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Galilei particles revisited

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Abstract

We revisit the classifications of classical and quantum galilean particles: that is, we fully classify homogeneous symplectic manifolds and unitary irreducible projective representations of the Galilei group. Equivalently, these are coadjoint orbits and unitary irreducible representations of the Bargmann group, the universal central extension of the Galilei group. We provide an action principle in each case, discuss the nonrelativistic limit, as well as exhibit, whenever possible, the unitary irreducible representations in terms of fields on Galilei spacetime. Motivated by a forthcoming study of planons we pay close attention to the mobility of the less familiar massless Galilei particles.

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1 Introduction

Galilei, Minkowski and Carroll spacetimes [1,2] are distinguished Klein models for galilean, lorentzian and carrollian Cartan geometries, respectively. They are affine spaces admitting a transitive action of the Galilei, Poincaré and Carroll groups, respectively, which play the rôle of relativity groups for these spacetimes. The latter are the arena for both particle dynamics (classical and quantum) as well as quantum field theories: galilean, (the misnamed) relativistic and carrollian, respectively. Despite the physical spacetimes being the arena for dynamics, the actual degrees of freedom are often described by other homogeneous spaces of the groups in question: homogeneous symplectic manifolds in the case of classical particle dynamics, and momentum orbits in the case of quantum particles and fields.

These relativity groups are semidirect products $G = K \ltimes T$, where T is abelian (the translations) and K consists, roughly, of rotations and boosts. Of these relativity groups, the best understood is arguably the Poincaré group, where K is the Lorentz group. The Poincaré group has played a starring rôle in Physics for more than a century: it underlies Special Relativity and Relativistic Quantum Field Theory, and its associated Cartan geometry (i.e., lorentzian geometry) underlies General Relativity. Classical Poincaré particles correspond to coadjoint orbits of the Poincaré group and they were first classified by Arens in [3]. Its unitary irreducible representations were famously classified by Wigner [4], pioneering the method of induced representations which would find its most general expression in Mackey's theory [5].

By contrast, the Carroll group is more recent. Introduced by Lévy-Leblond in [6] and independently by Sen Gupta in [7], its coadjoint orbits were studied initially in the Appendix of [8] and more recently also in [9] in the context of fractons. The unitary irreducible representations of the Carroll group were recently determined in [10] using the method of induced representations (see also [6,11]).

The Galilei group is of course the oldest of these relativity groups and it is the subject of this paper. In contrast to both the Poincaré and Carroll groups, the Galilei group has nontrivial symplectic cohomology in the language of Souriau [12]. This means that its homogeneous symplectic manifolds are not all coadjoint orbits of the Galilei group, but of its one-dimensional central extension: the eponymous group introduced by Bargmann in [13] in a quantum mechanical context. In that paper, Bargmann showed that the unitary irreducible ray representations of the Galilei group are honest unitary irreducible representations of the Bargmann group, but stopped short of their classification. The coadjoint orbits of the Bargmann group were initially studied by Souriau [12, §14] and are also discussed by Guillemin and Sternberg in [14, §54], but to the best of our knowledge a full analysis of the resulting particle dynamics has not been done before. The story of the unitary irreducible representations is more tortuous. It was Inönü and Wigner [15] who first considered the unitary irreducible representations of the Galilei group. In that paper they consciously restricted to honest representations of the connected Galilei group, thus missing massive representations, and moreover they did not consider the simply-connected cover of the Galilei group, thus missing representations with half-integer spin. Brennich [16] classified the unitary irreducible ray representations of the simply-connected Galilei group and also of its extension by parity and time reversal automorphisms. His labeling of representations is somewhat redundant, as we shall see. The unitary irreducible ray representations were also determined by Lévy-Leblond in [17] with one minor omission, as we shall discuss below. These works pay no attention to identifying the homogeneous vector bundles whose sections carry the representations.

In recent years novel quasiparticles with the distinctive property of having only restricted mobility (see [18–20] for reviews) have challenged, and hence advanced, our understanding of conventional quantum field theories [21,22]. One way to understand them [23,24] is coming from theories with higher moment conservation laws like, e.g., dipole moment conservation. These exotic symmetries can be studied systematically and it was observed [25] that the symmetries of theories with conserved dipole and trace of the quadrupole moment closely resemble the Bargmann algebra (similar to the relation between fractons and Carroll particles [9, 10, 26, 27]). It is then natural to ask if and how their particles are related, which leads to study the classical and quantum elementary and composite systems. These particles indeed play a rôle in applications (e.g., [28]) and to be able to contrast the planons with the Galilei particles [29] we find it useful to review the Galilei particles independently.

We make no strong claims of originality, but we are unaware of a resource that uniformly covers the topics of this article at the same level of completeness. We therefore think it may be useful to collect these results in a uniform way, using contemporary language and paying close attention to the geometrical nature of the representations. In addition we discuss the particle dynamics associated to the different coadjoint orbits as well as, whenever possible, a description of the unitary irreducible representations as fields in Galilei spacetime.

This paper is organised as follows. In Section 2 we introduce the Bargmann group. We do not define it as a central extension of the Galilei group, but rather as the subgroup of the Poincaré group in one dimension higher which stabilises a nonzero null translation generator and only then show that its Lie algebra is a central extension of the Galilei algebra. In Section 3 we discuss the adjoint and coadjoint representations of the Bargmann group and in Section 4 we write explicit expressions for the left-invariant Maurer–Cartan one-form on the Bargmann group, for later use in deriving action functionals for classical Galilei particles. Until this point we have been working in generic dimension, but starting from Section 5 we restrict our attention to the case of four-dimensional Galilei spacetime. In Section 6 we determine for later use the automorphisms of the Bargmann algebra which act trivially on the rotational subalgebra and we give an expression for the group automorphisms which integrate them. In Section 7 we determine the coadjoint orbits of the Bargmann group. They are summarised in Table 1, which gives equations for each of the orbits. We then discuss how this classification changes when we extend the Bargmann group by parity and time reversal. Finally, we discuss the geometric structure of the coadjoint orbits as bundles over the momentum orbits, which is summarised in Table 2. In Section 8 we study the particle actions associated to each of the coadjoint orbits and determine the corresponding dynamics (with further details delegated to Appendix A). In Section 9 we discuss a group-theoretical approach to particle dynamics and compare with the results in Section 8. In Section 10 we contrast the energy-momentum orbits and the limit from Poincaré to Galilei (see Figure 1). In Section 11 we classify the unitary irreducible representations of the Bargmann group using the method of induced representations. The results are summarised in Table 6. We compare our classification with those of Inönü–Wigner [15], Brennich [16] and Lévy-Leblond [17]. Finally, in Section 12 we discuss some realisations of these unitary irreducible representations in terms of fields in Galilei spacetime.

2 The Bargmann group and its Lie algebra

In this section we define the Bargmann group and work out its Lie algebra.

The $(n+1)$-dimensional Bargmann group is the subgroup G of the $(n+2)$-dimensional Poincaré group which leaves invariant a null translation under the adjoint representation on its Lie algebra. The Poincaré group sits inside the affine group, which in turn embeds inside the linear group in one higher dimension. Therefore the $(n+1)$-dimensional Bargmann group sits naturally inside $\mathrm{GL}(n+3,\mathbb{R})$.

The $(n+2)$-dimensional Poincaré group, by which we mean the subgroup of isometries of $(n+2)$-dimensional Minkowski spacetime, is the subgroup of $\mathrm{GL}(n+3,\mathbb{R})$ given by the set of matrices

$$\left\{ \begin{pmatrix} L & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{R}^{n+2}, \qquad L \in \mathrm{GL}(n+2,\mathbb{R}) \qquad \text{and} \qquad L^T \eta L = \eta \right\}, \tag{1}$$

where

$$\eta = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ 1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & I_n \end{pmatrix}, \qquad \text{with } I_n \text{ the identity matrix of size } n. \tag{2}$$

Notice that we have chosen a Witt frame $(\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_a)$ for the lorentzian vector space $(\mathbb{R}^{n+2}, \eta)$, in such a way that $\eta(\mathbf{e}_+, \mathbf{e}_-) = 1$, $\eta(\mathbf{e}_a, \mathbf{e}_b) = \delta_{ab}$ and all other inner products vanish. The Bargmann group is therefore given by the set of matrices

$$\left\{ \begin{pmatrix} L & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{R}^{n+2}, \qquad L \in \mathrm{GL}(n+2,\mathbb{R}), \qquad L^T \eta L = \eta \qquad \text{and} \qquad L\mathbf{e}_+ = \mathbf{e}_+ \right\}. \tag{3}$$

The matrices $L \in \mathrm{GL}(n+2,\mathbb{R})$ in the Bargmann group can be seen to take the following form

$$L = \begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & \mathbf{v}^T R \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{v} & R \end{pmatrix}, \qquad \text{where } R \in \mathrm{O}(n) \text{ and } \mathbf{v} \in \mathbb{R}^n. \tag{4}$$

The subgroup $H \subset \mathrm{GL}(n+2,\mathbb{R})$ consisting of such matrices is isomorphic to the euclidean group $\mathbb{R}^n \rtimes \mathrm{O}(n)$. In other words, if is diffeomorphic to $\mathbb{R}^n \times \mathrm{O}(n)$ where the element of H corresponding to $(\mathbf{v}, R) \in \mathbb{R}^n \times \mathrm{O}(n)$ is given by the matrix L in equation (4). That element, which we denote[1] by $\mathsf{g}(\mathbf{v}, R)$, can be factorised as

$$\begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & \mathbf{v}^T R \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{v} & R \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & \mathbf{v}^T \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{v} & I_n \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & R \end{pmatrix}, \tag{5}$$

corresponding to the multiplication on $\mathbb{R}^n \times \mathrm{O}(n)$ defined by

$$\mathsf{g}(\mathbf{v}_1, R_1)\mathsf{g}(\mathbf{v}_2, R_2) = \mathsf{g}(\mathbf{v}_1 + R_1\mathbf{v}_2, R_1 R_2). \tag{6}$$

The Bargmann subgroup $G \subset \mathrm{GL}(n+3,\mathbb{R})$ thus consists of matrices of the form

$$\begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & \mathbf{v}^T R & a_+ \\ 0 & 1 & \mathbf{0}^T & a_- \\ \mathbf{0} & -\mathbf{v} & R & \mathbf{a} \\ 0 & 0 & \mathbf{0}^T & 1 \end{pmatrix}, \tag{7}$$

which factorises as

$$\begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & \mathbf{v}^T R & a_+ \\ 0 & 1 & \mathbf{0}^T & a_- \\ \mathbf{0} & -\mathbf{v} & R & \mathbf{a} \\ 0 & 0 & \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \mathbf{0}^T & a_+ \\ 0 & 1 & \mathbf{0}^T & a_- \\ \mathbf{0} & \mathbf{0} & I_n & \mathbf{a} \\ 0 & 0 & \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & \mathbf{v}^T & 0 \\ 0 & 1 & \mathbf{0}^T & 0 \\ \mathbf{0} & -\mathbf{v} & I_n & \mathbf{0} \\ 0 & 0 & \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathbf{0}^T & 0 \\ 0 & 1 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbf{0} & R & \mathbf{0} \\ 0 & 0 & \mathbf{0}^T & 1 \end{pmatrix}, \tag{8}$$

corresponding to

$$\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R) = \mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{0}, I_n)\mathsf{g}(0, 0, \mathbf{0}, \mathbf{v}, I_n)\mathsf{g}(0, 0, \mathbf{0}, \mathbf{0}, R). \tag{9}$$

[1]We use the notation $\mathsf{g}(\ldots)$, $\mathsf{A}(\ldots)$ and $\mathsf{M}(\ldots)$ to represent group elements, Lie algebra elements and elements of the dual of the Lie algebra ("moments") parametrised by the data inside the parentheses.

The Bargmann group is thus diffeomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathrm{O}(n)$, with multiplication defined by

$$\begin{aligned} &\mathrm{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)\mathrm{g}(\alpha_+, \alpha_-, \alpha, \beta, \Sigma) \\ &\quad = \mathrm{g}(a_+ + \alpha_+ + \mathbf{v}\cdot R\alpha - \tfrac{1}{2}\alpha_-\|\mathbf{v}\|^2, a_- + \alpha_-, \mathbf{a} + R\alpha - \alpha_-\mathbf{v}, \mathbf{v} + R\beta, R\Sigma)\,, \end{aligned} \tag{10}$$

from where we see that the identity element corresponds to $\mathrm{g}(0,0,\mathbf{0},\mathbf{0},I_n)$. From here we can work out the group inversion:

$$\mathrm{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)^{-1} = \mathrm{g}(-a_+ + \mathbf{a}\cdot\mathbf{v} + \tfrac{1}{2}a_-\|\mathbf{v}\|^2, -a_-, -R^T(\mathbf{a} + a_-\mathbf{v}), -R^T\mathbf{v}, R^T)\,, \tag{11}$$

where we have used that $R^T = R^{-1}$ in $\mathrm{O}(n)$.

The Lie algebra $\mathfrak{g}$ of the Bargmann group embeds in $\mathfrak{gl}(n+3,\mathbb{R})$ with image consisting of matrices of the form

$$\mathsf{A}(x_+, x_-, \mathbf{x}, \mathbf{y}, X) := \begin{pmatrix} 0 & 0 & \mathbf{y}^T & x_+ \\ 0 & 0 & \mathbf{0}^T & x_- \\ \mathbf{0} & -\mathbf{y} & X & \mathbf{x} \\ 0 & 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad \text{where } X^T = -X\,, \tag{12}$$

from where we can easily work out the Lie bracket:

$$\begin{aligned} &\left[\mathsf{A}(x_+^1, x_-^1, \mathbf{x}_1, \mathbf{y}_1, X_1), \mathsf{A}(x_+^2, x_-^2, \mathbf{x}_2, \mathbf{y}_2, X_2)\right] \\ &\quad = \mathsf{A}(\mathbf{y}_1\cdot\mathbf{x}_2 - \mathbf{y}_1\cdot\mathbf{x}_1, 0, X_1\mathbf{x}_2 - X_2\mathbf{x}_1 + x_-^1\mathbf{y}_2 - x_-^2\mathbf{y}_1, X_1\mathbf{y}_2 - X_2\mathbf{y}_1, [X_1, X_2])\,. \end{aligned} \tag{13}$$

If we introduce a basis L_{ab}, B_a, P_a, H, M for the Bargmann algebra in such a way that[2]

$$\mathsf{A}(x_+, x_-, \mathbf{x}, \mathbf{y}, X) = x_+ M - x_- H + x^a P_a + y^a B_a + \tfrac{1}{2}X^{ab}L_{ab}\,, \tag{14}$$

we read off the following nonzero Lie brackets:

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} - \delta_{ac}L_{bd} - \delta_{bd}L_{ac} + \delta_{bd}L_{ac}\,, \\ [L_{ab}, B_c] &= \delta_{bc}B_a - \delta_{ac}B_b\,, \\ [L_{ab}, P_c] &= \delta_{bc}P_a - \delta_{ac}P_b\,, \\ [B_a, H] &= P_a\,, \\ [B_a, P_b] &= \delta_{ab}M\,, \end{aligned} \tag{15}$$

which exhibits the Bargmann algebra as a central extension of the Galilei algebra. Indeed, the coadjoint orbits of the Bargmann group coincide, up to covering, with the homogeneous symplectic manifolds of the Galilei group.

The other homogeneous space of the Galilei group we shall be interested in is Galilei spacetime itself. It admits an effective transitive action of the Galilei group and hence the Bargmann group too acts transitively, but not effectively. It is simple to describe this action. Let M denote the Galilei spacetime: it is an affine space diffeomorphic to G/G_0 with $G_0 = K \times Z$, where Z is the one-dimensional centre and K is the homogeneous Galilei group. Its Lie algebra $\mathfrak{g}_0$ is spanned by L_{ab}, B_a, M. We choose a coset representative $\zeta : M \to G$ defined by $\zeta(t,\mathbf{x}) = \exp(tH + \mathbf{x}\cdot P)$. The action of G on M is induced by left-multiplication on G:

$$\mathrm{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)\zeta(t,\mathbf{x}) = \zeta(t', \mathbf{x}')h\,, \tag{16}$$

[2] The sign in the $-x_-H$ term is so that the Lie brackets are the ones we are familiar with.

for some $h \in G_0$, which depends in principle on $a_+, a_-, \mathbf{a}, \mathbf{v}, R, t, \mathbf{x}$. One can calculate the above product and arrives at

$$\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R) : \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \mapsto \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} t + a_- \\ R\mathbf{x} - t\mathbf{v} + \mathbf{a} \end{pmatrix}, \tag{17}$$

from where we see that the central subgroup Z acts trivially and hence the action factors through the Galilei group G/Z. The action is via a sequence of affine transformations: rotation followed by a galilean boost and followed by translations in both space and time.

3 The adjoint and coadjoint representations

The adjoint representation of G on $\mathfrak{g}$ is the derivative of the conjugation action of G on itself at the identity. Conjugation is easily worked out from the formulae for multiplication (10) and inversion (11). We find that

$$\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)\mathsf{g}(\alpha_+, \alpha_-, \alpha, \beta, \Sigma)\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)^{-1} = \mathsf{g}(\alpha'_+, \alpha'_-, \alpha', \beta', \Sigma'), \tag{18}$$

where

$$\begin{aligned} \alpha'_+ &= \alpha_+ + \mathbf{a}\cdot\mathbf{v} - \tfrac{1}{2}\alpha_-\|\mathbf{v}\|^2 + \tfrac{1}{2}a_-(\|\mathbf{v}\|^2 + \|\mathbf{v} + R\beta\|^2) + \mathbf{v}\cdot R\alpha - (\mathbf{v} + R\beta)\cdot R\Sigma R^T(\mathbf{a} + a_-\mathbf{v}), \\ \alpha'_- &= \alpha_-, \\ \alpha' &= R\alpha + \mathbf{a} - \alpha_-\mathbf{v} + a_-\mathbf{v} + a_- R\beta - R\Sigma R^T\mathbf{a} - a_- R\Sigma R^T\mathbf{v}, \\ \beta' &= R\beta + \mathbf{v} - R\Sigma R^T\mathbf{v}, \\ \Sigma' &= R\Sigma R^T. \end{aligned} \tag{19}$$

Differentiating at the identity, we obtain the adjoint action of G on $\mathfrak{g}$. We promote $(\alpha_+, \alpha_-, \alpha, \beta, \Sigma)$ to a curve via the identity and simply compute the velocity at the identity of the curve obtained after conjugation. Doing so, we find

$$\mathrm{Ad}_{\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)} \mathsf{A}(x_+, x_-, \mathbf{x}, \mathbf{y}, X) = \mathsf{A}(x'_+, x'_-, \mathbf{x}', \mathbf{y}', X'), \tag{20}$$

where

$$\begin{aligned} x'_+ &= x_+ - \tfrac{1}{2}x_-\|\mathbf{v}\|^2 + \mathbf{v}\cdot R\mathbf{x} - \mathbf{a}\cdot R\mathbf{y} + \mathbf{v}\cdot RXR^T\mathbf{a}, \\ x'_- &= x_-, \\ \mathbf{x}' &= R\mathbf{x} - x_-\mathbf{v} - RXR^T(\mathbf{a} + a_-\mathbf{v}) + a_- R\mathbf{y}, \\ \mathbf{y}' &= R\mathbf{y} - RXR^T\mathbf{v}, \\ X' &= RXR^T. \end{aligned} \tag{21}$$

Let us introduce a basis $\lambda^{ab}, \beta^a, \pi^a, \eta, \mu$ for $\mathfrak{g}^*$ canonically dual to L_{ab}, B_a, P_a, H, M. We now parametrise the dual $\mathfrak{g}^*$ of the Bargmann Lie algebra by[3]

$$\mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, J) = m\mu + E\eta - p_a\pi^a + k_a\beta^a + \tfrac{1}{2}J_{ab}\lambda^{ab}. \tag{22}$$

It follows that if $\alpha = \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, J)$, then the linear function on $\mathfrak{g}^*$ defined by H takes the value E on α. The dual pairing is given by

$$\langle \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, J), \mathsf{A}(x_+, x_-, \mathbf{x}, \mathbf{y}, X)\rangle = mx_+ - Ex_- - \mathbf{p}\cdot\mathbf{x} + \mathbf{k}\cdot\mathbf{y} + \tfrac{1}{2}\operatorname{Tr}(J^T X). \tag{23}$$

[3] The choice of sign is such that later the momentum of a particle of mass m moving with velocity $\mathbf{v}$ is given by the familiar $\mathbf{p} = m\mathbf{v}$.

