Current status of inclusive hadronic τ decay determinations of $|V_{us}|$

K. Maltman[1,2★], P. Boyle[3], R. J. Hudspith[4], T. Izubuchi[5,6], A. Jüttner[7], C. Lehner[5], R. Lewis[4], H. Ohki[6,8], A. Portelli[9], M. Spraggs[9], J. M. Zanotti[10]

1 Mathematics and Statistics, York University, Toronto, Canada
2 CSSM, University of Adelaide, Adelaide, Australia
3 SAP, School of Physics, University of Edinburgh, Edinburgh, UK
4 Physics and Astronomy, York University, Toronto, Canada
5 Physics Department, Brookhaven National Lab, Upton, NY, USA
6 RIKEN-BNL Research Center, Brookhaven National Lab, Upton, NY USA
7 School of Physics and Astronomy, University of Southampton, Southampton, UK
8 Physics Department, Nara Women's University, Nara, Japan
9 School of Physics and Astronomy, University of Edinburgh, Edinburgh, UK
10 Department of Physics, University of Adelaide, Adelaide, Australia

★ kmaltman@yorku.ca

Proceedings for the 15th International Workshop on Tau Lepton Physics, Amsterdam, The Netherlands, 24-28 September 2018
doi:10.21468/SciPostPhysProc.1

Abstract

We review the status of the determination of $|V_{us}|$ from both flavor-breaking finite-energy sum rules based on inclusive non-strange and strange hadronic τ decay data and the recent lattice-based analysis of inclusive strange hadronic τ decay data. In particular, we update the results from these analysis frameworks taking into account recent improvements to a number of strange branching fractions reported by HFLAV at CKM2018 and this meeting. We find that inclusive τ decay data yields results for $|V_{us}|$ compatible within errors with the expectations of three-family unitarity.

Received 14-11-2018
Accepted 09-01-2019
Published 18-02-2019
doi:10.21468/SciPostPhysProc.1.006

Contents

1 Introduction

As is well known, the conventional implementation of the flavor-breaking (FB) finite-energy sum rule (FESR) determination of $|V_{us}|$, employing (i) inclusive non-strange and strange branching fractions (BFs) and (ii) certain assumptions about dimension $D = 6$ and 8 OPE contributions on the theory side of the relevant FESR [1, 2], yields results $\sim 3\sigma$ low compared to the expectations of three-family unitarity. The most recent such determination, by the HFLAV collaboration, reported at this meeting, for example, yields 0.2195(18) [3].

Problems with the theory-side OPE assumptions in this conventional implementation have been identified in Ref. [4], and an alternate implementation of the FB FESR framework proposed, in which, rather than making assumptions about their values, the relevant effective $D > 4$ OPE condensates are obtained from fits to data. This alternate implementation was shown to resolve self-consistency problems associated with the theory-side assumptions underlying the conventional implementation. The results for $|V_{us}|$ obtained obtained from this improved implementation were found lie ~ 0.0020 higher than those obtained from the same data when conventional implement assumptions are employed.

Recently, an improved, lattice-based, method for obtaining $|V_{us}|$ from inclusive hadronic τ decay data has been proposed [5]. In this approach, high-precision lattice data for the spin $J = 0, 1$ hadronic vacuum polarizations (HVPs) of the flavor *us* vector (V) and axial-vector (A) current-current two-point functions is used in place of the OPE approximation on the theory side of appropriately-weighted dispersive sum rules. Weights were shown to exist which successfully reduce the relative importance of spectral contributions from the region of the inclusive *us* V+A spectral distribution where experimental errors are large without at the same time blowing up lattice systematic or statistical errors. The $|V_{us}|$ obtained from the version of this analysis reported in Ref. [5] have smaller errors than those obtained using FB FESRs, and are in agreement within errors with the expectations of three-family unitarity.

The alternate FB FESR and lattice-based analyses reported in Refs. [4, 5] were both based on earlier versions of the HFAG/HFLAV compilation of the exclusive mode strange BFs needed to set the overall scale for the low-multiplicity exclusive mode unit-normalized distributions measured by BaBar and Belle. Because a non-trivial tension existed between the HFAG 2016 result [6] and dispersively constrained expectations [7] for $B\left[\tau^- \to K^-\pi^0\nu_\tau\right]$, both of these analyses considered two cases, one based entirely on 2016 HFAG strange exclusive BFs, and one in which both the 2016 HFAG exclusive-mode $K\pi$ BFs were replaced by the expectations for these BFs obtained in the dispersive analysis of Ref. [7].

