaning it is proportional to the size of subsystem's boundary.

The entanglement energy spectrum provides a detailed view of the quantum correlations between subsystems, with the entanglement entropy serving as a condensed summary of the information contained within this spectrum. The entanglement energy ξ_i is the ith energy eigenvalue of a hypothetical Hamiltonian, called entanglement Hamiltonian H_E, which is defined by regarding the reduced density matrix as a thermal density matrix of the entanglement Hamiltonian at unity temperature, $\rho_A = e^{-H_E}/Z$. Where Z is the partition function that ensures $\text{Tr}[\rho_A] = 1$. Although the exact form of H_E is generally unknown, its eigenvalues can be obtained by taking logarithm on the eigenvalues of the reduced density matrix, $\xi_i = -\ln \omega_i - \ln Z$. The entanglement energy spectrum provides deep insights into topological systems and can be considered as a kind of 'fingerprint' of these systems. [5] This 'fingerprint' means that even when the entire system is divided into two halves, a topological system generates a gapless edge state, and this signature can be observed in the low-energy portion of the entanglement energy spectrum. [5–8] For example in the 2-dimensional topological systems, such as the fractional quantum Hall systems, the low-energy portion of momentum-resolved entanglement spectrum presents the same state counting with the low-energy spectrum of the edge Hamiltonian. This relationship is known as the renowned Li-Haldane conjecture [9] or the edge-entanglement spectrum correspondence. [8]

Interesting phenomena related to the entanglement Hamiltonian H_E can also arise when subsystem B is considered a copy of subsystem A, and a strong enough interaction H_{AB} is introduced to create a nearly maximally entangled ground-state between A and B. For example, in an antiferromagnetic spin ladder where the rung coupling is much stronger than the leg coupling, the ground-state forms multiple rung-singlets, resulting in a nearly maximally entangled state between the two legs [10–18]. In contrast to the Li-Haldane conjecture in topological systems, the entire entanglement spectrum has some similarity to the energy spectrum of subsystem A. Remarkably, under carefully selected parameters, the entanglement Hamiltonian $H_E \approx \beta H_A$ can be proportional to the Hamiltonian of subsystem A by a constant β. In other words, the finite temperature properties of an isolated system A can be approximated by the reduced density matrix ρ_A obtained from the ground-state of an enlarged system at zero temperature,

$$\rho_A \approx \frac{1}{Z} \exp[-\beta H_A], \tag{1}$$

where β is the inverse temperature as a function of system parameters.

It is worth noting that the finite-temperature Density Matrix Renormalization Group (DMRG) is a well-established numerical method for obtaining the finite-temperature properties of a system [19]. This method relies on the ancilla trick [20], which involves pairing the original system with an auxiliary system, known as the ancilla, to create an enlarged system. In this framework, the initial state is intentionally selected as a maximally entangled pure state shared between the original system and the ancilla. This state represents the system at infinite temperature. By performing imaginary time evolution on this initial state, the system can be effectively cooled to finite temperatures. This process enables the calculation of thermal expectation values.

In contrast, Eq. (1) provides an alternative approach that avoids imaginary time evolution. By examining the ground state of an enlarged system, it is possible to directly approximate the original system's finite-temperature properties, offering a simpler method for studying finite-temperature behavior from ground-state entanglement alone. This approach could also inspire experimental designs for simulating finite-temperature properties in real systems.

In this paper, we show that Eq. (1) remains valid for non-Hermitian Hamiltonians. In the next section, we briefly introduce non-Hermitian quantum mechanics and describe the non-Hermitian Hamiltonian for the spin-1/2 ladder.