We then define the coadjoint representation as the dual representation to the adjoint representation:

$$\begin{aligned}&\left\langle \mathrm{Ad}^*_{\mathsf{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)} \mathsf{M}(m,E,\mathbf{p},\mathbf{k},J), \mathsf{A}(x_+,x_-,\mathbf{x},\mathbf{y},X)\right\rangle \\ &\quad = \left\langle \mathsf{M}(m,E,\mathbf{p},\mathbf{k},J), \mathrm{Ad}_{\mathsf{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)^{-1}} \mathsf{A}(x_+,x_-,\mathbf{x},\mathbf{y},X)\right\rangle .\end{aligned} \tag{24}$$

We calculate

$$\begin{aligned}&\mathrm{Ad}_{\mathsf{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)^{-1}} \mathsf{A}(x_+,x_-,\mathbf{x},\mathbf{y},X) \\ &\quad = \mathrm{Ad}_{\mathsf{g}(-a_++\mathbf{a}\cdot\mathbf{v}+\frac{1}{2}a_-\|\mathbf{v}\|^2,-a_-,-R^T(\mathbf{a}+a_-\mathbf{v}),-R^T\mathbf{v},R^T)} \mathsf{A}(x_+,x_-,\mathbf{x},\mathbf{y},X),\end{aligned} \tag{25}$$

to give

$$\mathrm{Ad}_{\mathsf{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)^{-1}} \mathsf{A}(x_+,x_-,\mathbf{x},\mathbf{y},X) = \mathsf{A}(x'_+,x'_-,\mathbf{x}',\mathbf{y}',X'), \tag{26}$$

where

$$\begin{aligned} x'_+ &= x_+ - \tfrac{1}{2}x_-\|\mathbf{v}\|^2 - \mathbf{v}\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{y} + a_-\mathbf{v}\cdot\mathbf{y} + \mathbf{v}\cdot X\mathbf{a}, \\ x'_- &= x_-, \\ \mathbf{x}' &= R^T(\mathbf{x} + X\mathbf{a} + x_-\mathbf{v} - a_-\mathbf{y}), \\ \mathbf{y}' &= R^T(\mathbf{y} + X\mathbf{v}), \\ X' &= R^T X R. \end{aligned} \tag{27}$$

The dual pairing then gives

$$\mathrm{Ad}^*_{\mathsf{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)} \mathsf{M}(m,E,\mathbf{p},\mathbf{k},J) = \mathsf{M}(m',E',\mathbf{p}',\mathbf{k}',J'), \tag{28}$$

where

$$\begin{aligned} m' &= m, \\ E' &= E + R\mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}m\|\mathbf{v}\|^2, \\ \mathbf{p}' &= R\mathbf{p} + m\mathbf{v}, \\ \mathbf{k}' &= R\mathbf{k} + m\mathbf{a} + a_-(R\mathbf{p} + m\mathbf{v}), \\ J' &= RJR^T + \mathbf{a}(R\mathbf{p})^T - (R\mathbf{p})\mathbf{a}^T + (R\mathbf{k})\mathbf{v}^T - \mathbf{v}(R\mathbf{k})^T - m\mathbf{v}\mathbf{a}^T + m\mathbf{a}\mathbf{v}^T. \end{aligned} \tag{29}$$

4 Maurer–Cartan one-form

In this section we let $g = \mathsf{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)$ be a generic element of the Bargmann group and we will compute the pull-back of the left-invariant Maurer–Cartan one-form to the parameter space:

$$\begin{aligned} g^{-1}dg &= \begin{pmatrix} 1 & -\frac{1}{2}\|\mathbf{v}\|^2 & -\mathbf{v}^T & -a_+ + \mathbf{a}\cdot\mathbf{v} + \frac{1}{2}a_-\|\mathbf{v}\|^2 \\ 0 & 1 & \mathbf{0}^T & -a_- \\ 0 & R^T\mathbf{v} & R^T & -R^T(\mathbf{a}+a_-\mathbf{v}) \\ 0 & 0 & \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{v}\cdot d\mathbf{v} & d\mathbf{v}^T R + \mathbf{v}^T dR & da_+ \\ 0 & 0 & \mathbf{0}^T & da_- \\ 0 & -d\mathbf{v} & dR & d\mathbf{a} \\ 0 & 0 & \mathbf{0}^T & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & d\mathbf{v}^T R & da_+ - \mathbf{v}^T d\mathbf{a} - \frac{1}{2}\|\mathbf{v}\|^2 da_- \\ 0 & 0 & \mathbf{0}^T & da_- \\ 0 & -R^T d\mathbf{v} & R^T dR & R^T\mathbf{v}da_- + R^T d\mathbf{a} \\ 0 & 0 & \mathbf{0}^T & 0 \end{pmatrix} \\ &= \mathsf{A}(da_+ - \mathbf{v}^T d\mathbf{a} - \tfrac{1}{2}\|\mathbf{v}\|^2 da_-, da_-, R^T\mathbf{v}da_- + R^T d\mathbf{a}, R^T d\mathbf{v}, R^T dR). \end{aligned} \tag{30}$$

Pairing with $\alpha = \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, J)$, we find

$$\left\langle \alpha, g^{-1}dg \right\rangle = m da_+ - (E + \tfrac{1}{2}m\|\mathbf{v}\|^2 + (R\mathbf{p})^T\mathbf{v})da_- - (R\mathbf{p} + m\mathbf{v})^T d\mathbf{a} + (R\mathbf{k})^T d\mathbf{v} + \tfrac{1}{2}\operatorname{Tr} J^T R^T dR\,. \tag{31}$$

5 The case of $n = 3$

When $n = 3$, the vector and adjoint representations of SO(3) are isomorphic. As described in [9], the isomorphism $\varepsilon : \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by $\varepsilon(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}$, which obeys $[\varepsilon(\mathbf{a}), \varepsilon(\mathbf{b})] = \varepsilon(\mathbf{a} \times \mathbf{b})$ and also $\varepsilon(R\mathbf{a}) = R\varepsilon(\mathbf{a})R^T$ and hence $\mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T = \varepsilon(\mathbf{b} \times \mathbf{a})$. It also relates the inner products so that $\frac{1}{2}\operatorname{Tr}\varepsilon(\mathbf{a})^T\varepsilon(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$.

It follows that in this dimension, letting $J = \varepsilon(\mathbf{j})$, equation (29) becomes

$$\begin{aligned}
m' &= m\,,\\
E' &= E + R\mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}m\|\mathbf{v}\|^2\,,\\
\mathbf{p}' &= R\mathbf{p} + m\mathbf{v}\,,\\
\mathbf{k}' &= R\mathbf{k} + m\mathbf{a} + a_-(R\mathbf{p} + m\mathbf{v})\,,\\
\mathbf{j}' &= R\mathbf{j} - \mathbf{a} \times (R\mathbf{p} + m\mathbf{v}) + \mathbf{v} \times R\mathbf{k}\,,
\end{aligned} \tag{32}$$

and equation (31) becomes

$$\left\langle \alpha, g^{-1}dg \right\rangle = m da_+ - (E + R\mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}m\|\mathbf{v}\|^2)da_- - (R\mathbf{p} + m\mathbf{v})\cdot d\mathbf{a} + R\mathbf{k}\cdot d\mathbf{v} + \mathbf{j}\cdot\varepsilon^{-1}(R^{-1}dR)\,. \tag{33}$$

6 Automorphisms

Let A denote the group of automorphisms of the Bargmann Lie algebra $\mathfrak{g}$ which act trivially on the rotational subalgebra. An easy calculation shows that any $\varphi \in A$ is parametrised by $\alpha, \beta \in \mathbb{R}^\times$ and $\lambda, \mu \in \mathbb{R}$ and acts on the generators via

$$M \mapsto \alpha^2\beta M\,, \qquad H \mapsto \beta H + \lambda M\,, \qquad B_a \mapsto \alpha B_a + \mu P_a\,, \qquad \text{and} \qquad P_a \mapsto \alpha\beta P_a\,. \tag{34}$$

It is not hard to determine the dual action of φ on $\mathfrak{g}^*$: $\varphi \cdot \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, \mathbf{j}) = \mathsf{M}(m', E', \mathbf{p}', \mathbf{k}', \mathbf{j}')$ where

$$\begin{aligned}
m' &= \alpha^{-2}\beta^{-1}m\,,\\
E' &= \beta^{-1}E - \alpha^{-2}\beta^{-2}\lambda m\,,\\
\mathbf{p}' &= \alpha^{-1}\beta^{-1}\mathbf{p}\,,\\
\mathbf{k}' &= \alpha^{-1}\mathbf{k} + \alpha^{-2}\beta^{-1}\mu\mathbf{p}\,,\\
\mathbf{j}' &= \mathbf{j}\,.
\end{aligned} \tag{35}$$

The Lie algebra automorphism φ integrates to an automorphism Φ of the Bargmann Lie group G via $\Phi(e^X) = e^{\varphi(X)}$ for $X \in \mathfrak{g}$, which is well defined because the exponential map on the nilpotent subgroup generated by $\mathbf{B}, \mathbf{P}, H, M$ is a diffeomorphism. A simple calculation yields

$$\Phi(\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)) = \mathsf{g}(\alpha^2\beta a_+ - \lambda a_- + \tfrac{1}{2}\alpha\mu v^2, \beta a_-, \alpha\beta\mathbf{a} + \mu\mathbf{v}, \alpha\mathbf{v}, R)\,. \tag{36}$$

Unlike the case of the Carroll group treated in [9], Bargmann automorphisms do not relate different types of coadjoint orbits nor can they be used to relate different classes of unitary irreducible representations. We only list them here for completeness.

7 Coadjoint orbits for $n = 3$

From now on we will let G denote the identity component of the Bargmann group. As such, G is diffeomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathrm{SO}(3)$ and at the group level all that happens is that the orthogonal transformation R now has determinant 1.

There are some obvious Casimirs of the Bargmann algebra. Clearly M is one, which defines an invariant linear function on $\mathfrak{g}^*$. It is therefore constant on coadjoint orbits, as we see from equation (32). There is also a quadratic Casimir $\delta^{ab}P_aP_b - 2HM$, which says that $\|\mathbf{p}\|^2 - 2Em$ is a constant on the orbits. This provides a check of equation (32):

$$\begin{aligned} \|\mathbf{p}'\|^2 - 2E'm' &= \|R\mathbf{p} + m\mathbf{v}\|^2 - 2(E + R\mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}m\|\mathbf{v}\|^2)m \\ &= \|\mathbf{p}\|^2 + m^2\|\mathbf{v}\|^2 + 2mR\mathbf{p}\cdot\mathbf{v} - 2Em - 2mR\mathbf{p}\cdot\mathbf{v} - m^2\|\mathbf{v}\|^2 \\ &= \|\mathbf{p}\|^2 - 2Em\,. \end{aligned} \tag{37}$$

Similarly, there is a quartic Casimir $\delta^{ab}W_aW_b$, where $W_a = J_aM + \epsilon_{abc}P_bB_c$, which says that $\|m\mathbf{j} + \mathbf{p}\times\mathbf{k}\|^2$ is constant on the orbits. Again this provides another check of equation (32):

$$\begin{aligned} m'\mathbf{j}' - \mathbf{p}'\times\mathbf{k}' &= m(R\mathbf{j} - \mathbf{a}\times(R\mathbf{p} + m\mathbf{v}) + \mathbf{v}\times R\mathbf{k}) - (R\mathbf{p} + m\mathbf{v})\times(R\mathbf{k} + m\mathbf{a} + a_-(R\mathbf{p} + m\mathbf{v})) \\ &= mR\mathbf{j} - m\mathbf{a}\times(R\mathbf{p} + m\mathbf{v}) + m\mathbf{v}\times R\mathbf{k} - R\mathbf{p}\times R\mathbf{k} - m\mathbf{v}\times R\mathbf{k} - m(R\mathbf{p} + m\mathbf{v})\times\mathbf{a} \\ &= R(m\mathbf{j} - \mathbf{p}\times\mathbf{k})\,, \end{aligned} \tag{38}$$

where we have used that for $R \in \mathrm{SO}(3)$, $R\mathbf{p}\times R\mathbf{k} = R(\mathbf{p}\times\mathbf{k})$. Since the vectors are related by a rotation, their norms agree.

We separate orbits into two kinds depending on the value of the linear Casimir.

7.1 Coadjoint orbits with $m \neq 0$

In this case, we may choose $\mathbf{a} = -\frac{1}{m}R\mathbf{k}$ and $\mathbf{v} = -\frac{1}{m}R\mathbf{p}$ in order to set $\mathbf{p}' = \mathbf{k}' = \mathbf{0}$. Doing so, we bring $\mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, \mathbf{j})$ to

$$\mathsf{M}(m, \tfrac{1}{2m}(\|\mathbf{p}\|^2 - 2Em), \mathbf{0}, \mathbf{0}, \tfrac{1}{m}R(m\mathbf{j} - \mathbf{p}\times\mathbf{k}))\,, \tag{39}$$

where we recognise the values of the quadratic Casimir and the vector whose norm is the quartic Casimir. For each pair (E_0, w) consisting of a real number E_0 and non-negative number w we have a coadjoint orbit with representative covector

$$\mathsf{M}(m, E_0, \mathbf{0}, \mathbf{0}, \frac{w}{m}\mathbf{e}_3) \in \mathfrak{g}^*\,, \tag{40}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the standard orthonormal basis for $\mathbb{R}^3$, and where $E_0 = \frac{1}{2m}\|\mathbf{p}\|^2 - E$ and $w^2 = \|m\mathbf{j} + \mathbf{p}\times\mathbf{k}\|^2$ are the values of the quadratic and quartic Casimirs, respectively.

7.2 Coadjoint orbits with $m = 0$

In this case, the coadjoint action simplifies:

$$\begin{aligned} E' &= E + R\mathbf{p}\cdot\mathbf{v}\,, \\ \mathbf{p}' &= R\mathbf{p}\,, \\ \mathbf{k}' &= R(\mathbf{k} + a_-\mathbf{p})\,, \\ \mathbf{j}' &= R\mathbf{j} - \mathbf{a}\times R\mathbf{p} + \mathbf{v}\times R\mathbf{k}\,. \end{aligned} \tag{41}$$

The squared norm $\|\mathbf{p}\|^2$ is now invariant and we have two cases, depending on whether it is zero or nonzero.

7.2.1 The case $\mathbf{p} = 0$

In this case both E and $\|\mathbf{k}\|^2$ are invariant and we must distinguish between two cases:

1. If $\mathbf{k} = 0$, then either $\mathbf{j} = 0$ or else we may rotate it to any desired direction and we have orbits with representatives $\mathsf{M}(0, E, \mathbf{0}, \mathbf{0}, j\mathbf{e}_3)$, where $j \geq 0$ and $E \in \mathbb{R}$.

2. If $\mathbf{k} \neq 0$, we may rotate it so that it points in the direction $\mathbf{e}_3$ we can then use $\mathbf{v}$ to make $\mathbf{j}'$ collinear to $\mathbf{e}_3$. In other words, we have orbits with representatives $\mathsf{M}(0, E, \mathbf{0}, k\mathbf{e}_3, j\mathbf{e}_3)$, where $k > 0$ and $E, j \in \mathbb{R}$.

7.2.2 The case $\mathbf{p} \neq 0$

If $\mathbf{p} \neq \mathbf{0}$, we may use $\mathbf{v}$ to set $E' = 0$ and a_- to make $\mathbf{k}'$ perpendicular to $\mathbf{p}$. We may use $\mathbf{a}$ in order to make $\mathbf{j}'$ collinear with $\mathbf{p}$ but then we may use the component of $\mathbf{v}$ perpendicular to $\mathbf{p}$ to bring $\mathbf{j}'$ to $\mathbf{0}$. Finally, using the rotations which fix $\mathbf{p}$ we can bring $\mathbf{k}$ to any desired direction perpendicular to $\mathbf{e}_3$. In other words, we have orbits with representatives $\mathsf{M}(0, 0, p\mathbf{e}_3, k\mathbf{e}_2, \mathbf{0})$, for $p > 0$ and $k \geq 0$.

7.3 Summary

We summarise the above discussion in Table 1, which lists the orbits together with the dimension and a set of equations which determine the orbit as an algebraic submanifold of $\mathfrak{g}^*$.

Table 1: Coadjoint orbits of the Bargmann group. This table lists the different coadjoint orbits by exhibiting an orbit representative $\alpha \in \mathfrak{g}^*$, its stabiliser subgroup G_α inside the Bargmann group, the dimension $\dim \mathcal{O}_\alpha$ of the orbit and the equations defining the orbit. In the last case, the second invariant is $\|\mathbf{p} \times \mathbf{k}\|$, which takes the value $p_0 k_0$ only in the chosen representative. In all cases one can check that $\dim \mathcal{O}_\alpha = \dim G - \#\{\text{equations}\}$.

#	Orbit representative $\alpha = \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, \mathbf{j})$	Stabiliser G_α	$\dim \mathcal{O}_\alpha$	Equations for orbits
1	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$\{\mathsf{g}(a_+, a_-, \mathbf{0}, \mathbf{0}, R)\}$	6	$m = m_0 \neq 0, \frac{1}{2m}(\|\mathbf{p}\|^2 - 2Em) = E_0, m\mathbf{j} - \mathbf{p} \times \mathbf{k} = \mathbf{0}$
2	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, s\mathbf{e}_3)$	$\{\mathsf{g}(a_+, a_-, \mathbf{0}, \mathbf{0}, R) \mid R\mathbf{e}_3 = \mathbf{e}_3\}$	8	$m = m_0 \neq 0, \frac{1}{2m}(\|\mathbf{p}\|^2 - 2Em) = E_0, \|m\mathbf{j} - \mathbf{p} \times \mathbf{k}\| = s > 0$
3	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	G	0	$m = 0, E = E_0, \mathbf{p} = \mathbf{k} = \mathbf{j} = \mathbf{0}$
4	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, s\mathbf{e}_3)$	$\{\mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R) \mid R\mathbf{e}_3 = \mathbf{e}_3\}$	2	$m = 0, E = E_0, \mathbf{p} = \mathbf{k} = \mathbf{0}, \|\mathbf{j}\| = s > 0$
5	$\mathsf{M}(0, E_0, \mathbf{0}, k_0\mathbf{e}_3, h\mathbf{e}_3)$	$\{\mathsf{g}(a_+, a_-, \mathbf{a}, v\mathbf{e}_3, R) \mid R\mathbf{e}_3 = \mathbf{e}_3\}$	4	$m = 0, E = E_0, \mathbf{p} = \mathbf{0}, \|\mathbf{k}\| = k_0 > 0, \mathbf{j} \cdot \mathbf{k} = hk_0$
6	$\mathsf{M}(0, 0, p_0\mathbf{e}_3, \mathbf{0}, \mathbf{0})$	$\{\mathsf{g}(a_+, 0, a\mathbf{e}_3, \mathbf{v}, R) \mid R\mathbf{e}_3 = \mathbf{e}_3, \mathbf{v} \cdot \mathbf{e}_3 = 0\}$	6	$m = 0, \|\mathbf{p}\| = p_0 > 0, \mathbf{p} \times \mathbf{k} = \mathbf{0}$
7	$\mathsf{M}(0, 0, p_0\mathbf{e}_3, k_0\mathbf{e}_2, \mathbf{0})$	$\{\mathsf{g}(a_+, 0, a\mathbf{e}_3, v\mathbf{e}_2, I)\}$	8	$m = 0, \|\mathbf{p}\| = p_0 > 0, \|\mathbf{p} \times \mathbf{k}\| = p_0 k_0 > 0$

7.4 Coadjoint orbits of the full Bargmann group

The full Bargmann group has two connected components since now $R \in \mathrm{O}(3)$. As a Lie group, $\mathrm{O}(3) = \mathrm{SO}(3) \cup \mathrm{SO}(3)P$, where the parity P can be thought of as space inversion, sending $\mathbf{x} \mapsto -\mathbf{x}$ in $\mathbb{R}^3$. However it follows from the explicit form of the orbit representatives, that their image under space inversion, which sends $\mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, \mathbf{j}) \mapsto \mathsf{M}(m, E, -\mathbf{p}, -\mathbf{k}, \mathbf{j})$ lies in the SO(3)-orbit. Therefore the coadjoint orbits in Table 1 are also the coadjoint orbits of the full Bargmann group. We may also extend the Bargmann group by time reversal. As shown, e.g., in [17, Section II.D], time reversal acts on the Galilei generators as $(H, P_a, B_a, L_a) \mapsto (-H, P_a, -B_a, L_a)$, which extends by $M \mapsto -M$ to an automorphism of the Bargmann Lie algebra. This integrates to an automorphism of the Bargmann Lie group and we can ask how it acts on its coadjoint orbits. As shown, e.g., in [9, App. A.6], it maps coadjoint orbits symplectomorphically into coadjoint orbits. Under time reversal, $\mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, \mathbf{j}) \mapsto \mathsf{M}(-m, -E, \mathbf{p}, -\mathbf{k}, \mathbf{j})$. It is then a simple matter to use the equations characterising the coadjoint orbits in Table 1 to see that time reversal leaves invariant the orbits of types #6, 7 and pairs up two orbits of the other types:

$$\begin{aligned} &\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{j}), && \text{and} && \mathsf{M}(-m_0, -E_0, \mathbf{0}, \mathbf{0}, \mathbf{j}), \\ &\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{j}), && \text{and} && \mathsf{M}(0, -E_0, \mathbf{0}, \mathbf{0}, \mathbf{j}), \\ &\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{k}, \mathbf{j}), && \text{and} && \mathsf{M}(0, -E_0, \mathbf{0}, \mathbf{k}, -\mathbf{j}). \end{aligned} \tag{42}$$

It is natural to ask whether galilean field theories obey the CPT theorem and the answer is negative [30].

