Implications of tau data for CP violation in K decays

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Abstract

The $D = 6$ contribution of the Operator Product Expansion (OPE) of the VV−AA correlator of quark currents can be related to hadronic matrix elements associated to CP violation in non-leptonic kaon decays. We use those relations to find an updated value for $\langle(\pi\pi)_{I=2}|Q_8|K\rangle$ in the chiral limit using the updated ALEPH spectral function. Taking instead values of the matrix elements from the lattice to obtain the $D = 6$ vacuum elements provides a new short-distance constraint that allows for an inclusive determination of f_π and an updated value for the $D = 8$ condensate.

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1 Introduction

Non-leptonic kaon decays are a challenging laboratory to study the interplay between weak, electromagnetic and strong interactions at low energies [1]. There is still a long path to reduce the large theory uncertainties due to the complex hadron dynamics, so that they can become comparable to the experimental ones. One of the most controversial observables involving them is the CP violating ratio ε'/ε. While some analytical and lattice studies report SM predictions below the experimental measurements [2,3], a recent SM re-analysis, based on a framework that properly accounts for the large absorptive corrections due to the pion re-scattering in the final states [4–6], found a value compatible with the experimental one [7].

Starting from the SM Lagrangian at the electroweak scales and using Renormalization Group Equations to resum large logarithms one obtains the following Effective $\Delta S = 1$ Lagrangian in the three-flavour theory [8]:

$$\mathcal{L}_{\text{eff}}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu)\, \mathcal{Q}_i(\mu)\,. \tag{1}$$

The Wilson Coefficients, $C_i(\mu)$, encode the short-distance dynamics and can be computed with perturbative methods. The nonperturbative hadronic dynamics is captured by four-quark operators, $\mathcal{Q}_i(\mu)$. One of the leading contributions to the ε'/ε ratio comes from matrix elements associated to the electroweak penguin operator contributions:

$$\langle \mathcal{Q}_7 \rangle_\mu \equiv \langle (\pi\pi)_{I=2} | \mathcal{Q}_7 | K^0 \rangle_\mu = \langle (\pi\pi)_{I=2} | \bar{s}_a \Gamma_L^\mu d_a (\bar{u}_b \Gamma_\mu^R u_b - \frac{1}{2} \bar{d}_b \Gamma_\mu^R d_b - \frac{1}{2} \bar{s}_b \Gamma_\mu^R s_b) | K^0 \rangle_\mu\,, \tag{2}$$

$$\langle \mathcal{Q}_8 \rangle_\mu \equiv \langle (\pi\pi)_{I=2} | \mathcal{Q}_8 | K^0 \rangle_\mu = \langle (\pi\pi)_{I=2} | \bar{s}_a \Gamma_L^\mu d_b (\bar{u}_b \Gamma_\mu^R u_a - \frac{1}{2} \bar{d}_b \Gamma_\mu^R d_a - \frac{1}{2} \bar{s}_b \Gamma_\mu^R s_a) | K^0 \rangle_\mu\,, \tag{3}$$

with $\Gamma_\mu^{L(R)} = \gamma_\mu (1 \mp \gamma_5)$. Even when there are no known first-principle computations of the different hadronic matrix elements with analytic methods for $N_C = 3$, one can connect the matrix elements of Eq. (2) and (3) to two vacuum condensates by using iteratively the soft-meson limit [9]. In the chiral limit, *i.e.*, at zero momenta, one has:

$$\langle \mathcal{Q}_7 \rangle_\mu = -\frac{2}{F^3} \langle \mathcal{O}_1 \rangle_\mu\,, \tag{4}$$

$$\langle \mathcal{Q}_8 \rangle_\mu = -\frac{2}{F^3} \left(\frac{1}{2} \langle \mathcal{O}_8 \rangle_\mu + \frac{1}{N_c} \langle \mathcal{O}_1 \rangle_\mu \right)\,, \tag{5}$$

with

$$\langle \mathcal{O}_1 \rangle_\mu \equiv \frac{1}{2} \langle 0 | \, \bar{d}\, \Gamma_\mu^L u \, \bar{u} \Gamma_R^\mu d \, | 0 \rangle_\mu\,, \tag{6}$$