More recently, HFLAV has updated its combined strange branching fraction fit, taking into account new, and significantly improved, BaBar results for the BFs $B\left[\tau^- \to K^- n\pi^0\nu_\tau\right]$ with $n = 0, 1, 2, 3$ [3]. The BaBar results resolve the tension between the HFAG 2016 results and dispersively constrained $B\left[\tau^- \to K^-\pi^0\nu_\tau\right]$ expectations. New versions of the analyses of Refs. [4, 5] with an updated, now-unique set of strange BFs are thus now possible, and we report on the results of these below.

The rest of this paper is organized as follows. In Section 2, we review the FESR approach, set notation, and specify the experimental input employed in the updated analyses. In Section 3, we outline the conventional implementation of the FB FESR framework, the problems associated with the OPE assumptions it employs, and the alternate FB FESR implementation which solves these problems, before providing the updated results of this alternate FB FESR analysis. In Section 4, we briefly review the new lattice-based approach before providing the updated results obtained from it. Finally, in Section 5 we present our conclusions and the discuss prospects for future improvements.

2 Background: hadronic τ decays in the Standard Model

In the Standard Model, with s the hadronic invariant mass-squared, the differential versions, $dR_{ij;V/A}/ds$, of the ratios

$$R_{ij;V/A} \equiv \Gamma[\tau^- \to \nu_\tau \,\text{hadrons}_{ij;V/A}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)] \tag{1}$$

involving the widths of hadronic τ decays mediated by the flavor $ij = ud, us$, vector (V) or axial-vector (A) currents, are related to the spectral functions, $\rho^{(J)}_{V/A;ij} = \frac{1}{\pi}\,\mathrm{Im}\,\Pi^{(J)}_{ij;V/A}(s)$, of the $J = 0,1$ HVPs, $\Pi^{(J)}_{ij;V/A}$, of the corresponding current-current two-point functions, by [8]

$$\begin{aligned} \frac{dR_{ij;V/A}}{ds} &= \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_\tau^2}\left[w_\tau(y_\tau)\rho^{(0+1)}_{ij;V/A}(s) - w_L(y_\tau)\rho^{(0)}_{ij;V/A}(s)\right] \\ &\equiv \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_\tau^2}\,(1-y_\tau)^2\,\tilde{\rho}_{ij;V/A}(s)\,, \end{aligned} \tag{2}$$

where V_{ij} is the flavor ij CKM matrix element, S_{EW} is a known short-distance electroweak correction [9], $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, and $w_L(y) = 2y(1-y)^2$. The π and K contributions, which are accurately known and chirally unsuppressed, dominate $\rho^{(0)}_{ud;A}(s)$ and $\rho^{(0)}_{us;A}(s)$, respectively. The remaining, continuum, $J = 0$, ij V and A contributions, are $\propto (m_i \mp m_j)^2$, and hence negligible for $ij = ud$. For $ij = us$, though not entirely negligible, they are small, and highly constrained through associated $ij = us$ scalar and pseudoscalar sum rules, allowing mildly model-dependent determinations over the kinematically allowed range $s \le m_\tau^2$ [10,11]. Subtracting the contributions of the resulting $J = 0$ pole-plus-continuum sums from the RHSs of Eq. (2), one obtains the $J = 0+1$ analogues, $dR^{(0+1)}_{ij;V/A}/ds$, of $dR_{ij;V/A}/ds$, and, from these, the products of the $J = 0+1$ spectral function combination and corresponding CKM factor, $|V_{ij}|^2\,\rho^{(0+1)}_{ij;V/A}(s)$.

The $J = 0$ (longitudinal) subtraction also allows re-weighted analogues of $R^{(0+1)}_{ij;V/A}$,

$$R^w_{ij;V/A}(s_0) \equiv \int_0^{s_0} ds\, \frac{w(s)}{w_\tau(s)}\, \frac{dR^{(0+1)}_{ij;V/A}(s)}{ds}\,, \tag{3}$$

to be constructed for arbitrary polynomial, w, and arbitrary $s_0 \le m_\tau^2$. Varying s_0 and w provides useful self-consistency constraints for FESR analyses in general and for the FB FESR analyses outlined below in particular.