7.5 Structure of the orbits

The Bargmann group is isomorphic to a semidirect product $(\mathrm{SO}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^5$, where the normal subgroup $\mathbb{R}^5$ is abelian and generated by M, H, P_a. We can therefore use the results in, e.g., Oblak's thesis [31], to describe the coadjoint orbits geometrically.

As described in [9], the coadjoint orbit $\mathcal{O}_\alpha$ of $\alpha = (\kappa, \tau) \in \mathfrak{k}^* \oplus \mathfrak{t}^* = \mathfrak{g}^*$ under $G = K \ltimes T$ with T abelian are fibred products

$$\begin{array}{ccc} \mathcal{O}_\alpha & \longrightarrow & T^*\mathcal{O}_\tau \\ \downarrow & & \downarrow \\ K \times_{K_\tau} \mathcal{O}_{\kappa_\tau} & \longrightarrow & \mathcal{O}_\tau \end{array} \tag{43}$$

where

- $\mathcal{O}_\tau$ is the K-orbit of $\tau \in \mathfrak{t}^*$;
- $K_\tau \subset K$ is the stabiliser of τ, so that $\mathcal{O}_\tau \cong K/K_\tau$;
- $\mathcal{O}_{\kappa_\tau}$ is the K_τ-coadjoint orbit of the restriction $\kappa_\tau \in \mathfrak{k}_\tau^*$ of $\kappa \in \mathfrak{k}^*$ to the Lie algebra $\mathfrak{k}_\tau$ of K_τ.

The standard notation for such fibred products is

$$\mathcal{O}_\alpha = T^*\mathcal{O}_\tau \times_{\mathcal{O}_\tau} \left(K \times_{K\tau} \mathcal{O}_{\kappa_\tau}\right), \tag{44}$$

whose dimension can be read off as follows:

$$\dim \mathcal{O}_\alpha = \dim T^*\mathcal{O}_\tau - \dim \mathcal{O}_\tau + \dim K - \dim K_\tau + \dim \mathcal{O}_{\kappa_\tau} = 2\dim \mathcal{O}_\tau + \dim \mathcal{O}_{\kappa_\tau}, \tag{45}$$

which is of course even dimensional since $\mathcal{O}_{\kappa_\tau}$ is itself a coadjoint orbit of K_τ. In Table 2 we deconstruct the coadjoint orbits in Table 1.

8 Actions of Galilei particles

In this section we discuss the actions for Galilei particles. As expected, we will recover the well-known particle action for massive Galilei particles, but we will also cover the possible less familiar massless case in full generality. These actions provide information concerning the mobility of the particles and are the starting point for many applications, e.g., for path integral quantisation [32].

Table 2: Deconstructing the coadjoint orbits. The manifold $\mathbb{R}^3$ in cases #1,2 is embedded in $\mathfrak{t}^* \cong \mathbb{R}^5$ as the paraboloid $\left\{\mathsf{M}(m, E + \frac{1}{2m}\|\mathbf{p}\|^2, \mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^3\right\}$. The stabiliser K_τ in cases #6,7 consists of elements $\mathsf{g}(\mathbf{v}, R) \in K$ where $\mathbf{v} \perp \mathbf{p}$ and $R\mathbf{p} = \mathbf{p}$. Since $\mathbf{p} \neq \mathbf{0}$, these are rotations about the axis defined by $\mathbf{p}$ and hence isomorphic to SO(2), so that the stabiliser is isomorphic to $\mathbb{R}^2 \rtimes \mathrm{SO}(2)$. The K_τ-coadjoint orbits should be self-explanatory. In cases #3,4,5, the stabiliser is $K \cong \mathrm{ISO}(3)$, whose coadjoint orbits can be read off from [9, App. B.5.2]: in case #3 we have the point-like orbit $\{(\mathbf{0}, \mathbf{0})\}$, in case #4 we have the 2-sphere of radius $\|\mathbf{j}\|$, and in case #5 we have the cotangent bundle of the sphere of radius $\|\mathbf{k}\|$. In cases #6,7, the stabiliser K_τ is isomorphic to ISO(2) and the coadjoint orbits again can be read off from [9, App. B.5.2]. In case #6 we have the point-like orbit $\{(\mathbf{0}, \mathbf{0})\}$ and in case #7 we have the cylinder $T^*S^1_{\|\mathbf{k}^\perp\|}$, where $\mathbf{k}^\perp$ is the component of $\mathbf{k}$ perpendicular to $\mathbf{p}$ – that being the restriction of κ to $\mathfrak{k}_\tau$.

#	$\alpha \in \mathfrak{g}^*$	$\tau \in \mathfrak{t}^*$	$\mathcal{O}_\tau$	K_τ	$\kappa \in \mathfrak{k}^*$	$\kappa_\tau \in \mathfrak{k}_\tau^*$	$\mathcal{O}_{\kappa_\tau}$	$\mathcal{O}_\alpha$
1	$\mathsf{M}(m, E, \mathbf{0}, \mathbf{0}, \mathbf{0})_{m\neq 0, E\in\mathbb{R}}$	$(m, E, \mathbf{0})$	$\mathbb{R}^3$	SO(3)	$(\mathbf{0}, \mathbf{0})$	$\mathbf{0}$	$\{\mathbf{0}\}$	$T^*\mathbb{R}^3$
2	$\mathsf{M}(m, E, \mathbf{0}, \mathbf{0}, \mathbf{j})_{m\neq 0, E\in\mathbb{R}, \mathbf{j}\neq\mathbf{0}}$	$(m, E, \mathbf{0})$	$\mathbb{R}^3$	SO(3)	$(\mathbf{0}, \mathbf{j})$	$\mathbf{j}$	$S^2_{\|j\|}$	$T^*\mathbb{R}^3 \times_{\mathbb{R}^3} (K \times_{K_\tau} S^2)$
3	$\mathsf{M}(0, E, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(0, E, \mathbf{0})$	$\{(0, E, \mathbf{0})\}$	K	$(\mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$	$\{(\mathbf{0}, \mathbf{0})\}$	$\{(0, E, \mathbf{0}, \mathbf{0}, \mathbf{0})\}$
4	$\mathsf{M}(0, E, \mathbf{0}, \mathbf{0}, \mathbf{j})_{E\in\mathbb{R}, \mathbf{j}\neq\mathbf{0}}$	$(0, E, \mathbf{0})$	$\{(0, E, \mathbf{0})\}$	K	$(\mathbf{0}, \mathbf{j})$	$(\mathbf{0}, \mathbf{j})$	$S^2_{\|\mathbf{j}\|}$	S^2
5	$\mathsf{M}(0, E, \mathbf{0}, \mathbf{k}, \mathbf{j})_{E\in\mathbb{R}, \mathbf{k}\times\mathbf{j}=\mathbf{0}, \mathbf{k}\neq\mathbf{0}}$	$(0, E, \mathbf{0})$	$\{(0, E, \mathbf{0})\}$	K	$(\mathbf{k}, \mathbf{j})$	$(\mathbf{k}, \mathbf{j})$	$T^*S^2_{\|\mathbf{k}\|}$	T^*S^2
6	$\mathsf{M}(0, 0, \mathbf{p}, \mathbf{0}, \mathbf{0})_{\mathbf{p}\neq\mathbf{0}}$	$(0, 0, \mathbf{p})$	$\mathbb{R} \times S^2_{\|\mathbf{p}\|}$	$\mathbb{R}^2 \rtimes \mathrm{SO}(2)$	$(\mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$	$\{(\mathbf{0}, \mathbf{0})\}$	$T^*(\mathbb{R} \times S^2)$
7	$\mathsf{M}(0, 0, \mathbf{p}, \mathbf{k}, \mathbf{0})_{\mathbf{k}\cdot\mathbf{p}=0, \mathbf{k}\times\mathbf{p}\neq\mathbf{0}}$	$(0, 0, \mathbf{p})$	$\mathbb{R} \times S^2_{\|\mathbf{p}\|}$	$\mathbb{R}^2 \rtimes \mathrm{SO}(2)$	$(\mathbf{k}, \mathbf{0})$	$(\mathbf{k}^\perp, \mathbf{0})$	$T^*S^1_{\|\mathbf{k}^\perp\|}$	$T^*(\mathbb{R} \times S^2) \times_{\mathbb{R}\times S^2} (K \times_{K_\tau} T^*S^1)$

As discussed above, classical Galilei particles correspond to homogeneous symplectic manifolds of the Galilei group and these in turn correspond to coadjoint orbits of the Bargmann group. Hence they can also be thought of as classical Bargmann particles. We view the Bargmann group as an auxiliary concept we are forced to introduce for mathematical consistency, but from a spacetime perspective it is the Galilei group which is the relativity group and hence we prefer to use the term Galilei particle, but in so doing we allow them to have a non-zero mass.

Let $\alpha \in \mathfrak{g}^*$ be an element in the dual of the Bargmann algebra and let $\mathcal{O}_\alpha \subset \mathfrak{g}^*$ denote its coadjoint orbit. It is G-equivariantly diffeomorphic to G/G_α, with G_α the stabiliser subgroup of α. Let $\pi_\alpha : G \to \mathcal{O}_\alpha$ denote the orbit map: $\pi_\alpha(g) = \mathrm{Ad}^*_g\, \alpha$. The G-invariant Kirillov–Kostant–Souriau symplectic structure ω_{KKS} on $\mathcal{O}_\alpha$ pulls back via the orbit map to a left-invariant presymplectic form $\pi^*_\alpha \omega_{\mathrm{KKS}} \in \Omega^2(G)$ on G, which is moreover exact:

$$\pi^*_\alpha \omega_{\mathrm{KKS}} = -d\, \langle \alpha, \vartheta \rangle\,, \tag{46}$$

where $\vartheta \in \Omega^1(G; \mathfrak{g})$ is the left-invariant Maurer–Cartan one-form on G. The primitive one-form $\langle \alpha, \vartheta \rangle$ defines a variational problem for curves $g : I \to G$ in the group:

$$S[g] := \int_I \langle \alpha, g^*\vartheta \rangle = \int_I \left\langle \alpha, g(\tau)^{-1} \dot{g}(\tau) \right\rangle d\tau\,, \tag{47}$$

where $\tau \in I$ is the parameter along the curve and should not be confused with the element of $\mathfrak{t}^*$ used in the previous section. This is the point of departure in this section for the study of the dynamical systems associated to each of the coadjoint orbits. Some of these actions have been discussed, e.g., in [33, §5.2].

We will now construct actions $S = \int L d\tau$ for the Galilei particles in 3 + 1 dimensions. Using (33) with the replacements $a_- \mapsto t$ and $\mathbf{a} \mapsto -\mathbf{x}$ leads to the general Lagrangian

$$\begin{aligned} L[a_+, t, \mathbf{x}, \mathbf{v}, R(\varphi)] &= m(\dot{a}_+ + \mathbf{v} \cdot \dot{\mathbf{x}} - \tfrac{1}{2}\|\mathbf{v}\|^2 \dot{t}) - E\dot{t} + R\mathbf{p} \cdot (\dot{\mathbf{x}} - \mathbf{v}\dot{t}) + R\mathbf{k} \cdot \dot{\mathbf{v}} + \mathbf{j} \cdot \varepsilon^{-1}(R^{-1}\dot{R}) & \text{(48a)} \\ &= m\dot{a}_+ - (E + R\mathbf{p} \cdot \mathbf{v} + \tfrac{m}{2}\|\mathbf{v}\|^2)\dot{t} + (R\mathbf{p} + m\mathbf{v}) \cdot \dot{\mathbf{x}} + R\mathbf{k} \cdot \dot{\mathbf{v}} + \mathbf{j} \cdot \varepsilon^{-1}(R^{-1}\dot{R})\,, & \text{(48b)} \end{aligned}$$

where the dot denotes derivatives with respect to the parameter τ. We parametrise the orbits by $\alpha = \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, J)$ which means they are not varied, while we vary with respect to the quantities in the square brackets of the Lagrangian. Since the action does not depend on the specific point α, but only on the coadjoint orbit itself, we are free to make a convenient choice. We will use the representatives in Table 1. Since the physics and degrees of freedom depend on the specific particle we will analyse them case by case, but first we discuss the global and gauge symmetries for the generic Lagrangian (48).

8.1 Symmetries

The action (48) has global Galilei symmetry since $g^{-1}\dot{g}$ is invariant under the τ-independent left action $g \mapsto hg$. Consequentially, the infinitesimal symmetries lead to Noether charges $\frac{d}{d\tau}X_Q = 0$ which are given by

$$\begin{aligned}
&\delta_{c_+} a_+ = c_+ && && && m_Q = m\,, && (49a)\\
&\delta_{c_t} t = c_t && && && E_Q = E + R\mathbf{p}\cdot\mathbf{v} + \tfrac{1}{2}m\|\mathbf{v}\|^2\,, && (49b)\\
&\delta_{c_x}\mathbf{x} = \mathbf{c}_x && && &\Rightarrow\; & \mathbf{p}_Q = R\mathbf{p} + m\mathbf{v}\,, && (49c)\\
&\delta_{c_v}\mathbf{v} = \mathbf{c}_v\,, && \delta_{c_v} a_+ = -\mathbf{x}\cdot\mathbf{c}_v\,, && \delta_{c_v}\mathbf{x} = t\mathbf{c}_v\,, && & \mathbf{k}_Q = R\mathbf{k} - m\mathbf{x} + t(R\mathbf{p} + m\mathbf{v})\,, && (49d)\\
&\delta_\omega R = \omega R\,, && \delta_\omega \mathbf{x} = \omega\mathbf{x}\,, && \delta_\omega \mathbf{v} = \omega\mathbf{v} && & \mathbf{j}_Q = R\mathbf{j} + \mathbf{x}\times(R\mathbf{p} + m\mathbf{v}) + \mathbf{v}\times R\mathbf{k}\,, && (49e)
\end{aligned}$$

where $\omega^T = -\omega$. This shows that the charges are given by the coadjoint action on α, cf., (32).

This action also has gauge freedom parametrised by the right action $g \mapsto gh(\tau)$, where h is now τ dependent and has to be in the stabiliser of α. The general infinitesimal gauge transformations for the case at hand are given by

$$\begin{aligned}
&\delta_{\lambda_+} a_+ = \lambda_+\,, && && && (50a)\\
&\delta_{\lambda_t} t = \lambda_t\,, && \delta_{\lambda_t} a_+ = -\tfrac{1}{2}\|\mathbf{v}\|^2\lambda_t\,, && \delta_{\lambda_t}\mathbf{x} = \mathbf{v}\lambda_t\,, && (50b)\\
&\delta_{\lambda_x}\mathbf{x} = R\boldsymbol{\lambda}_x\,, && \delta_{\lambda_x} a_+ = -\mathbf{v}\cdot R\boldsymbol{\lambda}_x\,, && && (50c)\\
&\delta_{\lambda_v}\mathbf{v} = R\boldsymbol{\lambda}_v\,, && && && (50d)\\
&\delta_{\lambda_\omega} R = R\lambda_\omega\,, && && && (50e)
\end{aligned}$$

where all λ are τ-dependent and $\lambda_\omega^T = -\lambda_\omega$.

8.2 Massive Galilei particles

In this subsection, we construct Lagrangians associated with massive orbits, both without spin (orbit #1) and with spin (orbit #2).

8.2.1 Orbit #1 (massive spinless)

The massive orbit without spin describes the most familiar type of galilean particle commonly encountered in textbooks. We will use this section to illustrate some known properties of these geometric actions (see, e.g., [9,31,34,35] and references therein) and provide further details in Appendix A.

According to Table 1, massive spinless Galilei particles can be characterised by the following representative

$$\alpha = \mathsf{M}(m, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})\,. \qquad (51)$$

Using this representative in (48) leads to the following action for the massive galilean particle

$$L[a_+, t, \mathbf{x}, \mathbf{v}, R(\varphi)] = m\dot{a}_+ - (E_0 + \tfrac{1}{2}m\|\mathbf{v}\|^2)\dot{t} + m\mathbf{v}\cdot\dot{\mathbf{x}}\,. \qquad (52)$$

The first term in (52) is a boundary term that ensures the existence of a non-vanishing conserved quantity m. This term and the variation with respect to $R(\varphi)$ do not contribute to the equations of motion and we will therefore omit it in the following and concentrate our discussion on the following Lagrangian

$$L[t,\mathbf{x},\mathbf{v}] = -(E_0 + \tfrac{1}{2}m\|\mathbf{v}\|^2)\dot{t} + m\mathbf{v}\cdot\dot{\mathbf{x}}\,. \tag{53}$$

To express the Lagrangian in canonical form we introduce the canonical momenta

$$p_t = \frac{\partial L}{\partial \dot{t}} = -(E_0 + \tfrac{1}{2}m\|\mathbf{v}\|^2)\,, \tag{54a}$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = m\mathbf{v}\,, \tag{54b}$$

$$\boldsymbol{p}_v = \frac{\partial L}{\partial \dot{\mathbf{v}}} = \mathbf{0}\,, \tag{54c}$$

which lead to the following constraints

$$\phi = p_t + E_0 + \tfrac{1}{2}m\|\mathbf{v}\|^2 \approx 0\,, \tag{55a}$$

$$\boldsymbol{\phi}^1 = \boldsymbol{p} - m\mathbf{v} \approx \mathbf{0}\,, \tag{55b}$$

$$\boldsymbol{\phi}^2 = \boldsymbol{p}_v \approx \mathbf{0}\,. \tag{55c}$$

We can use them to construct an action in Hamiltonian form, which is generically of the form

$$\int L_{\text{can}}[q,p,u,\phi]d\tau = \int (p\dot{q} - H_{\text{can}} - u\phi)d\tau\,, \tag{56}$$

with Poisson brackets given by $\{q,p\} = 1$. For the case at hand this leads to

$$L_{\text{can}}[t,p_t,\mathbf{x},\boldsymbol{p},\mathbf{v},\boldsymbol{p}_v,u,\boldsymbol{u}_1,\boldsymbol{u}_2] = p_t\dot{t} + \boldsymbol{p}\cdot\dot{\mathbf{x}} + \boldsymbol{p}_v\cdot\dot{\boldsymbol{v}} - u\phi - \boldsymbol{u}_i\boldsymbol{\phi}^i\,, \tag{57}$$

where we observe the vanishing of the canonical Hamiltonian H_{can} and the enforcement of constraints through the variation of the Lagrange multipliers u and $\boldsymbol{u}_i$. The set of constraints $(\boldsymbol{\phi}^1,\boldsymbol{\phi}^2)$ are of second-class. Indeed, they obey the following non-vanishing Poisson brackets

$$\{\phi^1_a,\phi^2_b\} = -m\delta_{ab}\,. \tag{58}$$

The second class constraints can be imposed to be strongly equal to zero. In particular, the constraint $\boldsymbol{\phi}^1 = 0$ can be conveniently solved as $\mathbf{v} = \boldsymbol{p}/m$. Thus, plugging it back in the action one obtains

$$L_{\text{can}}[t,p_t,\mathbf{x},\boldsymbol{p},u] = p_t\dot{t} + \boldsymbol{p}\cdot\dot{\mathbf{x}} - u\left(p_t + \frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right). \tag{59}$$

We could have circumvented the analysis of the second class constraints by realising that this part of the action (52) is already in first order form, i.e., we could have just redefined $\mathbf{v} = \boldsymbol{p}/m$ in (52).

On the other hand, the constraint ϕ is of first-class and generates the gauge symmetry of time reparametrisations. By solving this first-class constraint and applying the gauge-fixing condition $t = \tau$, the action can be written as

$$L_{\text{can}}[\mathbf{x},\boldsymbol{p}] = \boldsymbol{p}\cdot\dot{\mathbf{x}} - \left(\frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right), \tag{60}$$

where the derivatives are now with respect to t. To write this action in configuration space we can use the equation of motion obtained from the variation of $\boldsymbol{p}$ to obtain

$$L[\mathbf{x}]_{\text{red}} = \frac{m}{2}\|\dot{\mathbf{x}}\|^2 - E_0\,. \tag{61}$$

This is the standard action for a free (nonrelativistic) Galilei particle with mass m.

The dimension of the orbit #2 in Table 1 indeed agrees with the number of independent canonical variables of our actions. From (57) we obtain that they are $14-2\times 1-6=6$, where we have taken all canonical variables (14) and subtracted the constraints (first-class constraints count twice, e.g., Section 1.4.2. in [36]). This also agrees with the 6 canonical variables in (60), where all constraints have been resolved.

8.2.2 Orbit #2 (massive spinning)

The representative of this orbit is given by

$$\alpha = \mathsf{M}(m, E_0, \mathbf{0}, \mathbf{0}, \mathbf{j})\,. \tag{62}$$

From (48) one obtains the following Lagrangian

$$L[t, \mathbf{x}, \mathbf{v}, R(\varphi)] = -\left(E_0 + \tfrac{1}{2} m \|\mathbf{v}\|^2\right) \dot{t} + m\mathbf{v}\cdot\dot{\mathbf{x}} + \mathbf{j}\cdot\varepsilon^{-1}(R^{-1}\dot{R})\,. \tag{63}$$

An important property is that the last term, that describes the spin part of the particle, "decouples" from the rest of the action. This can be seen as a consequence of the fact that there is no Thomas precession for galilean particles (the same is true for carrollian particles [9]). On the other hand, the first two terms at the right-hand side are identical to those discussed in orbit #1.