$$\langle \mathcal{O}_8 \rangle_\mu \equiv \frac{1}{2} \langle 0 | \, \bar{d}\, \Gamma_\mu^L \lambda_i u \, \bar{u} \Gamma_R^\mu \lambda_i d \, | 0 \rangle_\mu\,, \tag{7}$$

where λ_i are color matrices. Rewriting those hadronic matrix elements in terms of those vacuum matrix elements is useful because we can relate them with the Operator Product Expansion (OPE) of the $VV - AA$ correlation function [10], $\Pi(s) \equiv \Pi_{ud,LR}^{(0+1)}(s) \equiv \Pi_{ud,LR}^{(0)}(s) + \Pi_{ud,LR}^{(1)}(s)$, with:

$$\begin{aligned} \Pi_{ud,LR}^{\mu\nu}(q) &\equiv i \int d^4x \, e^{iqx} \, \langle 0 | \, T \left(L_{ud}^\mu(x) R_{ud}^{\nu\dagger}(0) \right) | 0 \rangle \\ &= (-g^{\mu\nu} q^2 + q^\mu q^\nu)\, \Pi_{ud,LR}^{(1)}(q^2) + q^\mu q^\nu \, \Pi_{ud,LR}^{(0)}(q^2)\,, \end{aligned} \tag{8}$$

where $L^\mu_{ud}(x) \equiv \bar{u}(x)\gamma^\mu(1-\gamma_5)d(x)$ and $R^\mu_{ud}(x) \equiv \bar{u}(x)\gamma^\mu(1+\gamma_5)d(x)$, is given at NLO in QCD by:

$$\Pi^{(1+0)}(Q^2 = -q^2) = \sum_{p=D/2} \frac{a_p(\mu) + b_p(\mu)\ln\frac{Q^2}{\mu^2}}{Q^{2p}}. \tag{9}$$

b_p is α_s-suppressed with respect to a_p. The dimension 0 contributions vanish, since the correlator vanishes at all order in massless perturbative QCD. a_1 (and b_1) is suppressed by two powers of the light quark masses, and then is completely negligible. The leading contribution of a_2 is proportional to $\alpha_s \hat{m}\langle\bar{q}q\rangle$ and is also numerically negligible. The crucial point is that the leading short-distance contribution comes from the same vacuum condensates as in Eqs. (4) and (5) [11]:

$$a_3(\mu) = 2\left[2\pi\langle\alpha_s\mathcal{O}_8\rangle_\mu + A_8\langle\alpha_s^2\mathcal{O}_8\rangle_\mu + A_1\langle\alpha_s^2\mathcal{O}_1\rangle_\mu\ \right], \tag{10}$$
$$b_3(\mu) = 2[B_8\langle\alpha_s^2\mathcal{O}_8\rangle_\mu + B_1\langle\alpha_s^2\mathcal{O}_1\rangle_\mu]\ , \tag{11}$$

where A_i and B_i depend on the renormalization prescription and/or in the number of active flavors (they can be found in Ref. [11]). The OPE of the $V-A$ correlator is then connected to non-leptonic kaon decays.

On the other hand, experimental spectral functions coming from inclusive hadronic tau decays are directly connected to imaginary parts of two-point correlation functions (e.g. see [12]). This connection leads to very precise predictions. For example, a very nice test of asymptotic freedom, which can be translated into a determination of the strong coupling [13–17], can be performed with non-strange $V+A$ spectral function. Using also strange data, one can extract information on fundamental parameters such as m_s or V_{us} [15, 18–22].

In this work we use non-strange $V-A$ spectral functions, which, owing to its chiral suppression, are known to be a very nice probe of non-perturbative parameters [23–27]. Phenomenological implications of the relations of both inclusive hadronic tau-decay data and non-leptonic kaon ones with the $V-A$ correlator were studied using mostly tau-decay data in Refs. [9, 11, 28], where values for those $K \to \pi\pi$ matrix elements were obtained. Updated data sets [14] and further development of techniques to assess the so-called Duality Violation (DV) uncertainties [26, 27, 29–35] motivate a fresh numerical analysis.