The spectral function combinations $\rho(s) \equiv \rho^{(0+1)}_{ij;V/A}(s)$ and $s\rho^{(0)}_{ij;V/A}(s)$ correspond to HVP combinations, $\Pi(s) \equiv \Pi^{(0+1)}_{ij;V/A}(s)$ and $s\,\Pi^{(0)}_{ij;V/A}(s)$, which are free of kinematic singularities, and hence satisfy the FESR (Cauchy's theorem) relation

$$\int_0^{s_0} w(s)\rho(s)\,ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\Pi(s)\,ds \tag{4}$$

for arbitrary analytic $w(s)$ and arbitrary $s_0 \leq m_\tau^2$. For s_0 large enough, the operator product expansion (OPE) approximation can be used for $\Pi(s)$ on the RHS.

Both the FB FESR and lattice-based determinations of $|V_{us}|$ require as experimental input weighted integrals of the inclusive flavor us V+A distribution. The former also requires weighted integrals of the inclusive flavor ud V+A distribution.

The weighted us V+A integrals are obtained using the following experimental input

- either the measured $\tau \to K\nu_\tau$ BF, B_K, or the more precise Standard Model (SM) expectation for B_K implied by $K_{\mu 2}$ for the K pole contribution;
- Belle [12] and BaBar [13] input for the $K^-\pi^0$ and $\bar{K}^0\pi^-$ contributions;
- BaBar input [14] for the $K^-\pi^+\pi^-$ contribution;
- Belle input [15] for the $\bar{K}^0\pi^-\pi^0$ contribution;
- a combination of BaBar [16] and Belle [17] input for the very small $\bar{K}\bar{K}K$ contribution; and
- 1999 ALEPH input [18] for the combined "residual mode" contribution (the sum over contributions from exclusive strange modes not remeasured at the B-factories).

With BaBar and Belle results for strange exclusive-mode distributions given in unit-normalized form, measured BFs are required to set the overall scales. For these, we employ the results of the most recent HFLAV assessment [3], which takes into account significant recent BaBar improvements to the $K^- n\pi^0\nu_\tau$ BFs, reviewed at this conference by A. Lusiani. Of particular note is the new result for $B[K^- n\pi^0\nu_\tau]$, which supercedes the earlier BaBar result for this mode, and leads to a sizeable upward shift from the earlier 2016 HFAG average.

The weighted ud V+A integrals are obtained using $\pi_{\mu 2}$ and SM expectations for the π pole contribution and 2013 ALEPH results for the continuum ud V+A distribution [19]. A small rescaling ($\sim 0.5\%$ or less) is required to the convert the older inclusive, continuum ud V+A BF normalization used by ALEPH to the corresponding current value.

3 $|V_{us}|$ from FB FESRs

The FB FESR determination of $|V_{us}|$ is based on FESRs involving the FB spectral function combination, $\Delta\rho(s) \equiv \rho^{(0+1)}_{V+A;ud}(s) - \rho^{(0+1)}_{V+A;us}(s)$ and associated HVP difference, $\Delta\Pi(s) \equiv \Pi^{(0+1)}_{ud;V+A}(s)$ $- \Pi^{(0+1)}_{us;V+A}(s)$ [1,2]. Defining the w-weighted FB spectral integral differences, $\delta R^w_{V+A}(s_0)$, by

$$\delta R^w_{V+A}(s_0) \equiv \frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{V+A;us}(s_0)}{|V_{us}|^2}, \tag{5}$$

taking the superallowed nuclear β decay result for $|V_{ud}|$ [20] as external input, and using Eq. (4) to replace the LHS with its OPE representation, one finds, solving for $|V_{us}|$ [1,2],

$$|V_{us}| = \sqrt{R^w_{V+A;us}(s_0)/\left[\frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \delta R^{w,OPE}_{V+A}(s_0)\right]}, \tag{6}$$

a result necessarily independent of s_0 and w if all input is reliable. Varying s_0 and/or w thus allows any asssumptions employed in evaluating $\delta R^{w,OPE}_{V+A}(s_0)$ to be tested for self-consistency.