2 Non-Hermitian Hamiltonian

Non-Hermitian systems have become an important multidisciplinary field of study [21–23], spanning photonics [24], condensed matter physics [25–29], and quantum information science. [30–38] A novel approach to studying non-Hermitian systems on a quantum computer can be achieved by applying post-selection to qubits, enabling non-reciprocal and non-unitary evolution through controlled qubit operations [39]. Unlike traditional quantum systems, which are governed by Hermitian Hamiltonians that ensure real eigenvalues and physically observable energy levels, non-Hermitian systems are described by Hamiltonians that do not necessarily satisfy this condition. This leads to complex eigenvalues and unique physical phenomena, e.g. exceptional points [40–46] and non-Hermitian skin effect. [47–54] An exceptional point (EP) in non-Hermitian systems occurs when two or more eigenvalues and their corresponding eigenvectors coalesce, rendering the Hamiltonian non-diagonalizable. This leads to unique phenomena, such as enhanced sensitivity to external changes [24] and a negative divergence in real part of fidelity susceptibility [42–44] or quantum metric, [55–57] which measures how ground state changes under perturbations. The non-Hermitian skin effect refers to the phenomenon where a macroscopic fraction of the system's eigenstates accumulate at one edge under open boundary conditions, even in the absence of external fields or disorder [47]. This effect is also commonly observed in systems with nonreciprocal coupling, where particles or excitations have different probabilities of hopping forward versus backward [48, 49].

Mathematically, nonreciprocal coupling is often introduced into a system's Hamiltonian through asymmetric hopping terms. [25–27] For instance, in a one-dimensional lattice model, the hopping amplitude from site j to site $j + 1$ might differ from the amplitude from site $j + 1$ to site j. This asymmetry results in a complex band structure with eigenvalues that can form loops in the complex plane, leading to the skin effect [53]. In this paper, we study the following non-Hermitian Hamiltonian $H = H_A + H_B + H_{AB}$ for spin-1/2 ladder with the
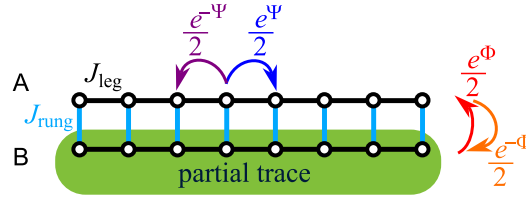
Figure 1: Schematic representation of the non-Hermitian spin-1/2 ladder with nonreciprocal couplings. All parameters are real and positive. The shaded region indicates the part of the system on which the partial trace is performed.

nonreciprocal coupling.

$$H_A = J_{\text{leg}} \sum_{j=1}^{N} \left[\frac{1}{2} \left(e^{\Psi} S_{j,A}^+ S_{j+1,A}^- + e^{-\Psi} S_{j,A}^- S_{j+1,A}^+ \right) + \Delta S_{j,A}^z S_{j+1,A}^z \right], \tag{2a}$$

$$H_B = J_{\text{leg}} \sum_{j=1}^{N} \left[\frac{1}{2} \left(e^{\Psi} S_{j,B}^+ S_{j+1,B}^- + e^{-\Psi} S_{j,B}^- S_{j+1,B}^+ \right) + \Delta S_{j,B}^z S_{j+1,B}^z \right], \tag{2b}$$

$$H_{AB} = J_{\text{rung}} \sum_{j=1}^{N} \left[\frac{1}{2} \left(e^{\Phi} S_{j,A}^+ S_{j,B}^- + e^{-\Phi} S_{j,A}^- S_{j,B}^+ \right) + \Delta S_{j,A}^z S_{j,B}^z \right], \tag{2c}$$

where N denotes the number of rungs, and Φ and Ψ are real parameters that control the nonreciprocal coupling between the legs and within each leg, respectively. J_{rung} and J_{leg} represent the coupling strengths between the legs and within the legs. Lastly, Δ denotes the XXZ anisotropy strength. Periodic boundary conditions are assumed. In the Hermitian limit, $\Phi = \Psi = 0$, the ground-state phase diagram has been studied in the literature [58]. We will mainly focus on the entanglement Hamiltonian in the rung-singlet phase. The schematic representation of the non-Hermitian spin-1/2 ladder is shown in Fig. 1.