Following [9], it is convenient to parametrise the rotation matrix as follows

$$R(\varphi) = e^{\varphi_1 \epsilon_1} e^{\varphi_2 \epsilon_2} e^{\varphi_3 \epsilon_3}\,, \tag{64}$$

where $(\epsilon_a)_{bc} = -\epsilon_{abc}$. Furthermore, if we choose the angular momentum to be aligned with the z-axis, i.e. $\mathbf{j} = (0, 0, s)$ with $s > 0$, then the Lagrangian can be written as

$$L[t, \mathbf{x}, \mathbf{v}, \varphi] = -\left(E_0 + \tfrac{1}{2} m \|\mathbf{v}\|^2\right) \dot{t} + m\mathbf{v}\cdot\dot{\mathbf{x}} + s\left(\dot{\varphi}_3 + \sin(\varphi_2)\,\dot{\varphi}_1\right)\,. \tag{65}$$

To express the Lagrangian in canonical form, we can use the momenta defined in (54a) and the momenta associated with the spin part, which are given by

$$\Pi_1 = \frac{\partial L}{\partial \dot{\varphi}_1} = s \sin\varphi_2\,, \qquad \Pi_2 = \frac{\partial L}{\partial \dot{\varphi}_2} = 0\,, \qquad \Pi_3 = \frac{\partial L}{\partial \dot{\varphi}_3} = s\,, \tag{66}$$

Following the same approach as for orbit #1 for the non-spinning part, the Lagrangian in canonical form can be written as

$$L_{\text{can}}[t, p_t, \mathbf{x}, \boldsymbol{p}, \varphi, \boldsymbol{\Pi}, u, u_2, u_3] = p_t \dot{t} + \boldsymbol{p}\cdot\dot{\mathbf{x}} + \boldsymbol{\Pi}\cdot\dot{\varphi} - u\left(p_t + \frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right) - u_2\Pi_2 - u_3(\Pi_3 - s)\,. \tag{67}$$

To emphasise the relevant physical degrees of freedom, we can solve the constraint and impose the gauge fixing condition $t = \tau$. Then, after eliminating some boundary terms, the Lagrangian in the reduced phase space takes the form

$$L_{\text{red}}[\mathbf{x}, \boldsymbol{p}, \varphi_1, \Pi_1] = \boldsymbol{p}\cdot\dot{\mathbf{x}} + \Pi_1\dot{\varphi}_1 - \left(\frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right)\,. \tag{68}$$

Alternatively, by eliminating the linear momentum using its equation of motion, we can express the Lagrangian for the spinning massive Galilei particle as follows:

$$L[\mathbf{x}, \varphi_1, \Pi_1] = \frac{m}{2}\|\dot{\mathbf{x}}\|^2 + \Pi_1\dot{\varphi}_1 - E_0\,. \tag{69}$$

Comparing our actions (68) or (69) with Section 8.2 we see that they have two additional canonical variables and hence in total 8 independent canonical variables, which agrees with Table 1.

8.3 Massless Galilei particles

In this subsection, we construct the Lagrangians and study dynamics of massless Galilei particles. The foundational aspects for the analysis were provided in [12, 14], and the dynamics associated with the orbit #6 was discussed in [12, 33, 37]. Here, we present a self-contained discussion of this case, while also extending our analysis to include those orbits that have not been explored previously.

From a galilean perspective, although massless Galilei particles may not seem to describe any known particle, they are however connected to geometrical optics [38] and they emerge as the most relevant entities in the context of their application to planons. In this scenario, they represent elementary dipoles with restricted motion. A comprehensive discussion of this approach will be presented in our forthcoming work [29].

8.3.1 Orbit #3 (vacuum)

According to Table 1 the representative of this orbit is given by

$$\alpha = \mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})\,. \tag{70}$$

The corresponding Lagrangian can then be directly obtained using the orbit representative in (48). It is given by

$$L[t] = -E_0 \dot{t}\,,$$

and is a pure boundary term. Considering the trivial dynamics and the fact that the stabiliser is the entire Bargmann group, one can interpret this orbit as the vacuum configuration.

8.3.2 Orbit #4 (spinning vacuum)

The orbit representative for this case is given by

$$\alpha = \mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{j})\,. \tag{71}$$

Thus, using (48) one finds

$$L[t, R(\varphi)] = -E_0 \dot{t} + \mathbf{j} \cdot \varepsilon^{-1}(R^{-1}\dot{R})\,. \tag{72}$$

The first term at the right-hand side is a boundary term that can be neglected, while the second one describes the spin degrees of freedom. Thus, if the angular momentum is aligned with the z-axis, i.e., $\mathbf{j} = (0, 0, s)$, and if one employs the same parameterisation for the rotations as introduced in Eq. (64), then the Lagrangian becomes

$$L[\varphi] = s\left(\dot{\varphi}_3 + \sin(\varphi_2)\,\dot{\varphi}_1\right)\,. \tag{73}$$

Therefore, this configuration may be interpreted as a spinning vacuum.

8.3.3 Orbit #5

This orbit is determined by the following representative:

$$\alpha = \mathsf{M}(0, E_0, \mathbf{0}, \mathbf{k}, \mathbf{j})\,, \tag{74}$$

where $||\mathbf{k}|| = k_0 > 0$ and $\mathbf{j} \cdot \mathbf{k} = h k_0$. Using (48) one finds the following Lagrangian

$$L[t, \mathbf{v}, R(\varphi)] = -E_0 \dot{t} + R\mathbf{k} \cdot \dot{\mathbf{v}} + \mathbf{j} \cdot \varepsilon^{-1}(R^{-1}\dot{R})\,. \tag{75}$$

It is convenient to consider the parametrisation in Eq. (64) for the rotation matrix, and to write $\mathbf{k} = k_0\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector defined by

$$\hat{\mathbf{n}} = (\sin\varphi_2, -\sin\varphi_1\cos\varphi_2, -\cos\varphi_1\cos\varphi_2) \,. \tag{76}$$

Therefore, when the angular momentum aligns with the z-axis ($\mathbf{j} = (0,0,h)$), the Lagrangian becomes

$$L[\varphi, \mathbf{v}, t] = -E_0\dot{t} + k_0\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + h(\dot{\varphi}_3 + \sin(\varphi_2)\dot{\varphi}_1) \,. \tag{77}$$

Indeed, up to boundary terms and the renaming $\mathbf{k} \to \mathbf{p}$ and $\mathbf{v} \to \mathbf{x}$, this action is identical to the one found in the study of coadjoint orbits of the Carroll group, referred to as "massless Carrollion" in Ref. [9]. The reason is that the space defining the orbits are the same in both cases.

The canonical form of the action is obtained by introducing the canonical momenta

$$\begin{aligned} p_t &= \frac{\partial L}{\partial \dot{t}} = -E_0 \,, & \mathbf{p_v} &= \frac{\partial L}{\partial \dot{\mathbf{v}}} = k_0\hat{\mathbf{n}} \,, \\ \Pi_1 &= \frac{\partial L}{\partial \dot{\varphi}_1} = h\sin\varphi_2 \,, & \Pi_2 &= \frac{\partial L}{\partial \dot{\varphi}_2} = 0 \,, & \Pi_3 &= \frac{\partial L}{\partial \dot{\varphi}_3} = h \,, \end{aligned} \tag{78}$$

which satisfy the following constraints

$$||\mathbf{p_v}||^2 - k_0^2 = 0 \,, \qquad k_0\Pi_1 - hp_v^1 = 0 \,, \qquad \Pi_2 = 0 \,, \qquad \Pi_3 - h = 0 \,. \tag{79}$$

These constraints are of first class. Therefore, neglecting boundary terms, the Lagrangian in canonical form can be written as

$$\begin{aligned} L_{\text{can}}\left[\varphi, \mathbf{\Pi}, \mathbf{v}, \mathbf{p_v}, t, E, u, u^1, \eta\right] = {} & -E\dot{t} + \mathbf{\Pi}\cdot\dot{\varphi} + \mathbf{p_v}\cdot\dot{\mathbf{v}} - u(E - E_0) - u_1\left(||\mathbf{p}_v||^2 - k_0^2\right) \\ & - \eta_1\left(k_0\Pi_1 - hp_v^1\right) - \eta_2\Pi_2 - \eta_3(\Pi_3 - h) \,. \end{aligned} \tag{80}$$

For simplicity, let us restrict to the case with vanishing spin ($h = 0$) and let us fix the gauge $t = \tau$. Then, the Lagrangian becomes

$$L_{\text{can}}[\mathbf{v}, \mathbf{p_v}, u_1] = \mathbf{p_v}\cdot\dot{\mathbf{v}} - u_1\left(||\mathbf{p_v}||^2 - k_0^2\right) - E_0 \,, \tag{81}$$

where a dot now stands for derivative with respect to the physical time t.

Next, we can solve for $\mathbf{p}_v$. By varying with respect to $\mathbf{p}_v$ and u_1, we find

$$||\mathbf{p}_v||^2 = k_0^2 \,, \qquad 2u_1\mathbf{p}_v = \dot{\mathbf{v}} \,. \tag{82}$$

These equations are solved by writing $\mathbf{p}_v = k_0\hat{\mathbf{n}}$, and $u_1 = \frac{1}{2k_0}||\mathbf{v}||$. Plugging back in the Lagrangian we obtain

$$L_{\text{red}}[\mathbf{v}] = -E_0 + k_0||\dot{\mathbf{v}}|| \,. \tag{83}$$

As a final remark, the counting of independent variables from the Hamiltonian analysis $14 - 2\times 5 = 4$ coincides with the dimension of this orbit.

8.3.4 Orbit #6

The representative of this orbit is given by

$$\alpha = \mathsf{M}(0, 0, \mathbf{p}, \mathbf{0}, \mathbf{0}) \,, \tag{84}$$

where $||\mathbf{p}|| = p_0 > 0$. From (48) one finds the following Lagrangian:

$$L[\varphi, \mathbf{v}, \mathbf{x}, t] = (R\mathbf{p}\cdot\mathbf{v})\,\dot{t} + (R\mathbf{p})\cdot\dot{\mathbf{x}} \,. \tag{85}$$

It can be written in canonical form as follows

$$L_{\text{can}}[t, p_t, \mathbf{x}, \pi, \mathbf{v}, \mathbf{p}_v, u, u_1, \boldsymbol{u}_2] = p_t \dot{t} + \pi \cdot \dot{\mathbf{x}} + \mathbf{p}_v \cdot \dot{\mathbf{v}} - u\phi - u_1\phi^1 - \boldsymbol{u}_2 \cdot \boldsymbol{\phi}^2 , \tag{86}$$

with constraints of the form

$$\phi = p_t - \pi \cdot \mathbf{v} , \qquad \phi^1 = \|\pi\|^2 - p_0^2 , \qquad \boldsymbol{\phi}^2 = \mathbf{p}_v . \tag{87}$$

The Lagrangian (86) gives the following dynamical equations of motion:

$$\dot{t} = u , \qquad \dot{p}_t = 0 , \qquad \dot{\mathbf{x}} = 2u_1\pi - u\mathbf{v} , \tag{88a}$$

$$\dot{\pi} = 0 , \qquad \dot{\mathbf{v}} = \mathbf{u}_2 , \qquad \dot{\mathbf{p}}_v = u\pi . \tag{88b}$$

The preservation in time of the constraints does not result in secondary constraints. However, it fixes some of the Lagrange multipliers, indicating the presence of second-class constraints. From the preservation of ϕ and ϕ^2 one finds that (the conservation of ϕ^1 does not yield further equations)

$$\mathbf{u}_2 \cdot \pi = 0 , \tag{89a}$$

$$u = 0 . \tag{89b}$$

There are two interesting properties that can be derived from the previous equations. From (88b) and (89a) one finds the following restriction on the dynamics

$$\pi \cdot \dot{\mathbf{v}} = 0 . \tag{90}$$

Consequently, the acceleration $\dot{\mathbf{v}}$ in the direction of the momentum π must vanish, and the acceleration in the direction of the plane orthogonal to the momentum will be part of the gauge freedom (since $\dot{\mathbf{v}} = \mathbf{u}_2$, where the transverse component of $\mathbf{u}_2$ with respect to π is arbitrary). This property will play a key role in the mobility restriction of planons that will be studied in [29].

The second important property that can be derived from (89b) is that the equation describing the evolution of the time variable becomes

$$\dot{t} = 0 . \tag{91}$$

This means that this type of galilean particle does not evolve in the physical time t, and the orbit is instantaneously defined at a certain fixed value of t. Indeed, it is not possible to choose a "gauge fixing" of the form $t = \tau$ as in the previous cases.

Let us now examine the structure of the constraints in more detail. The second-class constraints are given by

$$\phi = p_t - \pi \cdot \mathbf{v} , \qquad \chi := \pi \cdot \mathbf{p}_v . \tag{92}$$

In particular, its Poisson bracket yields

$$\{\phi, \chi\} = -\|\pi\|^2 = -p_0^2 . \tag{93}$$

On the other hand, the first-class constraints are given by

$$\|\pi\|^2 - p_0^2 \approx 0 , \qquad \mathbf{p_v}^{\perp} := \mathbf{p_v} - \frac{1}{p_0^2}(\pi \cdot \mathbf{p_v})\,\pi \approx 0 . \tag{94}$$

Note that there are 14 canonical variables, 3 first-class constraints and 2 second-class constraints. Thus, the number of independent variables is $14 - 2 \times 3 - 2 = 6$, which precisely coincides with the dimension of the orbit.

If we impose the second-class constraints to be strongly equal to zero, $\phi = \chi = 0$, and if in addition we solve the first-class constraint $\pi_{\mathbf{v}}{}^{\perp} = 0$, together with the gauge fixing condition $\mathbf{v}^{\perp} = 0$, then the Lagrangian takes the form

$$L = p_0 v^L \dot{t} + \pi \cdot \dot{\mathbf{x}} - \eta_1 \left(\|\pi\|^2 - p_0^2\right) , \tag{95}$$

where $v_L := \frac{\pi}{p_0} \cdot \mathbf{v}$ is the longitudinal component of the velocity.

8.3.5 Orbit #7

The orbit representative for this case is given by

$$\alpha = \mathsf{M}(0,0,\mathbf{p},\mathbf{k},\mathbf{0}), \tag{96}$$

where $\|\mathbf{p}\| = p_0 > 0$, $\|\mathbf{k}\| = k_0 > 0$ and $\|\mathbf{p}\times\mathbf{k}\| = p_0 k_0 > 0$. The Lagrangian is obtained by using the representative (96) in (48)

$$L = (R\mathbf{p}\cdot\mathbf{v})\,\dot{t} + (R\mathbf{p})\cdot\dot{\mathbf{x}} + (R\mathbf{k})\cdot\dot{\mathbf{v}}. \tag{97}$$

The Lagrangian in canonical form can then be written as

$$L_{\text{can}}[t,p_t,\mathbf{x},\pi,\mathbf{v},\mathbf{p}_v,u,u_1,u_2,u_3] = p_t\dot{t} + \pi\cdot\dot{\mathbf{x}} + \mathbf{p}_v\cdot\dot{\mathbf{v}} - u\phi - u_i\phi^i, \tag{98}$$

with

$$\phi = p_t - \pi\cdot\mathbf{v}, \qquad \phi^1 = \|\pi\|^2 - p_0^2, \qquad \phi^2 = \|\mathbf{p}_v\|^2 - k_0^2, \qquad \phi^3 = \pi\cdot\mathbf{p}_v. \tag{99}$$

The corresponding dynamical equations of motion are given by

$$\dot{t} = u, \qquad \dot{p}_t = 0, \qquad \dot{\pi} = 0, \tag{100}$$

$$\dot{\mathbf{x}} = -u\mathbf{v} + 2u_1\pi + u_3\mathbf{p}_v, \qquad \dot{\mathbf{v}} = 2u_2\mathbf{p}_v + u_3\pi, \qquad \dot{\mathbf{p}}_v = u\pi. \tag{101}$$

The preservation of the constraints under time evolution does not generate secondary constraints. Nevertheless, some of the Lagrange multipliers are determined by the equations of motion, indicating the presence of second-class constraints. In particular, the preservation in time of ϕ and ϕ_3 implies that

$$u = u_3 = 0. \tag{102}$$

Indeed, it is straightforward to show that the set (ϕ,ϕ_3) defines second-class constraints with a non-vanishing Poisson bracket given by

$$\{\phi,\phi_3\} = -p_0^2. \tag{103}$$

On the other hand, the first-class constraints are given by

$$\|\pi\|^2 - p_0^2 \approx 0, \qquad \|\mathbf{p}_v\|^2 - k_0^2 \approx 0. \tag{104}$$

There are 14 canonical variables, 2 first-class constraints and 2 second-class constraints. Consequently a direct counting of the degrees of freedom gives $14 - 2\times 2 - 2 = 8$ independent variables. This is precisely the dimension of the orbit.

Using (102) the dynamical equations of motion can be rewritten as follows

$$\dot{t} = 0, \qquad \dot{p}_t = 0, \qquad \dot{\pi} = 0, \tag{105a}$$

$$\dot{\mathbf{x}} = 2u_1\pi, \qquad \dot{\mathbf{v}} = 2u_2\mathbf{p}_v, \qquad \dot{\mathbf{p}}_v = 0. \tag{105b}$$

In particular, like in orbit #6, the condition $\dot{t} = 0$ implies that this specific type of galilean particle does not evolve in the physical time t. It is defined at a given instant of time and relates simultaneous events.

Additionally, one finds the following conditions:

$$\pi\cdot\dot{\mathbf{v}} = 0, \qquad \mathbf{p}_v\cdot\dot{\mathbf{x}} = 0. \tag{106}$$

Alongside the equations of motion, these conditions imply that the component of the acceleration $\dot{\mathbf{v}}$ parallel to the momentum π vanishes, the component of the acceleration orthogonal to π and $\mathbf{p}_v$ also vanishes, while the component perpendicular to π and parallel to $\mathbf{p}_v$ is pure gauge. Additionally, the component of $\dot{\mathbf{x}}$ parallel to $\mathbf{p}_v$ vanishes, the one orthogonal to $\mathbf{p}_v$ and π also vanishes, while the component orthogonal to $\mathbf{p}_v$ and parallel to π is pure gauge. This type of restriction in the dynamics plays a crucial role in the study of planons [29].

To solve the second-class constraint $\phi_3 = 0$ one can decompose $\mathbf{p}_v$ into its longitudinal and transverse components relative to π

$$\mathbf{p}_v = \mathbf{p}_v^L + \mathbf{p}_v^\perp , \tag{107}$$

where

$$\mathbf{p}_v^L = \hat{\mathbf{n}} \cdot \mathbf{p}_v , \qquad \mathbf{p_v}^\perp := \mathbf{p_v} - (\hat{\mathbf{n}} \cdot \mathbf{p_v})\, \hat{\mathbf{n}} , \tag{108}$$

with

$$\hat{\mathbf{n}} = \frac{\pi}{p_0} . \tag{109}$$

Hence, the second-class constraints (ϕ, ϕ_3) in Eq. (99) are solved by imposing that

$$p_t = p_0 v^L , \qquad \mathbf{p}_v^L = 0 . \tag{110}$$

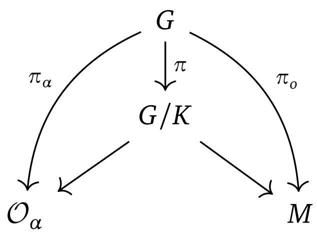
where $\mathbf{v}^L = \hat{\mathbf{n}} \cdot \mathbf{v}$. Therefore, the Lagrangian reduces to

$$L[t, \mathbf{x}, \pi, \mathbf{v}, \mathbf{p}_v, u_1, u_2] = p_0 v^L \dot{t} + \pi \cdot \dot{\mathbf{x}} + \mathbf{p}_v^\perp \cdot \dot{\mathbf{v}} - u_1 \left(\|\pi\|^2 - p_0^2 \right) - u_2 \left(\left\|\mathbf{p}_v^\perp\right\|^2 - k_0^2 \right) . \tag{111}$$

9 A geometrical approach to Galilei particle dynamics

In the previous section we have analysed the dynamics described by the action (47) associated to the coadjoint orbits $\mathcal{O}_\alpha$. In this section we will briefly outline a geometrical approach to studying the dynamics. The starting point is the action (47), which we can analyse for general α. As shown, e.g., in [9, Appendix A.4], its extrema are given by curves $g(\tau) = g_0 c(\tau)$, where $g_0 \in G$ and $c : I \to G_\alpha$ is an arbitrary curve in the stabiliser of α. Under the orbit map $\pi_\alpha : G \to \mathcal{O}_\alpha$, the curve is sent to the constant $\mathrm{Ad}^*_{g_0} \alpha$, which defines a point in $\mathcal{O}_\alpha$.