2 Dispersion relations with polynomial kernel

From tau data, one have access to:

$$\mathrm{Im}\Pi(s), \qquad s_{th} = 4m_\pi^2 < s \equiv q^2 < m_\tau^2\,. \tag{12}$$

However, the OPE of the correlator is defined at large Euclidean momentum:

$$\Pi(s) \approx \Pi^{\mathrm{OPE}}(s=-Q^2)\,, \quad \text{at} \quad Q^2 \gg \Lambda_{QCD}^2\,. \tag{13}$$

In order to relate both regions, one uses that $\Pi(s)$ is known to be an analytic function in the whole complex plane except for a cut in the positive real axis. Then, integrating the correlator times an analytic but otherwise arbitrary weight function $\omega(s)$ along the circuit of Figure 1, one finds [25]

$$\int_{s_{th}}^{s_0} \frac{ds}{s_0}\,\omega(s)\,\mathrm{Im}\,\Pi(s) - \frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\Pi(s) = 2\pi\frac{f_\pi^2}{s_0}\,\omega(m_\pi^2)\,. \tag{14}$$

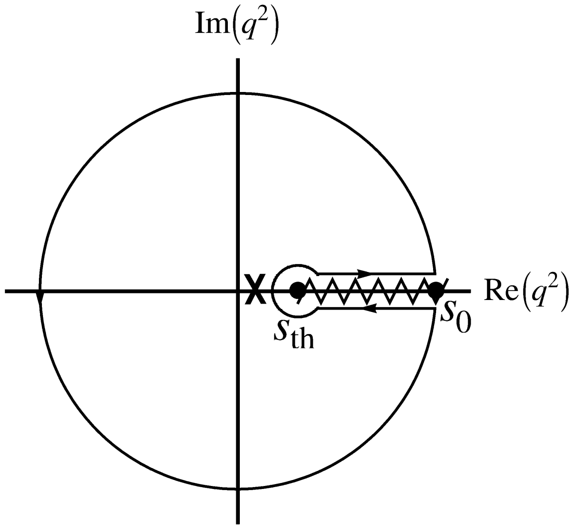

Figure 1: Circuit of integration in Eq. (14).

In the first term of Eq. (14) one can introduce data, while the second one can be evaluated with the analytic continuation of $\Pi^{\rm OPE}(s)$. The differences arising from using the OPE approximant instead of the physical correlator are known as quark-hadron Duality Violations (DVs) [26,27, 29–35]:

$$\delta_{\rm DV}[\omega(s),s_0] \equiv \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\left[\Pi(s)-\Pi^{\rm OPE}(s)\right] = \int_{s_0}^{\infty}\frac{ds}{s_0}\,\omega(s)(\rho-\rho^{OPE})(s)\,.$$

3 $\langle(\pi\pi)_{I=2}|Q_8|K^0\rangle$ in the chiral limit

Large experimental and DV uncertainties prevent us from working at NLO in α_s when extracting the dimensional OPE coefficients from tau data, since we are not able to fit both condensates entering at that order (they are suppressed both by 6 powers of the tau mass and by α_s). As a consequence, we add conservatively (owing to the large value of A_8), a 25% of uncertainty to the final result. At that order, a determination of $a_3(\mu)$ leads to a determination of $\langle\mathcal{O}_8\rangle_\mu$. In principle, it is not enough if one wants to extract $\langle Q_8\rangle_\mu$, since one also needs the contribution coming from $\langle\mathcal{O}_1\rangle_\mu$. However, this contribution is suppressed by two powers of $1/N_c$. Different phenomenological and lattice approaches confirm this strong suppression (e.g. see [11,36,37]). Then, one has:

$$\lim_{p,q,k=0}\ \langle(\pi\pi)_{I=2}|Q_8|K^0\rangle_\mu = -\frac{a_3(\mu)}{4\pi\alpha_s(\mu)F^3}\,. \tag{15}$$