When $s_0 = m_\tau^2$ and $w = w_\tau$, the spectral integrals on the RHS of Eq. (6) can be determined using the inclusive ud and us τ BFs. In contrast, the spectral integrals for any other choice of s_0 and/or w require a knowledge of the full inclusive $dR_{ud,us;V+A}/ds$ distributions. The conventional implementation of the FB FESR framework, in which a single s_0 ($s_0 = m_\tau^2$) and single weight ($w = w_\tau$), are employed [1,2], is motivated by this experimental simplification. The disadvantage of these restrictions is the absence of variable-s_0 and or -w self-consistency tests.

This disadvantage is relevant since, with w_τ having degree three, $\delta R^{w_\tau,OPE}_{V+A}(m_\tau^2)$ receives OPE contributions up to dimension $D = 8$. $D = 2$ (perturbative mass-squared-dependent) and $D = 4$ contributions to $\delta R^{w_\tau,OPE}_{V+A}(m_\tau^2)$ are fixed by α_s and the quark masses and condensates, all of which are known from external sources [21–25]. The effective $D = 6$ and 8 condensates, C_6 and C_8, needed to determine the $D = 6$ and 8 contributions, are, however, not known. With only the single $s_0 = m_\tau^2$, $w = w_\tau$-weighted spectral integral as experimental input, it is, of course, impossible to simultaneously determine $|V_{us}|$, C_6 and C_8.

The conventional implementation proceeds by estimating the $D = 6$ contribution using the vacuum saturation "approximation" (VSA) and, based on the smallness of that estimate, assuming the $D = 8$ contribution can be neglected. These assumptions are potentially dangerous since there is a very strong double cancellation in the FB V+A VSA estimate[1], and the VSA is known to be badly violated, in a channel-dependent manner, from studies in the non-strange sector [26]. The reliability of these assumptions can be tested using FESRs involving variable s_0 and different weights, $w(y) = \sum_{n=0} w_n y^n$ (with $y = s/s_0$), which, in general, produce unsuppressed $D > 4$ OPE contributions

$$-\frac{1}{2\pi i}\oint_{|s|=s_0} ds\, w(y)\left[\Pi^{(0+1)}_{ud-us;V+A}(Q^2)\right]_{D>4} = \textstyle\sum_{k=2}(-1)^k w_k \frac{C_{2k+2}}{s_0^k}, \tag{7}$$

where the C_D are higher dimension effective condensates defined via

$$\left[\Pi^{(0+1)}_{ud-us;V+A}(Q^2)\right]_{D>4} = \sum_{n=2}\frac{C_{2n}}{Q^{2n}}, \tag{8}$$

with $Q^2 = -s$.

A particularly useful test of this type is provided by the comparison of results for $|V_{us}|$ obtained from the variable-s_0 $w_\tau(y) = 1-3y^2+2y^3$ and $\hat{w}(y) = 1-3y+3y^2-y^3$ FESRs, both of which receive unsuppressed OPE contributions up to $D = 8$. The two $D = 6$ contributions differ only by a sign, while the $\hat{w}$ $D = 8$ contribution is $-1/2$ that of the w_τ FESR. Thus, if the VSA provides a reliable estimate of $D = 6$ contributions for the w_τ FESR, it must also do so for the $\hat{w}$ FESR. Similarly, if $D = 8$ contributions are negligible for the w_τ FESR, they must be even more negligible for the $\hat{w}$ FESR.

The $|V_{us}|$ results obtained from these two FESRs, assuming the conventional implementation $D > 4$ assumptions are reliable, should thus be s_0 independent, and in agreement with one another. The results of this test are shown in the left panel of Fig. 1. Additional evidence for the significance of the s_0- and w-instability problems seen in the left panel is provided in the right panel, which shows the difference between the $|V_{us}|$ obtained from the $\hat{w}$ FESR at variable s_0 and at fixed $s_0 = m_\tau^2$, as a function of variable s_0. These differences should, of course, be compatible with zero within errors if the conventional implementation assumptions underlying them are reliable. The experimental errors shown are obtained by full propagation of the experimental covariances. The results shown in the two panels clearly establish the breakdown of the conventional implementation assumptions, and preclude the use of this implementation going forward.

In Ref. [4] an alternate implementation of the FB FESR approach was developed to deal with the s_0- and w-instabilities encountered when conventional implementation assumptions

[1]A factor of ~ 3 reduction in the individual ud and us V+A sums, and a further factor of ~ 6 reduction when these are combined to form the FB $ud-us$ V+A difference.