Due to the non-Hermitian nature of the system, where $H^\dagger \neq H$, the time evolution of the wavefunctions is driven by both H and $H^\dagger$ simultaneously. This results in the following time evolution equations: $\frac{\partial}{\partial t}|\varphi^R(t)\rangle = -iH|\varphi^R(t)\rangle$, and $\frac{\partial}{\partial t}|\varphi^L(t)\rangle = -iH^\dagger|\varphi^L(t)\rangle$. Note that $\hbar \equiv 1$ is set. Consequently, in analogy to standard linear algebra, the eigenvectors of a non-Hermitian Hamiltonian are generalized into biorthogonal left and right eigenvectors, satisfying the following eigenvalue equations: $H^\dagger|\psi_n^L\rangle = E_n^*|\psi_n^L\rangle$ and $H|\psi_n^R\rangle = E_n|\psi_n^R\rangle$, with the biorthonormal condition: $\langle\psi_n^L|\psi_m^R\rangle = \delta_{nm}$. [23] In non-Hermitian quantum mechanics, observables are defined through the expectation value $\langle\psi^L|O|\psi^R\rangle$, involving both left and right eigenvectors. This biorthogonal framework naturally extends to the definition of the reduced density matrix in a bipartite system. For a system divided into subsystems A and B, the reduced density matrix (RDM) of the ground-state for subsystem A is defined as

$$\rho_A = \text{Tr}_B |\psi_0^R\rangle\langle\psi_0^L|. \tag{3}$$

Note that $\text{Tr}[\rho_A] = 1$, since $\langle\psi_0^L|\psi_0^R\rangle = 1$.

This biorthogonal definition of the RDM immediately raises a key issue: the RDM itself becomes non-Hermitian, meaning that its eigenvalues, ω_i, are generally complex. Even when the total Hamiltonian has PT symmetry and the ground-state energy is real [21], the eigenvalues of the RDM in typical cases remain complex. As a result, appropriate definitions of generic entanglement entropy of both von Neumann type and Rényi type have been proposed

by Tu, Tzeng and Chang to account for these complex eigenvalues. [30]

$$S_{\text{TTC}} = -\sum_i \omega_i \ln|\omega_i|,$$

$$S_{\text{TTC}}^{(n)} = \frac{1}{1-n} \ln\left(\sum_i \omega_i |\omega_i|^{n-1}\right). \tag{4}$$

These entropies Eq.(4) effectively capture the negative central charge in non-Hermitian critical systems through the logarithmic scaling. [30,31]

The entanglement Hamiltonian H_E defined by $\rho_A = e^{-H_E}/Z$ is also non-Hermitian, and the entanglement energy ξ_i is complex in general. The real part of the entanglement energy can be obtained directly as $\text{Re}[\xi_i] = -\ln|\omega_i| - \ln Z$, and the entanglement entropy Eq. (4) can be seen as the expectation value of the real part of the entanglement energy, $S_{\text{TTC}} = \sum_i \omega_i \text{Re}[\xi_i] + \ln Z$.[1] However, the imaginary part of the entanglement energy cannot be easily determined, as the logarithmic function becomes multi-valued. To gain further insight into this complexity, in the following section, we directly derive the entanglement Hamiltonian of the non-Hermitian spin-1/2 ladder using perturbation theory for the rung-singlet phase, where the ground state is adiabatically connected to the limit case of $J_{\text{rung}} \gg 1$.

3 Entanglement Hamiltonian

3.1 Perturbation theory

The non-Hermitian spin-1/2 ladder Hamiltonian is given by Eq.(2). In the limit of $J_{\text{rung}} \gg 1$, the interaction between legs A and B at each rung j defines the unperturbed Hamiltonian $H_0 = H_{AB} = \sum_{j=1}^N h_0^j$, where

$$h_0^j = J_{\text{rung}}\left[\frac{1}{2}\left(e^{\Phi} S_{j,A}^+ S_{j,B}^- + e^{-\Phi} S_{j,A}^- S_{j,B}^+\right) + \Delta S_{j,A}^z S_{j,B}^z\right], \tag{5}$$