At leading order in α_s, the determination of a_3 is equivalent to the determination of $\mathcal{O}_{D=6}$ of Ref. [27]. We have revisited it introducing some extra tests and trying to implement some small improvements. We proceed as follows:

- Taking two different weight functions, $\omega(s) = 1-\left(\frac{s}{s_0}\right)^2$ (one-pinched) and $\omega(s)=\left(1-\frac{s}{s_0}\right)^2$ (double-pinched), we observe good agreement for the obtained values of a_3 for $s_0\sim m_\tau^2$ (see Fig. 2). We also observe a stable plateau for the latter. Adding DV uncertainties based on the small fluctuations under the change of s_0 in a conservative interval, we obtain, preliminarily:

$$a_3 = (-2.8\pm 0.9)\cdot 10^{-3}\,{\rm GeV}^6\,. \tag{16}$$

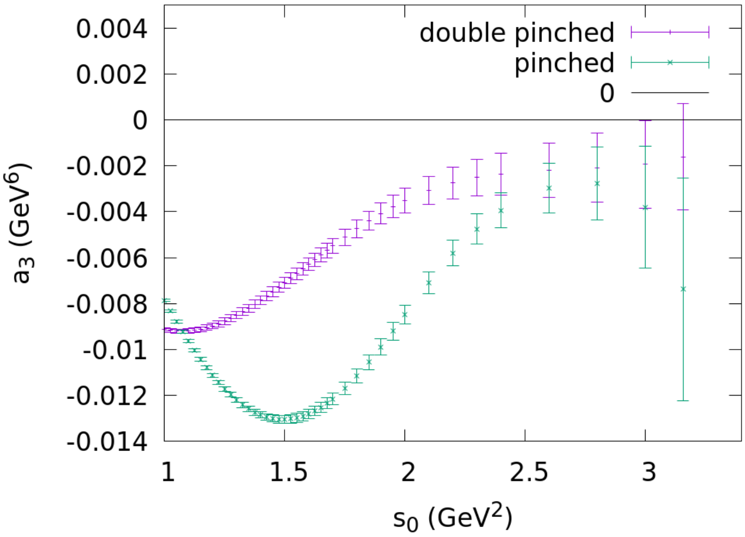

Figure 2: Pinched weight functions as a function of s_0 rescaled so that at s_0 large enough converge to $a_3(s_0)$.

- An alternative approach consists in trying to guess how the exact spectral function is at $s_0 > m_\tau^2$. One pays the price of having to choose a specific parametrization and then introducing some model-dependence. We try to relax it by allowing data not to obey extrictly the model but imposing they must obey WSRs. The ansatz we use is [30, 32, 33, 38–40]

$$\rho(s) = \frac{1}{\pi}\kappa e^{-\gamma s}\sin(\beta(s - s_z)) \quad s > \hat{s}_0 \,. \tag{17}$$

Following the procedure of Refs. [25, 27, 32, 33] we generate random tuples of parameters $(\kappa, \gamma, \beta, s_z)$, everyone of them representing a possible spectral function above a threshold $\hat{s}_0$. If we perform a fit with ALEPH data, we find that there are no significant deviations (p-value above a 5%) from this specific model above $\hat{s}_0 = 1.25$ GeV^2. However, the model is only motivated as an approximation at higher energies, where the hadronic multiplicity is higher. As a first constraint, as in Ref. [27], we accept only those tuples that are in the 90% C.L. region ($\chi^2 < \chi^2_{\text{min}} + 7.78$). In contrast with Ref. [27], we make a combined fit of the moment used to obtain a_3 with the WSRs, accepting only those tuples compatible with them (p-value larger than a 5%).[1]. We find

$$a_3(s_0) = (-3.7^{+1.3}_{-0.9}) \cdot 10^{-3}\,\text{GeV}^6\,, \tag{18}$$

in good agreement with the result of Ref. [27] and with Eq. (16).