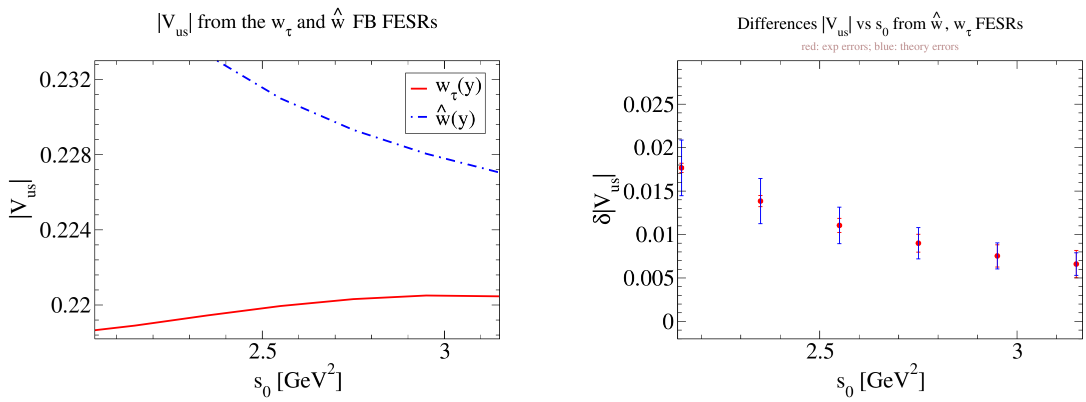

Figure 1: *Left panel: $|V_{us}|$ versus s_0 with conventional implementation $D = 6$ and 8 assumptions for the w_τ and $\hat{w}$ FB FESRs. Right panel: variable s_0 minus $s_0 = m_\tau^2$ $|V_{us}|$ differences from the $\hat{w}$ FB FESR, using conventional implementation assumptions.*

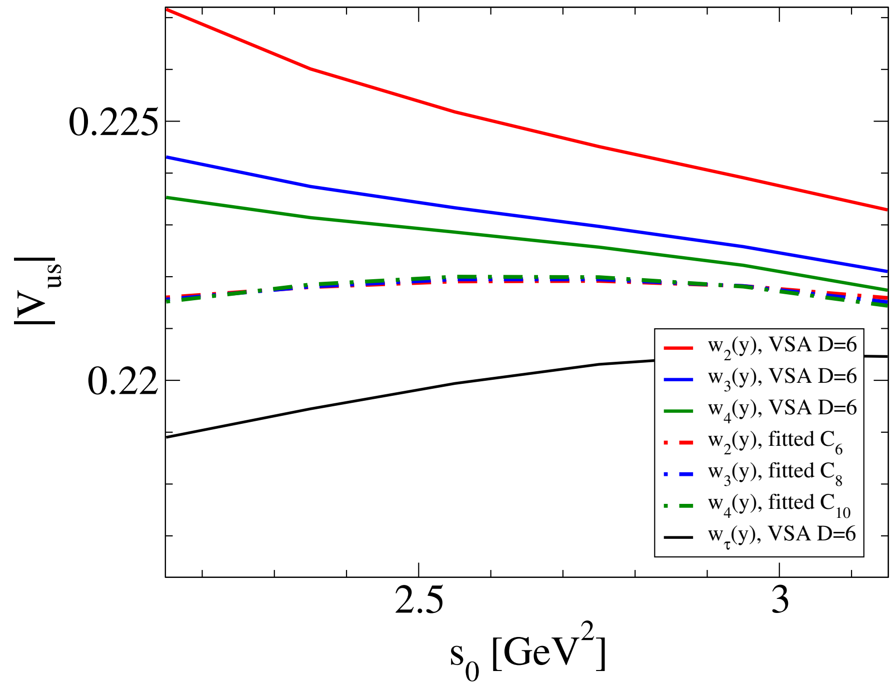

Figure 2: *s_0- and weight-stability test results. Solid lines show the s_0 dependence of $|V_{us}|$ obtained from the w_2, w_3, w_4 and w_τ FESRs with the conventional implementation assumptions $C_6 \simeq C_6^{VSA}$ and $C_8 = 0$. The dashed lines show the corresponding w_2, w_3 and w_4 results obtained using instead the central values for C_6, C_8 and C_{10} obtained from the alternate FB FESR fits.*