and the Hamiltonians of the legs A and B, $H_1 = H_A + H_B$, are treated as perturbation. The left and right eigenvectors of the single-rung Hamiltonian Eq.(5) are

$$|s_j^x\rangle = \frac{1}{\sqrt{2}}\left(e^{\sigma(x)\Phi} |\uparrow\rangle_{j,A} |\downarrow\rangle_{j,B} - |\downarrow\rangle_{j,A} |\uparrow\rangle_{j,B}\right), \tag{6a}$$

$$|t_j^{+x}\rangle = |t_j^+\rangle = |\uparrow\rangle_{j,A} |\uparrow\rangle_{j,B}, \tag{6b}$$

$$|t_j^{0x}\rangle = \frac{1}{\sqrt{2}}\left(e^{\sigma(x)\Phi} |\uparrow\rangle_{j,A} |\downarrow\rangle_{j,B} + |\downarrow\rangle_{j,A} |\uparrow\rangle_{j,B}\right), \tag{6c}$$

$$|t_j^{-x}\rangle = |t_j^-\rangle = |\downarrow\rangle_{j,A} |\downarrow\rangle_{j,B}, \tag{6d}$$

where $x = L, R$ labels the 'left' and 'right', respectively, and

$$\sigma(x) = \begin{cases} +1, & \text{if } x = R, \\ -1, & \text{if } x = L. \end{cases} \tag{7}$$

---

[1]In PT-symmetric non-Hermitian systems, if the bipartition does not break the PT symmetry, the reduced density matrix remains PT-symmetric, meaning that its eigenvalues ω_i are either real or come in complex conjugate pairs. Consequently, the partition function Z is real.

Let $|\psi_0^{L(0)}\rangle$ and $|\psi_0^{R(0)}\rangle$ denote the left and right ground states of the unperturbed Hamiltonian H_0, with the corresponding ground state energy $E_0^{(0)} = -NJ_{\text{rung}}(\frac{1}{2} + \frac{\Delta}{4})$. Here, we assume $\Delta > -1$. The ground state is a product of singlet states on each rung,

$$|\psi_0^{x(0)}\rangle = \bigotimes_j |s_j^x\rangle. \tag{8}$$

Using first-order perturbation theory, the corrected ground state can be written as,

$$|\psi_0^x\rangle \approx |\psi_0^{x(0)}\rangle + |\psi_0^{x(1)}\rangle, \tag{9}$$

where the left and right first-order correction terms are,

$$
\begin{aligned}
\langle\psi_0^{L(1)}| &= \sum_{j=1}^N \sum_{n\neq 0} \frac{\langle\psi_0^{L(0)}|H_1|\psi_n^{jR(0)}\rangle}{E_0^{(0)} - E_n^{(0)}} \langle\psi_n^{jL(0)}| \\
&= \frac{J_{\text{leg}}}{4J_{\text{rung}}} \sum_{j=1}^N \left[\frac{2e^{-\Phi}e^{-\Psi}}{1+\Delta} \ldots \langle t_j^+|\langle t_{j+1}^-|\ldots + \frac{2e^{-\Phi}e^{\Psi}}{1+\Delta} \ldots \langle t_j^-|\langle t_{j+1}^+|\ldots - \Delta \ldots \langle t_j^{0L}|\langle t_{j+1}^{0L}|\ldots \right],
\end{aligned}
\tag{10a}
$$

$$
\begin{aligned}
|\psi_0^{R(1)}\rangle &= \sum_{j=1}^N \sum_{n\neq 0} |\psi_n^{jR(0)}\rangle \frac{\langle\psi_n^{jL(0)}|H_1|\psi_0^{R(0)}\rangle}{E_0^{(0)} - E_n^{(0)}} \\
&= \frac{J_{\text{leg}}}{4J_{\text{rung}}} \sum_{j=1}^N \left[\frac{2e^{\Phi}e^{\Psi}}{1+\Delta} \ldots |t_j^+\rangle|t_{j+1}^-\rangle\ldots + \frac{2e^{\Phi}e^{-\Psi}}{1+\Delta} \ldots |t_j^-\rangle|t_{j+1}^+\rangle\ldots - \Delta \ldots |t_j^{0R}\rangle|t_{j+1}^{0R}\rangle\ldots \right].
\end{aligned}
\tag{10b}
$$