- When assuming a model, as in the previous bullet point, one is changing the assumption of convergence of data to its OPE approximant at $s_0 \sim m_\tau^2$, capturing most of the possible DV tails by adding a systematic uncertainty based on fluctuations under the change of s_0, by the assumption of convergence of data at a lower energy[2] to a specific parametrization for the difference between the spectral function and its OPE approximant. A priori, it is unclear to us which procedure should be preferred. One minimal reliability test one

[1] In this way, large correlations between experimental uncertainties when imposing the WSRs and the moment used to extract a_3 are taken into account.

[2] This is unfortunately needed in order to fit the free parameters.

Table 1: Value of a_3 obtained with our tuple procedure for different $\hat{s}_0$.

$\hat{s}_0(\mathrm{GeV}^2)$	1.25	1.4	1.55	1.7	1.9
$a_3(10^{-3}\mathrm{GeV}^6)$	$-5.3^{+0.7}_{-0.5}$	$-5.1^{+0.7}_{-0.5}$	$-5.3^{+0.5}_{-0.3}$	$-3.7^{+1.3}_{-0.9}$	$-3.8^{+1.8}_{-1.0}$

should ask to any model, in analogy with the reliability test of independence of the result on s_0 when directly assuming good convergence of data to its OPE approximant, is a soft dependence in the choice of threshold $\hat{s}_0$. By changing $\hat{s}_0$ in the large interval $\hat{s}_0 \in [1.25, 1.9]\,\mathrm{GeV}^2$ we have tested that results display a decent stability (see Table 1).

Combining Eqs. (16) and (18) and introducing it into Eq. (15), we obtain at zero momenta:

$$\langle(\pi\pi)_{I=2}|\mathcal{Q}_8|K^0\rangle_{2\,\mathrm{GeV}} = (1.14 \pm 0.53)\,\mathrm{GeV}^3\,, \tag{19}$$

where uncertainties are dominated by uncertainties in a_3, followed by perturbative ones, estimated as explained above. The value is in good agreement with the ones obtained by similar approaches [9, 11, 28]. It is also in agreement with the result obtained using factorization of currents in the large-N_c limit:

$$\langle(\pi\pi)_{I=2}|\mathcal{Q}_8|K^0\rangle^{N_c\to\infty}_{2\,\mathrm{GeV}} = 2FB_0^2 = 2\frac{M_{K_0}^4 F}{(m_d+m_s)^2} \approx 1.2\,\mathrm{GeV}^3\,, \tag{20}$$

and also with previous lattice results (e.g. see [36, 37]).

4 Using kaon matrix elements from the lattice to improve other tau-based results

Instead of using inclusive hadronic tau-decay data to obtain $K\to\pi\pi$ matrix elements, one can take advantage of the very precise values for the matrix elements of Eqs. (2) and (3) given by the lattice in Ref. [37] to obtain the coefficients $a_3(\mu)$ and $b_3(\mu)$. One has in Naive Dimensional Regularization (NDR) $\bar{MS}$ for 4 active flavors:

$$\langle Q_7\rangle_{3\,\mathrm{GeV}} = 0.36 \pm 0.03\,\mathrm{GeV}^3\,, \tag{21}$$

$$\langle Q_8\rangle_{3\,\mathrm{GeV}} = 1.6 \pm 0.1\,\mathrm{GeV}^3\,. \tag{22}$$

Now we can work at NLO in α_s for the $D=6$ contribution. In order to avoid large logarithms we run from $\mu = 3\,\mathrm{GeV}$ to $\mu = \sqrt{s_0}$ and then apply Eqs. (10) and (11) to obtain, respectively, $a_3(\mu=\sqrt{s_0})$ and $b_3(\mu=\sqrt{s_0})$. Using that input and taking $\omega(s) = \left(1-\frac{s}{s_0}\right)^2$ in Eq. (14), one can obtain a very powerful short-distance constraint for hadronic tau-decay data:

- Experimental uncertainties, typically dominated by the region near s_0 are reduced for that weight function.
- The first unknown OPE contribution is suppressed both by 8 powers of the tau mass and by α_s.
- Duality Violations are very suppressed for this moment. One would need a very artificial DV shape to make it noticeable. Different model estimates, for example using the tuple corresponding to the minimum in Eq. (17), typically predict they are one order of magnitude below experimental uncertainties at $s_0 \sim m_\tau^2$.