We may interpret $\mathrm{Ad}^*_{g_0} \alpha$ as the momentum of a particle moving in any homogeneous spacetime of G whose trajectory is given by composing the curve $g(\tau)$ with the orbit map associated to the spacetime. Let M denote a homogeneous spacetime and let $o \in M$ be a choice of origin. Then M is G-equivariantly diffeomorphic to G/G_o, with G_o the stabiliser subgroup of o. We let $\pi_o : G \to M$ denote the associated orbit map. Let $K := G_\alpha \cap G_o$ and consider the homogeneous space G/K. The following commutative diagram summarises the relations between these spaces:



(112)

In particular, the identity coset in G/K maps to both $\alpha \in \mathcal{O}_\alpha$ and $o \in M$. Let $\pi : G \to G/K$ denote the orbit map relative to the identity coset. Let $g(\tau)$ be a curve in G which extremises the action (47). We have seen that it maps to a point in $\mathcal{O}_\alpha$, which can be interpreted as the momentum of the particle trajectory $\pi_o(g(\tau))$ in M. We can work this out by first considering $\pi(g(\tau))$ as a trajectory in G/K and then mapping that trajectory to M. This amounts to writing

$$g(\tau) = g_0 \gamma(\tau) k(\tau) , \tag{113}$$

Table 3: Stabilisers associated to Galilei particle dynamics.

#	$\alpha = \mathsf{M}(m, E, \mathbf{p}, \mathbf{k}, \mathbf{j}) \in \mathfrak{g}^*$	$\mathfrak{g}_\alpha$	$\mathfrak{g}_\alpha \cap \mathfrak{g}_o$	$\mathfrak{m}$
1	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$\langle M, H, L_a \rangle$	$\langle M, L_a \rangle$	$\langle H \rangle$
2	$\mathsf{M}(m_0, E_0, \mathbf{0}, \mathbf{0}, s\mathbf{e}_3)$	$\langle M, H, L_3 \rangle$	$\langle M, L_3 \rangle$	$\langle H \rangle$
3	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$\mathfrak{g}$	$\langle M, B_a, L_a \rangle$	$\langle H, P_a \rangle$
4	$\mathsf{M}(0, E_0, \mathbf{0}, \mathbf{0}, s\mathbf{e}_3)$	$\langle M, H, P_a, B_a, L_3 \rangle$	$\langle M, B_a, L_3 \rangle$	$\langle H, P_a \rangle$
5	$\mathsf{M}(0, E_0, \mathbf{0}, k_0\mathbf{e}_3, h\mathbf{e}_3)$	$\langle M, H, P_a, B_3, L_3 \rangle$	$\langle M, B_3, L_3 \rangle$	$\langle H, P_a \rangle$
6	$\mathsf{M}(0, 0, p_0\mathbf{e}_3, \mathbf{0}, \mathbf{0})$	$\langle M, P_3, B_1, B_2, L_3 \rangle$	$\langle M, B_1, B_2, L_3 \rangle$	$\langle P_3 \rangle$
7	$\mathsf{M}(0, 0, p_0\mathbf{e}_3, k_0\mathbf{e}_2, \mathbf{0})$	$\langle M, P_3, B_2 \rangle$	$\langle M, B_2 \rangle$	$\langle P_3 \rangle$

where $\tau \mapsto k(\tau)$ is a curve in K and $\tau \mapsto \gamma(\tau)$ depends on a choice of coset representative for G_α/K. Then the particle trajectory on M is simply $g_0\gamma(\tau)\cdot o$. The action of g_0 is a global G-transformation which amounts to a change of "inertial frame" (G is the relativity group, after all), so that to understand the particle trajectory on M all we need to do is to understand $\gamma(\tau)\cdot o$.

Table 1 lists the stabiliser subgroups G_α for each coadjoint orbit. Galilei spacetime is described by a Klein pair $(\mathfrak{g}, \mathfrak{g}_o)$ with $\mathfrak{g}_o = \langle L_{ab}, B_a, M \rangle$.

It is a simple matter to list generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha \cap \mathfrak{g}_o$, as well as its complementary space $\mathfrak{m}$ in $\mathfrak{g}_\alpha$; that is, $\mathfrak{g}_\alpha = \mathfrak{m} \oplus (\mathfrak{g}_\alpha \cap \mathfrak{g}_o)$. These results are summarised in Table 3.

From this table and in particular from $\mathfrak{m}$, we can deduce the following, which are in agreement with the analysis in Section 8:

- For the massive orbits (those of types #1, 2), Galilei particles can be chosen not to move in space. This may sound surprising, but remember that all statements here are modulo the action of the relativity group. In this case, this simply means that any motion in space is an artefact of the choice of inertial frame; or in other words, that we can always boost to the rest frame.

- For massless orbits of types #3, 4, 5, there is no rest frame and motion in both space and time is physical.

- Finally, for massless orbits of type #6, 7, Galilei particles do not evolve in time: their trajectories instead relate simultaneous events.

It may be worth comparing this with the case of Carroll particles treated in [9]. We let $\mathfrak{c} = \langle L_a, B_a, P_a, H \rangle$ denote the Carroll algebra, $\mathfrak{c}_o = \langle L_a, B_a \rangle$ the stabiliser Lie algebra of a point in Carroll spacetime and $\mathfrak{c}_\alpha$ the stabiliser Lie algebra of $\alpha \in \mathfrak{c}^*$. We again let $\mathfrak{m}$ denote a choice of complement of $\mathfrak{c}_\alpha \cap \mathfrak{c}_o$ in $\mathfrak{c}_\alpha$. The results are summarised in Table 4. We see that Carroll particles with nonzero energy do have a rest frame, which explains why they were referred to as "massive" in [9]. They always evolve in time. Since $\mathrm{ad}_H^* = 0$ in the Carroll algebra, $H \in \mathfrak{c}_\alpha$ for all α and thus also $H \in \mathfrak{m}$ in all cases, hence any coadjoint orbit $\mathcal{O}_\alpha$ contains momenta of Carroll particles which evolve in time, but orbits with zero energy also contain momenta of Carroll particles which do not.

10 From Poincaré to Galilei particles

Let us show how to recover a Galilei particle from the $c \to \infty$ limit of a relativistic one. As a first step, we will show how to recover the Bargmann algebra from a one-dimensional extended Poincaré algebra. Let us start with the generators of the Poincaré algebra L_{ab}, K_a, T_a and T

Table 4: Stabilisers associated to particle dynamics for Carroll particles.

#	$\alpha = \mathsf{M}(\mathbf{j}, \mathbf{k}, \mathbf{p}, E) \in \mathfrak{c}^*$	$\mathfrak{c}_\alpha$	$\mathfrak{c}_\alpha \cap \mathfrak{c}_o$	$\mathfrak{m}$
1	$\mathsf{M}(\mathbf{0}, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$\langle H, L_a \rangle$	$\langle L_a \rangle$	$\langle H \rangle$
2	$\mathsf{M}(s\mathbf{e}_3, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$\langle H, L_3 \rangle$	$\langle L_3 \rangle$	$\langle H \rangle$
3	$\mathsf{M}(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	$\mathfrak{c}$	$\mathfrak{c}_o$	$\langle P_a, H \rangle$
4	$\mathsf{M}(j\mathbf{e}_3, \mathbf{0}, \mathbf{0}, 0)$	$\langle L_3, B_a, P_a, H \rangle$	$\langle L_3, B_a \rangle$	$\langle P_a, H \rangle$
5	$\mathsf{M}(h\mathbf{e}_3, k\mathbf{e}_3, \mathbf{0}, 0)$	$\langle L_3, B_3, P_a, H \rangle$	$\langle L_3, B_3 \rangle$	$\langle P_a, H \rangle$
6	$\mathsf{M}(h\mathbf{e}_3, \mathbf{0}, p\mathbf{e}_3, 0)$	$\langle L_3, B_a, P_3, H \rangle$	$\langle L_3, B_a \rangle$	$\langle P_3, H \rangle$
$7_\pm$	$\mathsf{M}(h\mathbf{e}_3, \pm k\mathbf{e}_3, p\mathbf{e}_3, 0)$	$\left\langle L_3, B_3, P_a - \frac{p}{k} B_a, H \right\rangle$	$\langle L_3, B_3 \rangle$	$\left\langle P_a - \frac{p}{k} B_a, H \right\rangle$
8	$\mathsf{M}(\mathbf{0}, k\cos\theta\,\mathbf{e}_3 + k\sin\theta\,\mathbf{e}_2, p\mathbf{e}_3, 0)$	$\left\langle B_2 - \frac{k}{p}\cos\theta P_2, B_3 + \frac{k}{p}\sin\theta P_2, P_3, H \right\rangle$	$\langle \sin\theta B_2 + \cos\theta B_3 \rangle$	$\left\langle P_2 - \frac{p}{k}(\cos\theta B_2 - \sin\theta B_3), P_3, H \right\rangle$

verifying

$$
\begin{aligned}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{bd} L_{ac} \,, \\
[L_{ab}, K_c] &= \delta_{bc} K_a - \delta_{ac} K_b \,, \\
[L_{ab}, T_c] &= \delta_{bc} T_a - \delta_{ac} T_b \,, \\
[K_a, K_b] &= L_{ab} \,, \\
[K_a, T_b] &= \delta_{ab} T \,, \\
[K_a, T] &= T_a \,.
\end{aligned}
\tag{114}
$$

Since the Bargmann algebra has one additional dimension and since contractions leave the Lie algebra dimension invariant we need to add an additional element to the Poincaré algebra. We will call this trivial central extension M and define the new generators from the relativistic ones

$$
K_a = cB_a \,, \qquad T_a = cP_a \,, \qquad T = c^2 M + H \,. \tag{115}
$$

We will assume that all powers of c appear explicitly. In terms of the new generators, the Lie brackets of the Poincaré algebra become

$$
\begin{aligned}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{bd} L_{ac} \,, \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \,, \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \,, \\
[B_a, B_b] &= \frac{1}{c^2} L_{ab} \,, \\
[B_a, P_b] &= \delta_{ab} M + \frac{1}{c^2} \delta_{ab} H \,, \\
[B_a, H] &= P_a \,,
\end{aligned}
\tag{116}
$$

where in the last bracket, we used the fact that M is a central element. It is easy to see that the $c \to \infty$ limit reproduces the Bargmann algebra (15) where M is now a nontrivial central extension.

Let us now analyse the implications for the Casimir elements of the centrally extended Poincaré algebra. First, the quadratic mass-squared Casimir of Poincaré now reads

$$
T_\mu T^\mu = T_a T^a - T^2 = c^2 P_a P^a - (c^2 M + H)(c^2 M + H) = c^2 P_a P^a - c^4 M^2 - 2c^2 MH - H^2 \,. \tag{117}
$$

Upon rescaling by the appropriate power of c, the limit $c \to \infty$ yields

$$
\lim_{c \to \infty} \frac{1}{c^4} T_\mu T^\mu = -M^2 \,, \tag{118}
$$

which we already know to be a Galilei Casimir, since M is central. To gain additional information, one can subtract this contribution of M^2 and go to the sub-leading order in $\frac{1}{c}$. Considering the following limit

$$\lim_{c\to\infty}\frac{1}{c^2}\left(T_\mu T^\mu + c^4 M^2\right) = P_a P^a - 2HM\,, \tag{119}$$

we recognise the expression of the quadratic Casimir of the Bargmann algebra, cf., Section 5. The interpretation of this in terms of the non-relativistic limit of the Poincaré momentum orbit is that the quadratic Casimir of the Bargmann algebra sits at sub-leading order in the quadratic Casimir of the Poincaré algebra (extended by M), and can be attained once the divergent mass contribution is properly removed. Finally, the relativistic Pauli-Lubanski vector gives, in the $c\to\infty$ limit

$$\lim_{c\to\infty}\frac{1}{c^2}W_a = \epsilon_{abc}(P^bB^c + L^{bc}M)\,, \qquad \lim_{c\to\infty}\frac{1}{c^2}W_0 = 0\,, \tag{120}$$

which is a spatial vector affected by spatial rotations only. Its norm is a conserved quantity

$$\lim_{c\to\infty}\frac{1}{c^4}W_\mu W^\mu = \epsilon_{abc}(P^bB^c + L^{bc}M)\epsilon^{ade}(P_dB_e + L_{de}M)\,, \tag{121}$$

in agreement with the purely bargmannian analysis of Section 7.

One should thus be able to obtain all Galilei from Poincaré orbits, at the expense of adding a central generator to the Poincaré algebra from the get-go. Let us illustrate how this works at the level of the momentum orbit. Expanding the Poincaré energy in the same fashion as the generator of time translations in (115), we obtain $E_{\text{P}} = mc^2 + E$ and the mass-shell condition becomes

$$-m_0^2c^2 = \frac{1}{c^2}p_\mu p^\mu = -\frac{1}{c^2}E_{\text{P}} + \|\mathbf{p}\|^2 = -c^2m^2 - 2Em - \frac{E^2}{c^2} + \|\mathbf{p}\|^2\,, \tag{122}$$

where m_0^2 labels orbits in the one-dimensional extension of Poincaré. Since M is a Casimir, the associated moment m is conserved along the orbit, and the $c\to\infty$ limit of $T_\mu T^\mu$ identifies m^2 with m_0^2. Turning now to the sub-leading contribution

$$\lim_{c\to\infty}\frac{1}{c^2}\left(p_\mu p^\mu + m_0^2c^4\right) = \|\mathbf{p}\|^2 - 2Em_0 := \mu_0\,, \tag{123}$$

we obtain the second constant along the orbit, which we will denote by μ_0 and which is identified as the value of the quadratic Casimir of the Bargmann algebra, $P_aP^a - 2HM$. It can assume any real value. Finally, the spin part gives rise to the eigenvalue of the last, quartic Casimir

$$\lim_{c\to\infty}\frac{1}{c^4}w_\mu w^\mu = \|m_0\mathbf{j} + \mathbf{p}\times\mathbf{k}\|^2 := S_0\,, \tag{124}$$

where w^μ is the Pauli-Lubanski vector. Note that the right-hand side is always a non-negative number for unitary representations of the Galilei group, while in the usual parameterisation for the norm of the Pauli-Lubanski vector for UIRs of Poincaré, this number is non-positive for massless and massive orbits, and is given by $-m_0^2c^4s_0(s_0+1)$ with $s_0\in\mathbb{N}$. Nevertheless, starting from irreducible (not necessarily unitary) representations of the Poincaré group with central extension, we can choose to write $w_\mu w^\mu$ as c^4S_0 where S_0 is any real number, yielding (124).

In order to have a matching between the coadjoint orbits of the Poincaré and the Galilei groups, at least at the level of momentum orbit, one should study the contractions of massless or massive orbits of the extended Poincaré group. Tachyonic orbits are problematic, because in (118) the eigenvalue of the Casimir of the right-hand side is always a non-positive number for unitary representations, as was already noticed by Souriau.

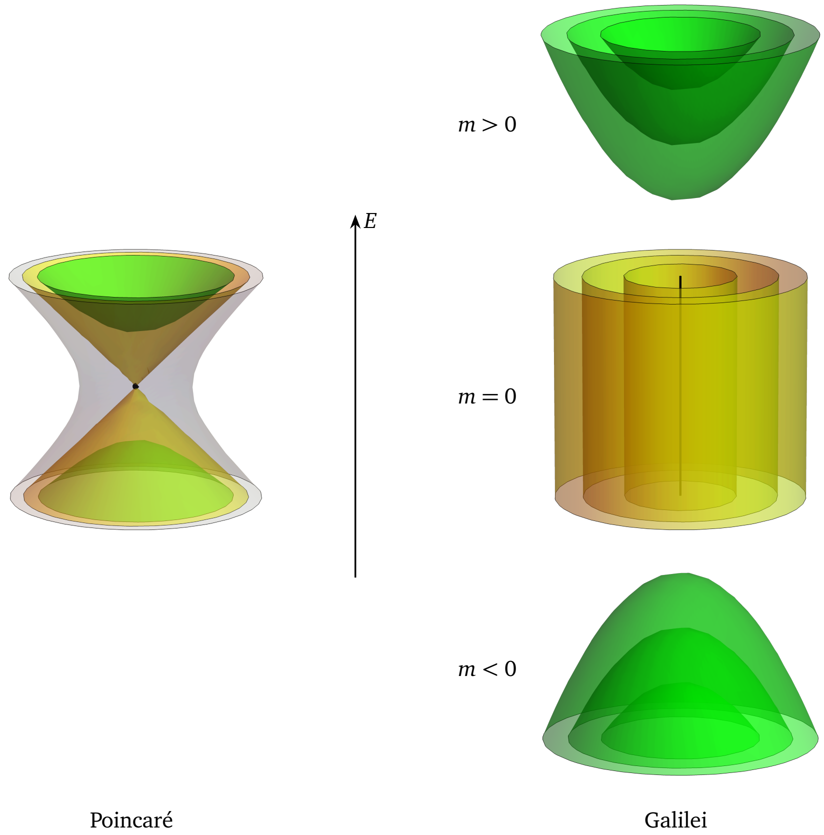


Figure 1: This figure contrasts the energy-momentum orbits $(E, \mathbf{p})$ of the Poincaré (left) and Galilei (right) group. For further details we refer to Section 9, cf., also with Table 1.
The Poincaré orbits are foliated by hypersurfaces of the form $E^2 - ||\mathbf{p}||^2 = m^2$. For $m^2 > 0$ this leads to the massive orbits with positive and negative energy (green), for $m = 0$ to the massless orbits (yellow) and for $m^2 < 0$ to tachyonic orbits (gray).
For the case of Galilei the energy-momentum orbits depend on the mass m and we picture three plots for fixed positive, vanishing and negative mass. For positive and negative m the energy-momentum orbits are hypersurfaces $E - \frac{||\mathbf{p}||^2}{2m} = E_0$, where E_0 shifts the parabolas along the energy axis. For vanishing mass $m = 0$ the Galilei orbits are foliated by cylinders $||\mathbf{p}|| = p_0 > 0$ and when $||\mathbf{p}|| = 0$ the orbits consist of disjoint points $E = E_0$ (pictured as a black line).

Starting from a massive orbit of extended Poincaré, we obtain massive orbits #1 or #2 of Galilei, depending on whether S_0 is zero or positive. Starting from a massless Poincaré orbit, we obtain Galilei orbits #3, #4 and #5 when μ_0 is zero (note that in that case, $\mathbf{p}$ is itself zero and therefore S_0 as well, these orbits corresponding to different types of Galilei vacua), and orbits #6 or #7 when μ_0 is non-zero (the difference between these last two orbits is that S_0 is zero in the former and non-zero in the latter). This is depicted in Figure 1, where massive orbits are depicted in green, and massless ones in yellow. Note that this Figure shows the correspondence between momentum orbits of Poincaré (without the one-dimensional extension) to momentum orbits of Bargmann, therefore a single orbit of the former corresponds to a family of orbits of the latter.

Table 5: K-orbits in $\mathfrak{t}^*$.

$\tau = (m, E, \mathbf{p}) \in \mathfrak{t}^*$	$\mathcal{O}_\tau$	K_τ
$(m, E, \mathbf{0})_{m\neq 0, E\in\mathbb{R}}$	$\{(m, E - \frac{1}{2}m\|\mathbf{v}\|^2, -m\mathbf{v}) \mid \mathbf{v} \in \mathbb{R}^3\} \cong \mathbb{R}^3$	$\mathrm{Spin}(3)$
$(0, E, \mathbf{0})_{E\in\mathbb{R}}$	$\{(0, E, \mathbf{0})\}$	K
$(0, 0, \mathbf{p})_{\mathbf{p}\neq\mathbf{0}}$	$\{(0, R\mathbf{p}\cdot\mathbf{v}, R\mathbf{p}) \mid \mathbf{v} \in \mathbb{R}^3, R \in \mathrm{Spin}(3)\} \cong S^2_{\|\mathbf{p}\|} \times \mathbb{R}$	$\{(\mathbf{v}, R) \mid R\mathbf{p} = \mathbf{p}, \mathbf{v} \perp \mathbf{p}\} \cong \mathrm{Spin}(2) \ltimes \mathbb{R}^2$

11 Unitary irreducible representations of the Bargmann group

In this section we classify the unitary irreducible representations (UIRs) of (the universal cover of) the Bargmann group via the method of induced representations. We are certainly not the first to do this. Earlier papers providing (partial) classifications are those of Inönü–Wigner [15], Bargmann [13], Lévy-Leblond [39], Brennich [17], who extended to representations of the full Bargmann group (including parity and time-reversal), culminating in the summary of Lévy-Leblond [16]. We will be able to compare these prior classifications with ours in the end.

11.1 K-orbits in $\mathfrak{t}^*$

We let G denote the universal cover of the identity component of the Bargmann group. It is isomorphic to $K \ltimes T$ where $K \cong \mathrm{Spin}(3) \ltimes \mathbb{R}^3$ is the subgroup generated by the rotations and the boosts and $T \cong \mathbb{R}^5$ is the abelian normal subgroup generated by the translations and the central element. The action of K on T, via conjugation in G, differentiates at the identity to an action of K on $\mathfrak{t}$ and this induces a dual action of K on $\mathfrak{t}^*$. We start by choosing $\tau \in \mathfrak{t}^*$ and let $\mathcal{O}_\tau$ denote its K-orbit. Letting $K_\tau \subset K$ denote the stabiliser, we have that $\mathcal{O}_\tau \cong K/K_\tau$, where the diffeomorphism is K-equivariant. We may also describe the orbit somewhat redundantly as $G/(K_\tau \ltimes T)$, where we have introduced a non-effective action of G on $\mathcal{O}_\tau$ which shall nevertheless prove to be very useful.