Where, the dots represent the singlet states on each rung. $|\psi_n^{jL(0)}\rangle$ and $|\psi_n^{jR(0)}\rangle$ denote the excited states of the unperturbed Hamiltonian $H_0 = H_{AB}$, and $E_n^{(0)}$ are the corresponding energies. Specifically, the excited states with non-zero contributions to the corrections are

$$
\begin{aligned}
|\psi_1^{j(0)}\rangle &= \ldots|t_j^+\rangle|t_{j+1}^-\rangle\ldots, \\
|\psi_2^{j(0)}\rangle &= \ldots|t_j^-\rangle|t_{j+1}^+\rangle\ldots, \\
|\psi_3^{jx(0)}\rangle &= \ldots|t_j^{0x}\rangle|t_{j+1}^{0x}\rangle\ldots,
\end{aligned}
\tag{11}
$$

where $x = L, R$, and the corresponding eigenenergies are

$$
\begin{aligned}
E_1^{(0)} = E_2^{(0)} &= J_{\text{rung}}\left[(1+\Delta) - N\left(\frac{1}{2} + \frac{1}{4}\Delta\right) \right], \\
E_3^{(0)} &= J_{\text{rung}}\left[2 - N\left(\frac{1}{2} + \frac{1}{4}\Delta\right) \right].
\end{aligned}
\tag{12}
$$

The reduced density matrix ρ_A defined in Eq. (3) can be approximated as,

$$
\begin{aligned}
\rho_A &\approx \rho_A^{(0)} + \rho_A^{(1)} = \text{Tr}_B \left(|\psi_0^{R(0)}\rangle\langle\psi_0^{L(0)}| + |\psi_0^{R(1)}\rangle\langle\psi_0^{L(0)}| + |\psi_0^{R(0)}\rangle\langle\psi_0^{L(1)}| \right) \\
&= \frac{1}{2^N}\left(1 - \frac{4J_{\text{leg}}}{J_{\text{rung}}(1+\Delta)} \sum_{j=1}^N \left[\frac{1}{2}\left(e^{\Psi}S_{j,A}^+S_{j+1,A}^- + e^{-\Psi}S_{j,A}^-S_{j+1,A}^+\right) + \frac{1}{2}\left(\Delta + \Delta^2\right)S_{j,A}^z S_{j+1,A}^z \right] \right).
\end{aligned}
\tag{13}
$$

Thus, the reduced density matrix can be written in terms of the Hamiltonian of subsystem A,

$$\rho_A \approx \frac{1}{Z}\exp[-\beta\tilde{H}_A] = \frac{1}{Z}\left(1 - \beta\tilde{H}_A + \frac{1}{2!}\beta^2\tilde{H}_A^2 - \frac{1}{3!}\beta^3\tilde{H}_A^3 + \cdots \right). \tag{14}$$

Compare Eq. (14) with Eq. (13), we obtain the *ad hoc* inverse temperature

$$\beta = \frac{4}{1+\Delta}\frac{1}{J_{\text{rung}}} \ll 1,\tag{15}$$

and the Hamiltonian of the subsystem A

$$\tilde{H}_A = J_{\text{leg}}\sum_{j=1}^{N}\left[\frac{1}{2}\left(e^{\Psi}S_{j,A}^{+}S_{j+1,A}^{-} + e^{-\Psi}S_{j,A}^{-}S_{j+1,A}^{+}\right) + \tilde{\Delta}S_{j,A}^{z}S_{j+1,A}^{z}\right],\tag{16}$$

which is in the form of XXZ interaction with a renormalized parameter $\tilde{\Delta} = \frac{1}{2}(\Delta + \Delta^2)$. The partition function is $Z = \text{Tr}[\exp(-\beta\tilde{H}_A)] = 2^N$.