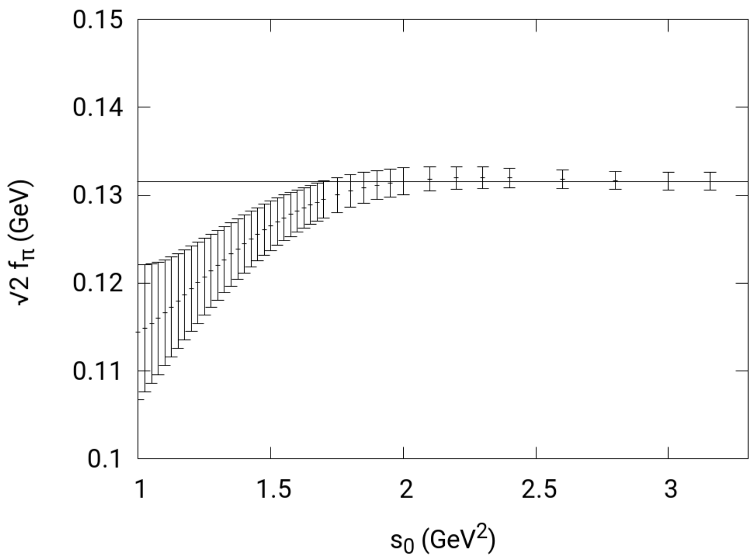

Figure 3: Equation (14) for $\omega(s) = \left(1 - \frac{s}{s_0}\right)^2$ rescaled so that at s_0 large enough converge to f_π. An horizontal line with the central value at $s_0 = m_\tau^2$ is displayed to guide the eye.

There are no unknown physical parameter entering into that expression. However, a good way of testing the power of this dispersion relation is simply translating it into a determination of f_π.[3] Even when it enters into the dispersion relation suppressed by two powers of the tau mass, a quite precise value is obtained in Figure 3. As expected, a stable plateau is observed. We find as preliminary result at $s_0 = m_\tau^2$:

$$\sqrt{2} f_\pi = (131.6 \pm 0.9_{\text{exp}} \pm 0.4_{\text{chiral}} \pm 0.1_{\text{latt}})\,\text{MeV} = (131.6 \pm 1.0)\,\text{MeV}\,, \tag{23}$$

where the first uncertainty is experimental, the second one due to the difference between physical matrix elements and the chiral limit values and the last one due to the uncertainty in the lattice input.

Finally, using the method of Section 3, but including the $D = 6$ contribution as an external input, we obtain a preliminary value for the $D = 8$ condensate:

$$a_4 = -(0.7 \pm 0.6)\,\text{GeV}^8\,, \tag{24}$$

in good agreement with previous works.

5 Conclusion

Relations in the chiral limit between kaon to two-pion matrix elements and vacuum condensates that can be related to inclusive tau data can be used to make precise predictions. From tau-decay data one finds at zero momenta:

$$\langle (\pi\pi)_{I=2} | \mathcal{Q}_8 | K^0 \rangle_{2\,\text{GeV}} = (1.14 \pm 0.53)\,\text{GeV}^3\,. \tag{25}$$

[3] One can also use it to find a New Physics bound [41].

Taking instead the $K \to \pi\pi$ input from the lattice, which nowadays turns out to be more precise, one still can make very precise predictions about inclusive tau decay data dominated by the non-perturbative $\sim 1\,\text{GeV}$ Minkowkian region. For example, one of them can be translated into a clean determination of f_π below the per cent level:

$$\sqrt{2} f_\pi = (131.6 \pm 1.0)\,\text{MeV}\,, \tag{26}$$

or to obtain information about a vacuum condensate

$$a_4 = -(0.7 \pm 0.6)\,\text{GeV}^8\,, \tag{27}$$

even when it is suppressed by 8 powers of the tau mass.

All the determinations studied here could be improved with future non-strange spectral functions, which in principle could be extracted from Belle-II [42].

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