There are three classes of orbits $\mathcal{O}_\tau$ of K on $\mathfrak{t}^*$ and these are summarised in Table 5, where we describe the orbit and also list the stabiliser of an orbit representative.

11.2 Invariant measures

Ignoring the point-like orbits, we now show that both types of three-dimensional orbits admit K-invariant measures. A K-invariant measure is given by integrating a K-invariant nowhere-vanishing 3-form. By the holonomy principle, K-invariant nowhere-vanishing 3-forms on $\mathcal{O}_\tau$ are in one-to-one correspondence with nonzero K_τ-invariant elements in $\wedge^3\mathfrak{k}^0_\tau$, where $\mathfrak{k}^0_\tau \subset \mathfrak{k}^*$ is the annihilator of $\mathfrak{k}_\tau$.

Let L_i, B_i denote a basis for $\mathfrak{k}$ and let λ^i, β^i denote the canonical dual basis for $\mathfrak{k}^*$. From the Lie brackets of $\mathfrak{k}$ in this basis

$$[L_i, L_j] = \epsilon_{ijk} L_k\,, \qquad [L_i, B_j] = \epsilon_{ijk} B_k\,, \qquad \text{and} \qquad [B_i, B_j] = 0\,, \tag{125}$$

we can work out the action of $\mathfrak{k}$ on k^*:

$$\begin{aligned} L_i \cdot \lambda^j &= \epsilon_{ijk}\lambda^k\,, \qquad & B_i \cdot \lambda^j &= 0\,, \\ L_i \cdot \beta^j &= \epsilon_{ijk}\beta^k\,, \qquad & B_i \cdot \beta^j &= \epsilon_{ijk}\lambda^k\,. \end{aligned} \tag{126}$$

For the orbit with representative $\tau = (m, \frac{1}{2m}E, \mathbf{0})$, with $m \neq 0$, the stabiliser Lie algebra $\mathfrak{k}_\tau$ is spanned by the L_a, so that its annihilator $\mathfrak{k}^0_\tau$ is spanned by the β^a. It follows from equation (126) that $\frac{1}{6}\epsilon_{ijk}\beta^i\wedge\beta^j\wedge\beta^k = \beta^1\wedge\beta^2\wedge\beta^3 \in \wedge^3\mathfrak{k}^0_\tau$ is $\mathfrak{k}_\tau$-invariant, but since K_τ is connected, it is also K_τ-invariant. We can determine the corresponding K-invariant volume form on the orbit relative to a chart, by choosing a coset representative $\sigma : \mathcal{O}_\tau \to K$, with $\sigma(\mathbf{p}) \in K$ such that $\sigma(\mathbf{p}) \cdot \tau = (m, \frac{1}{2m}(E - \|\mathbf{p}\|^2), \mathbf{p})$. A possible choice is $\sigma(\mathbf{p}) = \exp(-\frac{1}{m}\mathbf{p} \cdot \mathbf{B})$. The pull-back via σ of the the left-invariant Maurer–Cartan one-form on K is given by

$$\sigma^{-1} d\sigma = \tfrac{1}{m}\mathbf{p} \cdot \mathbf{B}\,, \tag{127}$$

and hence evaluating $\beta^1 \wedge \beta^2 \wedge \beta^3$ on $\sigma^{-1}d\sigma$ gives the volume form

$$\mathrm{dvol} = \tfrac{1}{m^3} dp^1 \wedge dp^2 \wedge dp^3\,. \tag{128}$$

The action of $e^{\mathbf{v}\cdot\mathbf{B}}R \in K$ on these coordinates is calculated by acting on the coset representative $\sigma(\mathbf{p})$:

$$e^{\mathbf{v}\cdot\mathbf{B}}R\sigma(\mathbf{p}) = e^{\mathbf{v}\cdot\mathbf{B}}\sigma(R\mathbf{p})R = \sigma(R\mathbf{p} - m\mathbf{v})R\,, \tag{129}$$

so that $\mathbf{p} \mapsto R\mathbf{p} - m\mathbf{v}$. This is a euclidean transformation under which the volume form dvol is clearly invariant.

For the orbit with representative $\tau = (0, 0, \mathbf{p})$ with $\mathbf{p} = (0, 0, p)$, say, the Lie algebra $\mathfrak{k}_\tau$ of the stabiliser K_τ is spanned by L_3, B_1, B_2, so that its annihilator is spanned by $\lambda^1, \lambda^2, \beta^3$. From equation (126) we see that $\lambda^1 \wedge \lambda^2 \wedge \beta^3$ is K_τ-invariant. Under the diffeomorphism $\mathcal{O}_\tau \cong S^2 \times \mathbb{R}$, we will see below that the invariant measure is the product of the measure defined by the area form of the round metric on S^2 and the translationally invariant measure on $\mathbb{R}$.

11.3 Inducing representations

We induce UIRs of G from UIRs of K_τ as (square-integrable) sections of homogeneous vector bundles over $\mathcal{O}_\tau$. Square-integrability is defined relative to a K-invariant measure on the orbit, as described in the previous section.

For the three-dimensional orbits with stabiliser Spin(3), every UIR is isomorphic to some V_s, the complex spin-s representation of Spin(3), for $2s \in \{0, 1, 2, \dots\}$.

For the point-like orbits, the inducing representations are representations of the euclidean group K, which is itself isomorphic to the semidirect product $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$ with an abelian normal subgroup. We may apply the method of induced representations to the euclidean group itself. This was done in [10, Section 3.3.1], for instance, in the context of the UIRs of the Carroll group. There are two kinds of UIRs of $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$:

- the complex spin-s representation V_s of Spin(3), for $2s \in \{0, 1, 2, \dots\}$ with the abelian normal subgroup acting trivially;
- and the square-integrable sections of the line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$ for any $n \in \mathbb{Z}$.

Finally, for the three-dimensional orbits with $\mathrm{Spin}(2) \ltimes \mathbb{R}^2$ stabilisers, the possible UIRs can be read off from [10, Section 3.3.2], which considers a trivial central extension of this group, by ignoring the central extension. We find that there are two possible UIRs of $\mathrm{Spin}(2) \ltimes \mathbb{R}^2$:

- one-dimensional representations $\mathbb{C}_n$ of $\mathrm{U}(1) \cong \mathrm{Spin}(2)$ with the normal subgroup $\mathbb{R}^2$ acting trivially;
- and the square-integrable spinor fields on the circle $L^2(S^1, \Sigma_\pm)$ with Σ_+ (resp. Σ_-) the spinor bundle corresponding to the Ramond (resp. Neveu–Schwarz) spin structure on the circle.

Table 6: Coadjoint orbits and UIRs of the Bargmann group. The table lists a representative α of each class of coadjoint orbit $\mathcal{O}_\alpha$, the base $\mathcal{O}_\tau$ of the fibration which describes $\mathcal{O}_\alpha$ and the little groups $K_\tau \subset K$ from which we induce the UIRs of the Bargmann group. In each row we also list the inducing representation of K_τ as well as the induced representation. The notation $\mathbb{C}_E$ denotes the copy of $\mathbb{C}$ on which the one-parameter subgroup generated by H acts via the character $\chi(e^{tH}) = e^{iEt}$ and also the trivial line bundle associated to that one-dimensional representation. Similarly the notation $\mathbb{C}_{p,k^\perp}$ is the copy of $\mathbb{C}$ on which the nilpotent subgroup generated by $\mathbf{B}, \mathbf{P}, M$ acts as in equation (145) below. The notation $L^2(X,V)$ means either L^2 functions $X \to V$, when V is a vector space, or L^2 sections of a vector bundle V over X. The notation $\widetilde{\mathscr{O}}(n)$ denotes the homogeneous line bundle over $\mathbb{R} \times S^2$ obtained by pulling back the line bundle $\mathscr{O}(n)$ over S^2 via the cartesian projection $\mathbb{R} \times S^2 \to S^2$. Finally, in the last row, $k^\perp > 0$ is defined by $\|\mathbf{p} \times \mathbf{k}\| = pk^\perp$, so that it is the norm of the component of $\mathbf{k}$ perpendicular to $\mathbf{p}$.

Class	#	$\alpha \in \mathfrak{g}^*$	$\mathcal{O}_\tau$	K_τ	inducing representation of K_τ	UIR of G
$\mathbb{II}(s=0,m,E)$	1	$(m,E,\mathbf{0},\mathbf{0},\mathbf{0})_{m\neq 0,E\in\mathbb{R}}$	$\mathbb{R}^3$	Spin(3)	$\mathbb{C}$	$L^2(\mathbb{R}^3,\mathbb{C}_E)$
$\mathbb{II}(s\neq 0,m,E)$	2	$(m,E,\mathbf{0},\mathbf{0},\mathbf{j})_{m\neq 0,E\in\mathbb{R},\mathbf{j}\neq\mathbf{0}}$	$\mathbb{R}^3$	Spin(3)	$V_{s\neq 0}$	$L^2(\mathbb{R}^3,V_s\otimes\mathbb{C}_E)$
$\mathbb{I}(s=0,E)$	3	$(0,E,\mathbf{0},\mathbf{0},\mathbf{0})_{E\in\mathbb{R}}$	$\{(0,E,\mathbf{0})\}$	K	$\mathbb{C}$	$\mathbb{C}_E$
$\mathbb{I}(s\neq 0,E)$	4	$(0,E,\mathbf{0},\mathbf{0},\mathbf{j})_{E\in\mathbb{R},\mathbf{j}\neq\mathbf{0}}$	$\{(0,E,\mathbf{0})\}$	K	$V_{s\neq 0}$	$V_s\otimes\mathbb{C}_E$
$\mathbb{III}(n,k,E)$	5	$(0,E,\mathbf{0},\mathbf{k},\mathbf{j})_{E\in\mathbb{R},\mathbf{k}\times\mathbf{j}=\mathbf{0},\mathbf{k}\neq\mathbf{0}}$	$\{(0,E,\mathbf{0})\}$	K	$L^2(S^2,\mathscr{O}(n))$	$L^2(S^2,\mathscr{O}(n)\otimes\mathbb{C}_E)$
$\mathbb{IV}(n,p)$	6	$(0,0,\mathbf{p},\mathbf{0},\mathbf{0})_{\mathbf{p}\neq\mathbf{0}}$	$\mathbb{R}\times S^2_{\|\mathbf{p}\|}$	$\mathrm{Spin}(2)\ltimes\mathbb{R}^2$	$\mathbb{C}_n$	$L^2(\mathbb{R}\times S^2,\widetilde{\mathscr{O}}(-n))$
$\mathbb{V}_\pm(p,k^\perp)$	7	$(0,0,\mathbf{p},\mathbf{k},\mathbf{0})_{\mathbf{k}\times\mathbf{p}\neq\mathbf{0}}$	$\mathbb{R}\times S^2_{\|\mathbf{p}\|}$	$\mathrm{Spin}(2)\ltimes\mathbb{R}^2$	$\mathscr{H}_\pm := L^2(S^1,\Sigma_\pm)$	$L^2_\pm(\mathbb{R}\times S^3,\mathbb{C}_{p,k^\perp})$

11.4 Induced representations

The induced representations are carried by square-integrable (with respect to a K-invariant measure) sections of homogeneous vector bundles over $\mathcal{O}_\tau$ associated to the inducing representations just described. Presumably, the induced representations are obtained by geometrically quantising the coadjoint orbits of the Bargmann group and one can hazard a correspondence between the class of orbits and the induced representations, which we summarise in Table 6 and upon which we elaborate below.

11.4.1 UIRs of class $\mathbb{I}(s,E)$ associated to orbits of types #3 and 4

These are what we could call the vacuum UIRs. They are induced from the finite-dimensional UIRs of $K \cong \mathrm{Spin}(3) \ltimes \mathbb{R}^3$. They are labelled by a non-negative half-integer spin s and a real number E. The underlying Hilbert space $\mathcal{H}$ is the complex $(2s+1)$-dimensional spin-s UIR of Spin(3) where H acts via the character $\chi(e^{-a_- H}) = e^{-ia_- E}$. In other words, $g = \mathrm{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)$ acts on $\psi \in \mathcal{H}$ as

$$g \cdot \psi = e^{iEa_-} \rho(R) \psi \,, \tag{130}$$

with ρ the spin-s representation of Spin(3). The inner product is any Spin(3)-invariant hermitian inner product on $\mathcal{H}$, which is unique up to scale. We label these representations $\mathbb{I}(s,E)$, with $E \in \mathbb{R}$ and $2s \in \{0,1,2,\dots\}$. They are the galilean analogue of the similarly labelled UIRs of the Carroll group in [10].

11.4.2 UIRs of class $\mathbb{II}(s,m,E)$ associated to orbits of types #1 and 2

These are the massive UIRs. They are induced from the UIRs V_s of Spin(3), which are labelled by their non-negative half-integer spin s and the underlying Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, V_s)$ are the square-integrable functions $\mathbb{R}^3 \to V_s$ relative to the standard euclidean measure on $\mathbb{R}^3$.

We take $\sigma(\mathbf{p}) = e^{-\frac{1}{m}\mathbf{p}\cdot\mathbf{B}}$ as coset representative and define $\psi(\mathbf{p}) = F(\sigma(\mathbf{p}))$ with $F : G \to V_s$ a Mackey function equivariant under $K_\tau \ltimes T$. The action of $g = \mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)$ on $\psi \in \mathcal{H}$ can be worked out as in the case of massive Carroll UIRs in [10] and one finds that

$$(g \cdot \psi)(\mathbf{p}) = e^{i(ma_+ + Ea_- + \mathbf{a}\cdot\mathbf{p})}\rho(R)\psi(R^{-1}(\mathbf{p} + m\mathbf{v}))\,, \tag{131}$$

which is unitary relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{R}^3} d^3p\, \langle \psi_1(\mathbf{p}), \psi_2(\mathbf{p})\rangle_{V_s}\,, \tag{132}$$

with $\langle -, -\rangle_{V_s}$ any Spin(3)-invariant hermitian inner product on V_s. These UIRs are labelled by $m \neq 0$, $E \in \mathbb{R}$ and s with $2s \in \{0, 1, 2, \dots\}$ and denoted $\mathbb{II}(s, m, E)$ and are the galilean analogue of the similarly-labelled massive Carroll UIRs in [10].

11.4.3 UIRs of class $\mathbb{III}(n, k, E)$ associated to orbits of type #5

Since the K-orbit is point-like, the induced representation shares the underlying Hilbert space with the inducing representation: $L^2(S^2, \mathscr{O}(-n))$ as in the Carroll UIRs of class $\mathbb{III}'(n, k)$ in [10]. We can read off the results from the Carroll case and we find that $g = \mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)$ acts on $\psi \in \mathcal{H}$, which we describe a complex-valued smooth function on the complex plane $\psi(z)$ with z a stereographic coordinate for the sphere, via

$$(g \cdot \psi)(z) = e^{i(Ea_- + \mathbf{v}\cdot\kappa(z))}\left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n}\psi\left(\frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right), \tag{133}$$

which is unitary under the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{C}} \frac{2i dz \wedge d\bar{z}}{(1 + |z|^2)^2}\overline{\psi_1(z)}\psi_2(z)\,. \tag{134}$$

Here $\kappa(z) = (\kappa_1(z), \kappa_2(z), \kappa_3(z))$ with

$$\kappa_1(z) = \frac{2k\,\mathrm{Re}(z)}{1 + |z|^2}\,, \qquad \kappa_2(z) = \frac{2k\,\mathrm{Im}(z)}{1 + |z|^2}\,, \qquad \text{and} \qquad \kappa_3(z) = \frac{(|z|^2 - 1)k}{1 + |z|^2}\,, \tag{135}$$

with $k = \|\mathbf{k}\|$, and where $R \in \mathrm{Spin}(3)$ is given under the isomorphism $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ by

$$R = \begin{pmatrix} \eta & \xi \\ -\bar{\xi} & \bar{\eta} \end{pmatrix} \in SU(2)\,. \tag{136}$$

These UIRs are labelled by $n \in \mathbb{Z}$, $k > 0$ and $E \in \mathbb{R}$ and denoted $\mathbb{III}(n, k, E)$. They are analogous to the Carroll UIRs of class $\mathbb{III}'(n, k)$ in [10].

11.4.4 UIRs of class $\mathbb{IV}(n, p)$ associated to orbits of type #6

These UIRs are the analogue of the Carroll UIRs of class $\mathbb{III}(n, p)$ in [10], with one main difference. Here they are carried by square-integrable sections of a line bundle over the cylinder $\mathbb{R} \times S^2$ and not over the sphere as in the Carroll case. Nevertheless we can re-use many of the calculations in [10]. Here $\tau = (0, 0, \mathbf{p})$ where $\mathbf{p} = (0, 0, p)$ with $p = \|\mathbf{p}\| > 0$. The stabiliser subgroup is $K_\tau \cong \mathrm{Spin}(2) \ltimes \mathbb{R}^2$ or, more invariantly, $\mathrm{Spin}(\mathbf{p}^\perp) \ltimes \mathbf{p}^\perp$ and the orbit is given by

$$\mathcal{O}_\tau = \left\{(0, s, \mathbf{p}) \mid s \in \mathbb{R}, \mathbf{p} \in S^2_p\right\} \cong \mathbb{R} \times S^2_p\,. \tag{137}$$

We define a coset representative $\sigma : \mathcal{O}_\tau \to K$ so that $\sigma(s,z)\cdot(0,\mathbf{p}) = (s,\pi(z))$, where z is a stereographic coordinate on S^2_p; that is,

$$\pi(z) = \frac{p}{1+|z|^2}\left(2\operatorname{Re} z, 2\operatorname{Im} z, |z|^2-1\right), \tag{138}$$

which lies in $S^2_p \subset \mathbb{R}^3$. A possible choice for $\sigma(s,z)$ is given by

$$\sigma(s,z) = e^{s/p^2\pi(z)\cdot\mathbf{B}}S(z), \qquad \text{where} \qquad S(z) = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} z & -1 \\ 1 & \bar{z}\end{pmatrix}. \tag{139}$$

Notice that we could also write it as

$$\sigma(s,z) = S(z)e^{s/p^2\mathbf{p}\cdot\mathbf{B}}, \tag{140}$$

using that $\pi(z) = S(z)\cdot\mathbf{p}$. Let $\chi : T \to \mathrm{U}(1)$ be the character associated to $\tau = (0,0,\mathbf{p})$ and let $\mathbb{C}_n$ denote the UIR of K_τ where Spin(2) acts with weight n and the translations in $\mathbf{p}^\perp$ act trivially. We make $\mathbb{C}_n$ into a UIR of $K_\tau \ltimes T$ with T acting via χ. Let $F : G \to \mathbb{C}_n$ be a $(K_\tau \ltimes T)$-equivariant Mackey function. We define $\psi(s,z) = F(\sigma(s,z))$ and we define the action of $g \in G$ on ψ via $(g\cdot\psi)(s,z) = F(g^{-1}\sigma(s,z))$, which we re-express in terms of ψ using equivariance.