3.2 Discussion

We make some remarks regarding the derivation: For the spin ladder Eq. (2) considered in this paper, the renormalized anisotropy parameter remains unchanged, i.e., $\tilde{\Delta} = \Delta$, when $\Delta = 1$ or 0. In these specific cases, the entanglement Hamiltonian is exactly equal to the subsystem Hamiltonian, $\tilde{H}_A = H_A$. When $\Psi = \Phi = 0$, our results are consistent with those from earlier studies [13], reproducing the known Hermitian case. Even when non-Hermitian couplings are introduced with non-zero Ψ and Φ, the overall behavior remains remarkably similar, confirming that the methods and conclusions from the Hermitian regime can be successfully extended to non-Hermitian systems without major deviations. In the general non-Hermitian case, if one wishes to ensure that $\tilde{H}_A = H_A$, a simple approach is to choose the inter-subsystem Hamiltonian H_{AB} as a Hermitian and isotropic Heisenberg interaction.

$$H_{AB} = J_{\text{rung}}\sum_{j=1}^{N}\vec{S}_{j,A}\cdot\vec{S}_{j,B}.\tag{17}$$

This allows the reduced density matrix to reflect the exact form of the subsystem Hamiltonian without any parameter renormalization. A particularly interesting scenario arises when H_A and H_B are both Hermitian, while only the inter-subsystem coupling H_{AB} is non-Hermitian, specifically with parameters $\Psi = 0$ and $\Phi \neq 0$. In this case, despite both the total Hamiltonian and the reduced density matrix being non-Hermitian, all the entanglement energies remain real. This situation is quite rare and demonstrates an unusual interplay between Hermitian and non-Hermitian components in the system.

4 Conclusion

The entanglement Hamiltonian, whose eigenvalue spectrum is known as the entanglement energy spectrum [5], plays a crucial role in revealing quantum correlations between subsystems in many-body systems. Understanding its analytical form is essential for gaining deeper insights into the nature of quantum entanglement and for facilitating entanglement Hamiltonian tomography [59]. However, obtaining the entanglement Hamiltonian is often challenging, especially in non-Hermitian systems, where it can be complex and non-Hermitian itself. While studies on non-interacting systems, such as the non-Hermitian Su-Schrieffer-Heeger (SSH) model [34, 60], have made progress in deriving the entanglement Hamiltonian, the challenge is even more pronounced in interacting systems, where many-body effects add significant complexity. Although the real part of the entanglement energy can be derived from the eigenvalues of the reduced density matrix as $\text{Re}[\xi_i] = -\ln|\omega_i| - \ln Z$, capturing the full entanglement Hamiltonian remains crucial for exploring the deeper properties of quantum many-body systems.

In this paper, we explore the entanglement Hamiltonian of a non-Hermitian spin ladder system using perturbation theory, providing an analytical approach to this difficult problem. Remarkably, we find that the entanglement Hamiltonian in the non-Hermitian case can be approximated by the Hamiltonian of subsystem A, indicating that the thermal density matrix of an isolated non-Hermitian system A can be derived by the partial trace of an enlarged system. Our work offers new insights into the study of quantum entanglement in non-Hermitian systems, potentially facilitating the development of advanced approaches for investigating their finite-temperature behavior.

Although non-Hermitian systems exhibit many phenomena absent in Hermitian systems, such as exceptional points (EP) and the non-Hermitian skin effect, many features of Hermitian systems persist in non-Hermitian counterparts when appropriately generalized. For instance, the entanglement entropy in critical systems still follows a logarithmic scaling [30,31], fidelity susceptibility diverges near phase transitions or EPs [42,43], and machine learning methods can be transferred from Hermitian to non-Hermitian systems [61]. In our work on entanglement Hamiltonians, we extend the Hermitian case to non-Hermitian systems and find that, for a nearly maximally entangled ground-state, the results remain consistent with the Hermitian case.

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