Write $g = \mathrm{g}(a_+,a_-,\mathbf{a},\mathbf{v},R) = e^{a_+M-a_-H+\mathbf{a}\cdot\mathbf{P}}e^{\mathbf{v}\cdot\mathbf{B}}R$, with $R \in \mathrm{SU}(2)$ given as in equation (136), and let us calculate

$$\begin{aligned}
g^{-1}\sigma(s,z) &= R^{-1}e^{-\mathbf{v}\cdot\mathbf{B}}e^{-a_+M+a_-H-\mathbf{a}\cdot\mathbf{P}}S(z)e^{s/p^2\mathbf{p}\cdot\mathbf{B}} \\
&= R^{-1}S(z)\underbrace{S(z)^{-1}e^{-\mathbf{v}\cdot\mathbf{B}}e^{-a_+M+a_-H-\mathbf{a}\cdot\mathbf{P}}S(z)}_{e^{-S(z)^{-1}\mathbf{v}\cdot\mathbf{B}}e^{-a_+M+a_-H-S(z)^{-1}\mathbf{a}\cdot\mathbf{P}}}e^{s/p^2\mathbf{p}\cdot\mathbf{B}} \\
&= R^{-1}S(z)e^{-(S(z)^{-1}\mathbf{v}-s/p^2\mathbf{p})\cdot\mathbf{B}}\underbrace{e^{-s/p^2\mathbf{p}\cdot\mathbf{B}}e^{-a_+M+a_-H-S(z)^{-1}\mathbf{a}\cdot\mathbf{P}}e^{s/p^2\mathbf{p}\cdot\mathbf{B}}}_{e^{-(a_+-s/p^2S(z)^{-1}\mathbf{a}\cdot\mathbf{p}-\frac{1}{2}a_-s^2/p^2)M+a_-H-(S(z)^{-1}\mathbf{a}+s/p^2a_-\mathbf{p})\cdot\mathbf{P}}} \\
&= R^{-1}S(z)e^{-(S(z)^{-1}\mathbf{v}-s/p^2\mathbf{p})\cdot\mathbf{B}}e^{-(a_+-s/p^2S(z)^{-1}\mathbf{a}\cdot\mathbf{p}-\frac{1}{2}a_-s^2/p^2)M+a_-H-(S(z)^{-1}\mathbf{a}+s/p^2a_-\mathbf{p})\cdot\mathbf{P}}.
\end{aligned}$$

Therefore, using that $S(z)^{-1}\mathbf{a}\cdot\mathbf{p} = \mathbf{a}\cdot S(z)\mathbf{p} = \mathbf{a}\cdot\pi(z)$, we have that

$$\begin{aligned}
F(g^{-1}\sigma(s,z)) &= F(R^{-1}S(z)e^{-(S(z)^{-1}\mathbf{v}-s/p^2\mathbf{p})\cdot\mathbf{B}}e^{-(a_+-s/p^2S(z)^{-1}\mathbf{a}\cdot\mathbf{p}-\frac{1}{2}a_-s^2/p^2)M+a_-H-(S(z)^{-1}\mathbf{a}+s/p^2a_-\mathbf{p})\cdot\mathbf{P}}) \\
&= e^{i(\mathbf{a}\cdot\pi(z)+sa_-)}F(R^{-1}S(z)e^{-(S(z)^{-1}\mathbf{v}-s/p^2\mathbf{p})\cdot\mathbf{B}}),
\end{aligned}$$

where we used the equivariance of F. Using equation (3.42) in [10] we have that for $R \in \mathrm{SU}(2)$ given by equation (136),

$$R^{-1}S(z) = \underbrace{S\left(\frac{\bar{\eta}-\xi}{\eta+\bar{\xi}z}\right)}_{S(R^{-1}z)}\underbrace{\frac{\eta+\bar{\xi}z}{|\eta+\bar{\xi}z|}}_{\lambda(S^{-1},z)\in\mathrm{U}(1)}, \tag{141}$$

where we identify U(1) with the diagonal matrices in SU(2), since those matrices stabilise $\mathbf{p}$. Therefore,

$$\begin{aligned}
F(R^{-1}S(z)e^{-(S(z)^{-1}\mathbf{v}-s/p^2\mathbf{p})\cdot\mathbf{B}}) &= F(S(R^{-1}z)\lambda(S^{-1},z)e^{-(S(z)^{-1}\mathbf{v}-s/p^2\mathbf{p})\cdot\mathbf{B}}) \\
&= F(S(R^{-1}z)e^{-((S(z)^{-1}\mathbf{v})_\parallel-s/p^2\mathbf{p})\cdot\mathbf{B}}\underbrace{\lambda(S^{-1},z)e^{-(S(z)^{-1}\mathbf{v})_\perp\cdot\mathbf{B}}}_{\in K_\tau}),
\end{aligned}$$

where we have used that U(1) preserves $\mathbf{p}$ and have broken up $S(z)^{-1}\mathbf{v}$ into a component along $\mathbf{p}$ (and hence preserved by U(1)) and a component perpendicular to $\mathbf{p}$. Using equivariance again, and the fact that $(S(z)^{-1}\mathbf{v})_{||} = S(z)^{-1}\mathbf{v}\cdot\mathbf{p}/p^2\mathbf{p} = \mathbf{v}\cdot\pi(z)\mathbf{p}/p^2$, we find

$$F(g^{-1}\sigma(s,z)) = e^{i(\mathbf{a}\cdot\pi(z)+sa_-)}\lambda(S^{-1},z)^{-n}F(\sigma(s-\mathbf{v}\cdot\pi(z),R^{-1}z)). \tag{142}$$

In summary, the action of g on $\psi(s,z)$ is given by

$$(g\cdot\psi)(s,z) = e^{-i(\mathbf{a}\cdot\pi(z)+sa_-)}\left(\frac{\eta+\overline{\xi}z}{|\eta+\overline{\xi}z|}\right)^{-n}\psi\left(s-\mathbf{v}\cdot\pi(z),\frac{\overline{\eta}z-\xi}{\eta+\overline{\xi}z}\right). \tag{143}$$

This representation is unitary relative to the inner product

$$(\psi_1,\psi_2) = \int_{\mathbb{R}\times\mathbb{C}}\frac{2ids\wedge dz\wedge d\overline{z}}{(1+|z|^2)^2}\overline{\psi_1(s,z)}\psi_2(s,z). \tag{144}$$

More invariantly, and as shown in [10, Section 3.3.1] for the case of Carroll UIRs, one can describe the Hilbert space $\mathcal{H}$ as the square-integrable sections of the line bundle $\widetilde{\mathcal{O}}(-n)$ over the cylinder $\mathbb{R}\times S^2_p$ obtain by pulling back the line bundle $\mathcal{O}(-n)$ over S^2_p via the cartesian projection $\mathbb{R}\times S^2_p \to S^2_p$. We denote these UIRs by $\mathbb{IV}(n,p)$ where $n\in\mathbb{Z}$ and $p>0$.

11.4.5 UIRs of class $\mathbb{V}_\pm(p,k^\perp)$ associated to orbits of type #7

These UIRs are the analogue of the Carroll UIRs of class $\mathbb{V}_\pm(p,k,\theta)$ in [10]. Their description is as sections of an infinite-dimensional Hilbert bundle over the K-orbit, but following similar steps as those in [10, Section 3.4.3], they can be seen to admit a simpler description.

Let N be the nilpotent subgroup of the Bargmann group generated by $\mathbf{B},\mathbf{P},M$. If $m=0$, N acts like an abelian group and its UIRs are therefore one-dimensional. We will consider the one-dimensional UIR $\mathbb{C}_{p,k^\perp}$ with character $\chi : N\to \mathrm{U}(1)$ given by

$$\chi(e^{a_+M+\mathbf{a}\cdot\mathbf{P}+\mathbf{v}\cdot\mathbf{B}}) = e^i(\mathbf{a}\cdot\mathbf{p}+\mathbf{v}\cdot\mathbf{k}), \tag{145}$$
$$\text{with}\qquad \mathbf{p}=(0,0,p),\qquad\text{and}\qquad \mathbf{k}=(0,k^\perp,0),\qquad\text{with both}\qquad p,k^\perp>0.$$

The homogeneous space G/N is diffeomorphic to $\mathbb{R}\times S^3$ and χ defines a trivial homogeneous line bundle L_χ over G/N, whose sections can be identified with functions $\mathbb{R}\times S^3\to\mathbb{C}_{p,k^\perp}$.

Let us choose a coset representative $\sigma:\mathbb{R}\times S^3\to G$ for G/N defined by $\sigma(s,S)=e^{sH}S$, where $S\in\mathrm{SU}(2)$ and where we have identified S^3 with SU(2). The UIR of the Bargmann group is carried by $\mathcal{H}=L^2(\mathbb{R}\times S^3,\mathbb{C}_\chi)$ relative to the inner product

$$(\psi_1,\psi_2) = \int_{\mathbb{R}\times S^3} dsd\mu(S)\overline{\psi_1(s,S)}\psi_2(s,S), \tag{146}$$

where $d\mu$ is a bi-invariant Haar measure on SU(2).

Let $g = \mathrm{g}(a_+,a_-,\mathbf{a},\mathbf{v},R)$ and let us calculate its action on $\psi\in\mathcal{H}$. As usual $\psi(s,S)=F(\sigma(s,S))$, with $F:G\to\mathbb{C}_\chi$ a $(K_\tau\ltimes T)$-equivariant Mackey function. Then

$$(g\cdot\psi)(s,S) = F(g^{-1}\sigma(s,S)), \tag{147}$$

which we must rewrite in terms of ψ using equivariance. We calculate

$$\begin{aligned} g^{-1}\sigma(s,S) &= R^{-1}e^{-\mathbf{v}\cdot\mathbf{B}}e^{-a_+M+a_-H-\mathbf{a}\cdot\mathbf{P}}e^{sH}S \\ &= R^{-1}e^{-\mathbf{v}\cdot\mathbf{B}}e^{(a_-+s)H}e^{-a_+M-\mathbf{a}\cdot\mathbf{P}}S \\ &= R^{-1}e^{(a_-+s)H}\underbrace{e^{-(a_-+s)H}e^{-\mathbf{v}\cdot\mathbf{B}}e^{(a_-+s)H}}_{e^{-\mathbf{v}\cdot\mathbf{B}-(a_-+s)\mathbf{v}\cdot\mathbf{P}}}e^{-a_+M-\mathbf{a}\cdot\mathbf{P}}S \\ &= e^{(a_-+s)H}R^{-1}Se^{-S^{-1}\mathbf{v}\cdot\mathbf{B}-(a_-+s)S^{-1}\mathbf{v}\cdot\mathbf{P}}e^{-a_+M-S^{-1}\mathbf{a}\cdot\mathbf{P}}. \end{aligned}$$

Table 7: Comparison with Carroll UIRs. This table provides a sort of dictionary between the UIRs of the Carroll group as determined in [10] and the UIRs of the Bargmann group we have just described. The correspondence is not perfect: there are labels in the Bargmann case which simply do not exist in the Carroll case and there are Carroll UIRs which have not counterpart among the Bargmann UIRs; namely, the (anti)parallel helicity representations of Carroll.

Carroll UIR in [10]	UIR in Table 6
$\mathbb{I}(s)$	$\mathbb{I}_+(s,E)$
$\mathbb{II}(s,m)$	$\mathbb{II}(s,m,E)$
$\mathbb{III}'(n,k)$	$\mathbb{III}(n,k,E)$
$\mathbb{III}(n,p)$	$\mathbb{IV}(n,p)$
$\mathbb{IV}_\pm(n,p,k)$	
$\mathbb{V}_\pm(p,k,\theta)$	$\mathbb{V}_\pm(p,k\sin\theta)$

Therefore, using equivariance, we see that

$$(g\cdot\psi)(s,S)=e^{i(\mathbf{v}\cdot S\mathbf{k}+(\mathbf{a}+(a_-+s)\mathbf{v})\cdot S\mathbf{p})}\psi(s+a_-,R^{-1}S), \tag{148}$$

where we have used that, say, $S^{-1}\mathbf{v}\cdot\mathbf{k}=\mathbf{v}\cdot S\mathbf{k}$. As in the case of the similar UIRs of the Carroll group, these representations are not irreducible, because of the action of the centre of SU(2). We define idempotents $\Pi_\pm:\mathcal{H}\to\mathcal{H}$ by

$$(\Pi_\pm\psi)(s,S)=\tfrac{1}{2}\left(\psi(s,S)\pm\psi(s,-S)\right). \tag{149}$$

Then $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$, with $\mathcal{H}_\pm$ the image of $\Pi_\pm$, is an orthogonal decomposition into UIRs of the Bargmann group. These UIRs are characterised by $p,k^\perp>0$ and the action of the centre of SU(2), which is a sign. We will denote them by $\mathbb{V}_\pm(p,k^\perp)$ by analogy with the Carroll UIRs in [10].

11.5 Comparison with Carroll UIRs

As we have been mentioning during the description of the Bargmann UIRs in the previous section, there are certain similarities between the Bargmann and Carroll UIRs that are worth highlighting. Table 7 summarises these similarities.

11.6 Comparison with prior classifications

The earliest classification of Galilei UIRs is that of Inönü–Wigner [15] who restricted themselves to honest (not ray) UIRs of the Galilei group, despite being aware (citing a private communication with none other than Bargmann himself!) of the need to consider projective representations. Moreover they classify UIRs of the connected component of the Galilei group, but not of its simply-connected double cover. Therefore their list should be compared with those UIRs with $m=0$ and with integer spin and helicity.

Restricting to the Galilei group has the technical advantage that the maximal abelian subgroup is now of larger dimension than in the Bargmann case. Letting G' denote the Galilei group, we can write $G'=K'\ltimes T'$, where $K'\cong\mathbb{R}\times\mathrm{SO}(3)$ is the connected subgroup generated by J_i,H, whereas $T'\cong\mathbb{R}^6$ is the abelian subgroup generated by B_i,P_i. The UIRs of T' are one-dimensional and defined by characters $\chi:T'\to\mathrm{U}(1)$ with

$$\chi(e^{\mathbf{v}\cdot\mathbf{B}+\mathbf{a}\cdot\mathbf{P}})=e^{i(\mathbf{v}\cdot\mathbf{k}+\mathbf{a}\cdot\mathbf{p})}, \tag{150}$$

Table 8: Comparison with Inönü–Wigner [15]. This table provides a dictionary between the representations of the Galilei group classified in [15] and the ones in our Table 6. The notation in the first column is the one adopted in [15]. In particular, their m is not the mass, but an integer helicity. Besides the massive representations, also missing are any Bargmann UIRs where the centre of SU(2) acts nontrivially. We reiterate that the missing UIRs were consciously and explicitly excluded in [15].

UIR in [15]	UIR in Table 6
$\mathbb{I}(p,s)$	$\mathbb{V}_+(p,s/p)$
$\mathbb{II}(m,p)$	$\mathbb{IV}(2m,p)$
$\mathbb{III}(m,k,e)$	$\mathbb{III}(2m,k,e)$
$\mathbb{IV}(\ell,e)$	$\mathbb{I}(2\ell,e)$

for some $(\mathbf{k},\mathbf{p}) \in \mathbb{R}^6$. The K'-action on the characters is such that

$$Re^{sH} \cdot (\mathbf{k},\mathbf{p}) = (R\mathbf{k} + sR\mathbf{p}, R\mathbf{p})\,. \tag{151}$$

There are four types of orbits depending on $\tau' = (\mathbf{k},\mathbf{p})$, in increasing dimension of the stabiliser $K'_{\tau'} \subset K'$ with the labels as in [15]:

(I) $\tau' = (\mathbf{k},\mathbf{p})$, with $\mathbf{k}\times\mathbf{p} \neq \mathbf{0}$. Let $p = \|\mathbf{p}\| > 0$ and $h = \|\mathbf{p}\times\mathbf{k}\| > 0$, which are the two invariants of the orbit. The stabiliser is trivial and hence the orbit is $\mathbb{R}\times \mathrm{SO}(3)$.

(II) $\tau' = (\mathbf{k},\mathbf{p})$, with $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{k}\times\mathbf{p} = \mathbf{0}$. Letting $p = \|\mathbf{p}\| > 0$, the orbit is now $\mathbb{R}\times S^2_p$ and the stabiliser is $K'_{\tau'} = \{R \mid R\mathbf{p} = \mathbf{p}\} \cong \mathrm{SO}(2)$.

(III) $\tau' = (\mathbf{k},\mathbf{0})$, with $\mathbf{k} \neq \mathbf{0}$. The orbit is the sphere of radius $k = \|\mathbf{k}\| > 0$. The stabiliser is $K'_{\tau'} = \left\{e^{aH}R \mid a\in\mathbb{R},\, R\mathbf{k} = \mathbf{k}\right\} \cong \mathbb{R}\times\mathrm{SO}(2)$.

(IV) $\tau' = (\mathbf{0},\mathbf{0})$. This is a point-like orbit with stabiliser all of K'.

The inducing UIRs of the stabilisers are easy to determine in all cases:

(I) The stabiliser is trivial, so there is the only UIR is the trivial one-dimensional representation $\mathbb{C}$.

(II) The stabiliser is SO(2) whose UIRs are one-dimensional and denoted $\mathbb{C}_n$ with $n\in\mathbb{Z}$, which we may identify with the helicity.

(III) The stabiliser is $\mathbb{R}\times\mathrm{SO}(2)$, whose UIRs are one-dimensional $\mathbb{C}_n\otimes\mathbb{C}_e$, with $n\in\mathbb{Z}$ and $e\in\mathbb{R}$.

(IV) The stabiliser is $\mathbb{R}\times\mathrm{SO}(3)$, whose UIRs are $V_\ell\otimes\mathbb{C}_e$, with the spin $\ell\in\mathbb{Z}$ and $e\in\mathbb{R}$.

It is then easy to compare their classification with ours and we give the dictionary in Table 8.

The UIRs classified in Lévy-Leblond [39] and Brennich [16], although expressed in the language of ray representations of the Galilei group instead of representations of the Bargmann group, are induced from characters of the abelian subgroup T generated by M, H, P_i together with a UIR of the stabiliser of the character, as we have done. This allows for an easier comparison than in the case of Inönü–Wigner. We give the dictionary in Tables 9 and 10.

Table 9: Comparison with Brennich [16]. This table provides a dictionary between the unitary irreducible ray representations of the Galilei group classified in [16] and the UIRs of the Bargmann group in Table 6. The labels $\mathbf{p}$ and e in the UIRs of classes I and III, respectively, in [16] are spurious, since they are not actually invariants. They correspond to a choice of inducing character. Also, the label s in Brennich's $\mathrm{III}(s,k,e)$ takes two possible values: $s=0$, corresponding to our V_+, and $s=\frac{1}{2}$, corresponding to our V_-. Other than those comments, there is a bijective correspondence between the UIRs in [16] and the ones in Table 6. Of course, Brennich also discusses the UIRs of the full Bargmann group, including inversion and time-reversal.

UIR in [16]	UIR in Table 6
$\mathrm{I}(s,m,e,\mathbf{p})$	$\mathrm{II}(s,m,e)$
$\mathrm{II}(s,p)$	$\mathrm{IV}(2s,p)$
$\mathrm{III}(s,p,k,e)$	$\mathrm{V}_{\pm(s)}(p,k)$
$\mathrm{IV}(s,e)$	$\mathrm{I}(s,e)$
$\mathrm{V}(s,k,e)$	$\mathrm{III}(2s,k,e)$

Table 10: Comparison with Lévy-Leblond [17]. This table provides a dictionary between the unitary irreducible ray representations of the Galilei group classified in [17] and the UIRs of the Bargmann group in Table 6. The UIR $\mathrm{V}_-(p,k^\perp)$ in Table 6 is missing from the list in [17], but otherwise we are in agreement. This can be explained by the fact that the euclidean group from which one induces the representation is actually the double cover of the euclidean group in [17] and hence the "little group" mentioned after equation (4.14) there is not actually the identity but the order-2 Galois group of the double cover.

UIR in [17]	UIR in Table 6
$\mathrm{I}(p,v)$	$\mathrm{V}_+(p,v)$
$\mathrm{II}(p,\sigma)$	$\mathrm{IV}(\sigma,p)$
$\mathrm{III}(E,k,\xi)$	$\mathrm{III}(2\xi,k,E)$
$\mathrm{IV}(E,\ell)$	$\mathrm{I}(\ell,E)$
$m(U,s)$	$\mathrm{II}(s,m,U)$

12 Galilean field-theoretical realisations

In this section we will realise some of the UIRs of the Bargmann group in terms of fields in Galilei spacetime. This follows the method explained in [10, Appendix A].

Of the UIRs of the Bargmann group, there are some which admit a description in terms of (finite-component) fields on Galilei spacetime. Galilei spacetime is a homogeneous space of the Bargmann group diffeomorphic to the space of cosets G/G_0, where $G_0 = K \times Z$ with Z the central subgroup generated by M. As explained, for example in [10, Appendix A], the first step in obtaining such a description is to embed the inducing representation of K_τ into a (finite-dimensional) representation of G_0. This is possible for all inducing representations except those associated with the coadjoint orbits of classes #5,7. Those representations associated to coadjoint orbits of classes #3,4 are finite-dimensional, so presumably they do not admit a nontrivial description as fields on Galilei spacetime. Thus we remain with the UIRs of classes $\mathrm{II}(s,m,E)$ associated with coadjoint orbits of types #1 and 2 and $\mathrm{IV}(n,p)$ associated with coadjoint orbits of type # 6.

12.1 Massive galilean fields

In Section 11.4.2 we described the momentum-space description of the UIRs of type $\mathbb{II}(s,m,E)$ and in this section we will realise these representations as fields in Galilei spacetime.

Galilei spacetime is diffeomorphic to the coset space G/G_0 with $G_0 = K \times Z$. Fields on Galilei spacetime are sections of homogeneous vector bundles associated to representations of G_0, so the first order of business is to *choose* a finite-dimensional representation of G_0 which embeds the inducing representation V of K_τ. Extending the representation V_s from K_τ to K is simply a matter of letting the boosts act trivially. We may also extend it to a representation of G_0 via

$$\mathsf{g}(a_+, 0, \mathbf{0}, \mathbf{v}, R) \cdot \psi = e^{ima_+} R \cdot \psi \,. \tag{152}$$

Let V denote this representation of G_0, sharing the same vector space with the representation V_s of K_τ. The action of G_0 on $\mathcal{O}_\tau$ is such that the central subgroup Z acts trivially and K acts via euclidean transformations, as seen above.

Next we "Fourier transform". We define $\widehat{F} : G \to V$ by[4]

$$\widehat{F}(g) := \int_{\mathbb{R}^3} d^3p \ \sigma(\mathbf{p}) \cdot F(g\sigma(\mathbf{p})) = \int_{\mathbb{R}^3} d^3p \ F(g\sigma(\mathbf{p})) \,, \tag{153}$$

since the boost $\sigma(\mathbf{p}) = e^{-\frac{1}{m}\mathbf{p}\cdot\mathbf{B}}$ acts trivially on V. As shown, for example, in [10, Appendix A], $\widehat{F}$ is G_0-equivariant and hence it defines a section of the homogeneous vector bundle over Galilei spacetime associated with the representation V.

Let $\zeta : G/G_0 \to G$ be a coset representative for Galilei spacetime, where $\zeta(t,\mathbf{x}) = \exp(tH + \mathbf{x}\cdot\mathbf{P})$ and define $\phi : G/G_0 \to V$ by

$$\phi(t,\mathbf{x}) = \widehat{F}(\zeta(t,\mathbf{x})) = \int_{\mathbb{R}^3} d^3p \ F(\zeta(t,\mathbf{x})\sigma(\mathbf{p})) \,. \tag{154}$$

We now calculate

$$\zeta(t,\mathbf{x})\sigma(\mathbf{p}) = \sigma(\mathbf{p}) \underbrace{\sigma(\mathbf{p})^{-1}\zeta(t,\mathbf{x})\sigma(\mathbf{p})}_{\in T} \,, \tag{155}$$

where, after a quick calculation, we find that

$$\sigma(\mathbf{p})^{-1}\zeta(t,\mathbf{x})\sigma(\mathbf{p}) = \zeta(t, \mathbf{x} + \tfrac{1}{m}\mathbf{p}t) e^{\frac{1}{m}(\mathbf{x}\cdot\mathbf{p} + \frac{1}{2m}\|\mathbf{p}\|^2)M} \,. \tag{156}$$

By equivariance,

$$\begin{aligned} F(\zeta(t,\mathbf{x})\sigma(\mathbf{p})) &= F(\sigma(\mathbf{p})\zeta(t, \mathbf{x} + \tfrac{1}{m}\mathbf{p}t) e^{\frac{1}{m}(\mathbf{x}\cdot\mathbf{p} + \frac{1}{2m}\|\mathbf{p}\|^2)M}) \\ &= e^{-i(\mathbf{x}\cdot\mathbf{p} + (Et + \frac{1}{2m}\|\mathbf{p}\|^2))} F(\sigma(\mathbf{p})) \,, \end{aligned} \tag{157}$$

and hence, integrating,

$$\phi(t,\mathbf{x}) = e^{-iEt} \int_{\mathbb{R}^3} d^3p \ e^{-i(\mathbf{x}\cdot\mathbf{p} + \frac{1}{2m}\|\mathbf{p}\|^2)} \psi(\mathbf{p}) \,, \tag{158}$$

which is up to the t-dependent phase in front of the integral, essentially the Fourier transform of the rescaled function $e^{-\frac{i}{2m}\|\mathbf{p}\|^2}\psi(\mathbf{p})$.

As shown in [10, Appendix A], the action of the Bargmann group on such a field ϕ is given by

$$(g \cdot \phi)(t,\mathbf{x}) = h^{-1} \cdot \phi(t', \mathbf{x}') \,, \tag{159}$$

[4] We tacitly restrict to Mackey functions F for which this integral converges.

where t', $\mathbf{x}'$ and $h \in G_0$ are defined by

$$g^{-1}\zeta(t,\mathbf{x}) = \zeta(t',\mathbf{x}')h\,. \tag{160}$$

Letting $g = \mathsf{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)$, we calculate

$$\begin{aligned} t' &= t - a_-\,, \\ \mathbf{x}' &= R^{-1}(\mathbf{x} - \mathbf{a} + (t-a_-)\mathbf{v})\,, \\ h &= \mathsf{g}(-a_+ - \mathbf{v}\cdot(\mathbf{x}-\mathbf{a}) - \tfrac{1}{2}(t-a_-)\|\mathbf{v}\|^2, 0, \mathbf{0}, -R^{-1}\mathbf{v}, R^{-1})\,, \end{aligned} \tag{161}$$

so that

$$h^{-1} = \mathsf{g}(a_+ + \mathbf{v}\cdot(\mathbf{x}-\mathbf{a}) + \tfrac{1}{2}(t-a_-)\|\mathbf{v}\|^2, 0, \mathbf{0}, \mathbf{v}, R)\,, \tag{162}$$

and hence

$$h^{-1}\cdot\phi(t',\mathbf{x}') = e^{im(a_+ + \mathbf{v}\cdot(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(t-a_-)\|\mathbf{v}\|^2)} R\cdot\phi(t-a_-, R^{-1}(\mathbf{x}-\mathbf{a}+(t-a_-)\mathbf{v}))\,. \tag{163}$$

In summary,

$$(g\cdot\phi)(t,\mathbf{x}) = e^{im(a_+ + \mathbf{v}\cdot(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(t-a_-)\|\mathbf{v}\|^2)} R\cdot\phi(t-a_-, R^{-1}(\mathbf{x}-\mathbf{a}+(t-a_-)\mathbf{v}))\,. \tag{164}$$

Breaking this transformation into its different components, we find that

- under translations,
$$(g\cdot\phi)(t,\mathbf{x}) = \phi(t-a_-, \mathbf{x}-\mathbf{a})\,; \tag{165}$$
- under rotations,
$$(g\cdot\phi)(t,\mathbf{x}) = R\cdot\phi(t, R^{-1}\mathbf{x})\,; \tag{166}$$
- under boosts,
$$(g\cdot\phi)(t,\mathbf{x}) = e^{im(\mathbf{v}\cdot\mathbf{x} + \frac{1}{2}t\|\mathbf{v}\|^2)}\phi(t, \mathbf{x}+t\mathbf{v})\,; \tag{167}$$
- and under the action of the Bargmann central element it transforms with a constant phase:
$$(g\cdot\phi)(t,\mathbf{x}) = e^{ima_+}\phi(t,\mathbf{x})\,. \tag{168}$$

12.2 Massless galilean fields

Now we will describe the massless UIRs of type $\mathbb{IV}(n,p)$ as galilean fields. These are honest (as opposed to projective) UIRs of the Galilei group. As described in Section 11.4.4, they are carried by square-integrable sections of a complex line bundle over $\mathbb{R}\times S^2$ obtained by pulling back the bundle $\mathscr{O}(-n)$ over S^2. They can be described locally by complex-valued functions $\psi(s,z)$ on $\mathbb{R}\times\mathbb{C}$ with z a stereographic coordinate on S^2. The treatment here is very similar to that of [10, Section 4.3] to which we will refer for the pertinent calculations. The inducing representation is a complex one-dimensional representation of $K_\tau \ltimes T$, with T acting via the unitary character associated to $\tau = (0,0,\mathbf{p})$ and $K_\tau \cong \mathrm{Spin}(\mathbf{p}^\perp)\ltimes\mathbf{p}^\perp$ acting in such a way that $\mathbf{p}^\perp$ acts trivially and $\mathrm{Spin}(\mathbf{p}^\perp)$ acts with weight $n\in\mathbb{Z}$. To describe the UIR as fields on Galilei spacetime, we need to embed this one-dimensional representation into an irreducible (without loss of generality) representation of K. We demand that the boosts act trivially, but must embed the weigh-n representation of Spin(2) into an irreducible representation of Spin(3). As was done in [10] for the Carroll particles, we may choose any complex irreducible representation of Spin(3) of spin $j \geq |n/2|$. The smallest such representation is that of spin $j = |n/2|$ into which the inducing representation embeds as the subspace with highest (if $n \geq 0$) or lowest (if $n \leq 0$) weight. Let us denote by V this complex $(|n|+1)$-dimensional representation with T acting via the character

$$\chi_\tau(e^{a_+M-a_-H+\mathbf{a}\cdot\mathbf{P}}) = e^{-i\mathbf{a}\cdot\mathbf{p}} . \tag{169}$$

Let $F : G \to V$ be a $K_\tau \ltimes T$-equivariant Mackey function and let $\zeta : G/G_0 \to G$ be the coset representative $\zeta(t,\mathbf{x}) = e^{tH+\mathbf{x}\cdot\mathbf{P}}$. Then the galilean field is given by

$$\phi(t,\mathbf{x}) = \widehat{F}(\zeta(t,\mathbf{x})) , \tag{170}$$

where $\widehat{F}$ is the group-theoretical Fourier transform

$$\widehat{F}(g) = \int_{\mathbb{R}\times\mathbb{C}} \frac{2i ds \wedge dz \wedge d\bar{z}}{(1+|z|^2)^2} \sigma(s,z) \cdot F(g\sigma(s,z)) , \tag{171}$$

where $\sigma(s,z)$ is given by equation (140). We calculate

$$\zeta(t,\mathbf{x})\sigma(s,z) = e^{tH+\mathbf{x}\cdot\mathbf{P}} S(z) e^{s/p^2 \mathbf{p}\cdot\mathbf{B}} = \sigma(s,z) e^{tH+(S(z)^{-1}\mathbf{x}-st/p^2\mathbf{p})\cdot\mathbf{P}} , \tag{172}$$

where we have ignored terms multiplying M since they act trivially on massless representations and we can essentially pretend that we are dealing with the Galilei group, where $\mathbf{B}$ and $\mathbf{P}$ commute. Using equivariance of F and the fact that $\psi(s,z) = F(\sigma(s,z))$ we arrive at

$$\phi(t,\mathbf{x}) = \int_{\mathbb{R}\times\mathbb{C}} \frac{2i ds \wedge dz \wedge d\bar{z}}{(1+|z|^2)^2} e^{i(ts-\mathbf{x}\cdot\pi(z))} \rho(S(z))\psi(s,z) , \tag{173}$$

where $\rho : \mathrm{Spin}(3) \to \mathrm{GL}(V)$ is the representation of Spin(3).

To describe the action of G on such fields, we let $g = \mathrm{g}(a_+, a_-, \mathbf{a}, \mathbf{v}, R)$ and we calculate

$$g^{-1}\zeta(t,\mathbf{x}) = R^{-1} e^{-\mathbf{v}\cdot\mathbf{B}} e^{-a_+M+(a_-+t)H+(\mathbf{x}-\mathbf{a})\cdot\mathbf{P}} \tag{174}$$

$$= \zeta(t+a_-, R^{-1}(\mathbf{x}-\mathbf{a}-(a_-+t)\mathbf{v})) R^{-1} e^{-\mathbf{v}\cdot\mathbf{B}} , \tag{175}$$

where we once again have ignored terms in M in the final calculation. Using equivariance of the Fourier-transformed Mackey function (171), we find that

$$(g\cdot\phi)(t,\mathbf{x}) = \widehat{F}(g^{-1}\zeta(t,\mathbf{x})) \tag{176}$$

$$= R\cdot\phi(t+a_-, R^{-1}(\mathbf{x}-\mathbf{a}-(t+a_-)\mathbf{v})) . \tag{177}$$

Since $\|\pi(z)\|^2 = p^2$, we may insert zero in the form $\|\pi(z)\|^2 - p^2$ in the integrand of equation (173) and using that any $\pi(z)$ in the integrand is the result of differentiating with $i\nabla$, we see that $\phi(t,\mathbf{x})$ obeys the Helmholtz equation

$$(\triangle + p^2)\phi(t,\mathbf{x}) = 0 , \tag{178}$$

with $\triangle$ the laplacian in three-dimensional euclidean space. This is the only equation for the inducing representation with $n = 0$, but for $n \neq 0$, we have additional equations. This is because the field ϕ is V-valued and in order to recover the UIR, we need to project to the inducing one-dimensional representation, which corresponds to the kernel of J_+ (if $n > 0$) or J_- (if $n < 0$), where

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad \text{and} \qquad J_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{179}$$

Of course $J_\pm$ live in the complexification of $\mathfrak{so}(3)$ and we extend the representation complex-linearly. We proceed as in [10, Section 4.3].

Let $n > 0$ for definiteness and consider

$$0 = \int_{\mathbb{R}\times\mathbb{C}} \frac{2ids \wedge dz \wedge d\bar{z}}{(1+|z|^2)^2} e^{i(ts - \mathbf{x}\cdot\pi(z))} \rho(S(z))\rho(J_+)\psi(s,z) \tag{180}$$

$$= \int_{\mathbb{R}\times\mathbb{C}} \frac{2ids \wedge dz \wedge d\bar{z}}{(1+|z|^2)^2} e^{i(ts - \mathbf{x}\cdot\pi(z))} \rho(J_+(z))\rho(S(z))\psi(s,z)\,, \tag{181}$$

where

$$J_+(z) = S(z)J_+S(z)^{-1} = \frac{1}{1+|z|^2}\begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}. \tag{182}$$

This is formally the same expression as in [10, Section 4.3] and we may borrow the results from that paper. For helicity $\pm\frac{1}{2}$ the equations are the massive Dirac equation in three-dimensional euclidean space:

$$(\partial\!\!\!/ \pm ip)\phi = 0\,, \tag{183}$$

where $\partial\!\!\!/ = \gamma^i\partial_i$ with γ^i the representation of $C\ell(0,3)$ given by

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{184}$$

Notice that either of these equations imply the Helmholtz equation (178). Similarly, as in [10, Section 4.3], for helicity ± 1 we obtain the field equation for topologically massive Maxwell theory [40, 41]

$$\pm p\phi_i = \epsilon_{ijk}\partial_j\phi_k\,, \tag{185}$$

which again implies the Helmholtz equation (178).

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A Symmetries of the massive spinless Galilei particle

In this appendix we provide further details concerning the symmetries of the massive spinless Galilei particle in Section 8.2. The starting point of our analysis was the following action

$$L[a_+, t, \mathbf{x}, \mathbf{v}, R(\varphi)] = m\dot{a}_+ - \left(E_0 + \tfrac{1}{2}m\|\mathbf{v}\|^2\right)\dot{t} + m\mathbf{v}\cdot\dot{\mathbf{x}}\,. \tag{A.1}$$

To obtain the global symmetries we restrict our generic symmetries (49) to the representative at hand, i.e., we set $\mathbf{p} = \mathbf{k} = \mathbf{j} = \mathbf{0}$ to obtain

$$\begin{aligned}
&\delta_{c_+}a_+ = c_+\,, && && && m_Q = m\,, && \text{(A.2a)}\\
&\delta_{c_t}t = c_t\,, && && && E_Q = \tfrac{1}{2}m\|\mathbf{v}\|^2 + E_0\,, && \text{(A.2b)}\\
&\delta_{c_x}\mathbf{x} = \mathbf{c}_x\,, && && &\Rightarrow\quad & \mathbf{p}_Q = m\mathbf{v}\,, && \text{(A.2c)}\\
&\delta_{c_v}\mathbf{v} = \mathbf{c}_v\,, && \delta_{c_v}a_+ = -\mathbf{x}\cdot\mathbf{c}_v\,, && \delta_{c_v}\mathbf{x} = t\mathbf{c}_v\,, && & \mathbf{k}_Q = m\mathbf{v}t - m\mathbf{x}\,, && \text{(A.2d)}\\
&\delta_\omega R = \omega R\,, && \delta_\omega\mathbf{x} = \omega\mathbf{x}\,, && \delta_\omega\mathbf{v} = \omega\mathbf{v} && & \mathbf{j}_Q = m\mathbf{x}\times\mathbf{v}\,. && \text{(A.2e)}
\end{aligned}$$

The gauge symmetries are given by the stabiliser of α which for the massive spinless particles is given by $\mathrm{g}(a_+, a_-, \mathbf{0}, \mathbf{0}, R)$. Infinitesimally they are given by the following transformations (taken from (50))

$$\delta_{\lambda_+} a_+ = \lambda_+ \,, \tag{A.3a}$$
$$\delta_{\lambda_t} t = \lambda_t \,, \qquad \delta_{\lambda_t} a_+ = -\tfrac{1}{2}\|\mathbf{v}\|^2 \lambda_t \,, \qquad \delta_{\lambda_t} \mathbf{x} = \mathbf{v}\lambda_t \,, \tag{A.3b}$$
$$\delta_{\lambda_\omega} R = R\lambda_\omega \,. \tag{A.3c}$$

One can explicitly show that they are indeed symmetries (up to boundary terms) of the action (A.1).

The global transformation of a_+ with parameter c_+ are the time-independent part of a gauge transformation. This piece of the action can also be written as $L[p_+, a_+, u] = p_+\dot{a}_+ - u\phi$ where u enforces the constraint $\phi = p_+ - m$. This action has no physical degrees of freedom, but the equations of motion $\dot{a}_+ = u$, $\dot{p}_+ = 0$ and $p_+ = m$, show that there exists a canonical variable p_+ that is constant along the trajectory and equal to m. Since p_+ commutes with the first-class constraint it is an observable.

After the canonical analysis we obtained the following action

$$L_{\text{can}}[t, p_t, \mathbf{x}, \boldsymbol{p}, u] = p_t \dot{t} + \boldsymbol{p} \cdot \dot{\mathbf{x}} - u\left(p_t + \frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right), \tag{A.4}$$

with variation

$$\begin{aligned}\delta L_{\text{can}} = (\dot{t} - u)\delta p_t + \dot{p}_t \delta t + \left(\dot{\mathbf{x}} - \tfrac{1}{m}u\boldsymbol{p}\right) \cdot \delta \mathbf{x} - \dot{\boldsymbol{p}} \cdot \delta \mathbf{x} \\ - \left(p_t + \frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right)\delta u + \frac{d}{d\tau}\left(p_t \delta t + \boldsymbol{p} \cdot \delta \mathbf{x}\right),\end{aligned} \tag{A.5}$$

and global symmetries

$$\delta_{c_t} t = c_t \,, \qquad\qquad E_Q = -p_t \approx \frac{1}{2m}\|\mathbf{p}\|^2 + E_0 \,, \tag{A.6a}$$
$$\delta_{c_x} \mathbf{x} = \mathbf{c}_x \,, \qquad \Rightarrow \qquad \mathbf{p}_Q = \mathbf{p} \,, \tag{A.6b}$$
$$\delta_{c_p} \mathbf{p} = m\mathbf{c}_p \,, \quad \delta_{c_p} p_t = -\tfrac{1}{m}\mathbf{p} \cdot \delta_{c_p} \mathbf{p} \,, \quad \delta_{c_p} \mathbf{x} = t\mathbf{c}_p \,, \qquad \mathbf{k}_Q = t\mathbf{p} - m\mathbf{x} \,, \tag{A.6c}$$
$$\delta_\omega \mathbf{x} = \omega \mathbf{x} \,, \quad \delta_\omega \mathbf{p} = \omega \mathbf{p} \qquad \mathbf{j}_Q = \mathbf{x} \times \mathbf{p} \,. \tag{A.6d}$$

The Poisson brackets are given by $\{t, p_t\} = 1$ and $\{x_i, p_j\} = \delta_{ij}$ and the gauge transformations generated by the gauge constraint via $\delta_\lambda F = \lambda\{F, \phi\}$ are given by

$$\delta_{\lambda_t} t = \lambda_t \,, \qquad \delta_{\lambda_t} \mathbf{x} = \frac{\mathbf{p}}{m}\lambda_t \,, \qquad \delta_{\lambda_t} u = \dot{\lambda}_t \,. \tag{A.7}$$

This is the remaining reparametrisation freedom in τ and we accompanied it by a transformation of the Lagrange multiplier u such that it is a symmetry of the action (A.6). A more geometric way to write these gauge transformations is by transforming all canonical variable as $\delta_\lambda z = \dot{z}\lambda$ and the Lagrange multiplier as $\delta_\lambda u = \frac{d}{d\tau}(u\lambda)$, where we see that the canonical variables transform as scalars while the u is a scalar density (see, e.g., Section 4.3.1. in [36]).

After gauge fixing the action has the following form

$$L_{\text{can}}[\mathbf{x}, \boldsymbol{p}] = \boldsymbol{p} \cdot \dot{\mathbf{x}} - \left(\frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0\right), \tag{A.8}$$

where the Hamiltonian is given by $E = \frac{1}{2m}\|\boldsymbol{p}\|^2 + E_0$ and symmetries are now given by

$$\delta_{c_x} \mathbf{x} = \mathbf{c}_x \,, \qquad\qquad \mathbf{p}_Q = \mathbf{p} \,, \tag{A.9a}$$
$$\delta_{c_p} \mathbf{p} = m\mathbf{c}_p \,, \qquad \delta_{c_p} \mathbf{x} = t\mathbf{c}_p \,, \qquad \Rightarrow \qquad \mathbf{k}_Q = t\mathbf{p} - m\mathbf{x} \,, \tag{A.9b}$$
$$\delta_\omega \mathbf{x} = \omega \mathbf{x} \,, \qquad \delta_\omega \mathbf{p} = \omega \mathbf{p} \qquad \mathbf{j}_Q = \mathbf{x} \times \mathbf{p} \,. \tag{A.9c}$$

We used $\mathbf{p} = m\dot{\mathbf{x}}$ to write the action in configuration space

$$L_{\text{red}}[\mathbf{x}] = \frac{m}{2}||\dot{\mathbf{x}}||^2 - E_0\,, \tag{A.10}$$

where it has the following symmetries

$$\begin{aligned} &\delta_{c_x}\mathbf{x} = \mathbf{c}_x\,, & & & &\mathbf{p}_Q = m\dot{\mathbf{x}}\,, && \text{(A.11a)}\\ &\delta_{c_p}\mathbf{x} = t\mathbf{c}_p\,, & & &\Rightarrow\quad &\mathbf{k}_Q = tm\dot{\mathbf{x}} - m\mathbf{x}\,, && \text{(A.11b)}\\ &\delta_\omega\mathbf{x} = \omega\mathbf{x}\,, & &\delta_\omega\mathbf{p} = \omega\mathbf{p} & &\mathbf{j}_Q = \mathbf{x}\times m\dot{\mathbf{x}}\,. && \text{(A.11c)}\end{aligned}$$

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