are used for the $D > 4$ OPE contributions. In this approach, the s_0 dependence of the weighted spectral integrals is used to provide additional experimental input, allowing not just $|V_{us}|$, but also the relevant $D > 4$ condensates, C_D, to be fit to data. The analysis employed FESRs involving the weights $w_N(y) = 1 - \frac{y}{N-1} + \frac{y^N}{N-1}$, with $N = 2, 3, 4$. These are convenient as the w_N FESR has a single unsuppressed $D > 4$ OPE contribution, with $D = 2N + 2$. $|V_{us}|$ and C_{2N+2} are fit simultaneously using the s_0 dependence of the w_N-weighted spectral integrals. Comparing the $|V_{us}|$ obtained from the different w_N FESRs provides a useful self-consistency test of the weight-independence type. Excellent agreement is observed [4].

One can also carry out an s_0-independence test by taking the central fit values for C_6, C_8 and C_{10}, obtained from the w_2, w_3 and w_4 FESRs, respectively, as input and working out the results for $|V_{us}|$ obtained from the same FESRs, as a function of s_0. If the results of the fits for C_{2N+2} are physically meaningful, the resulting $|V_{us}|$ should be, to a good approximation, s_0-independent and the same for the different w_N. The results of this exercise are shown in Fig. 2. The solid lines show, from top to bottom, the s_0-dependent $|V_{us}|$ results obtained from the w_2, w_3, w_4 and w_τ FESRs using conventional implementation assumptions for $C_{D>4}$. These are obviously far from stable with respect to changing either the choice of weight or s_0. The dashed lines, in contrast, show the s_0-dependent results obtained from the w_2, w_3 and w_4 FESRs when, rather than making conventional implementation assumptions, the central fitted results for the $C_{D>4}$ are used as input. These results obviously display vastly improved s_0-stability and extremely good weight-independence. The dashed curve obtained from the w_τ FESR is in excellent agreement with the other three dashed curves and omitted for visual clarity [2].

In view of the very good agreement between the results of the w_2, w_3 and w_4 FESRs, we follow Ref. [4] and obtain final results from a combined 3-weight fit. Following Ref. [2], we also evaluate π and K pole contributions using the more precise input implied by $\pi_{\mu 2}$, $K_{\mu 2}$ and SM expectations. The updated result, employing the most recent HFLAV strange BFs and the 3-loop-truncated fixed-order-perturbation-theory (FOPT) prescription favored by comparisons with lattice data [4] for the $D = 2$ OPE series, is[3]

$$|V_{us}| = 0.2219(22)\,. \tag{9}$$

This agrees well with the result, $|V_{us}| = 0.2233(6)$, obtained from $K_{\ell 3}$ using the most recent $n_f = 2+1+1$ lattice determination of $f_+(0)$ [27], and is within $\sim 1.6\sigma$ of the expectations of 3-family unitarity.

It is worth emphasizing that weighted us residual-mode contributions represent 5.3%, 6.0% and 6.6%, respectively, of the w_2-, w_3- and w_4-weighted us spectral integrals for $s_0 \simeq m_\tau^2$. With $> 25\%$ errors on these contributions, the uncertainties in the old, low-statistics residual-mode distribution represent a major stumbling block to any significant improvement in the experimental error on the FB FESR determination. With current experimental precision, as discussed in the next section, there are no obvious options available for significantly reducing the errors on the FB FESR determination. Significant improvement in the determination of $|V_{us}|$ using hadronic τ decay data thus, at present, requires an alternate approach. Such an approach is provided by the lattice-based analysis reviewed in the next section.

4 A lattice-based determination of $|V_{us}|$ from strange hadronic τ decay data

With the still-significant errors on the us residual-mode distribution representing a significant impediment to reducing the experimental errors on the FB FESR determination, it is useful to

[2]The agreement of the w_τ curve with the other dashed curves is as expected given the extremely good agreement of the dashed w_2 and w_3 curves and the fact that w_τ a linear combination of w_2 and w_3; for this reason, the agreement of the w_τ curve with the others should not be interpreted as providing a further independent self-consistency check on the alternate analysis.

[3]For completeness, we note that, if one employs instead the 3-loop-truncated contour-improved-perturbation-theory (CIPT) prescription for the integrated $D = 2$ series, one obtains $|V_{us}| = 0.2218(22)$. Similarly, if one uses the less precise single-prong $\tau \to \pi(K)\nu_\tau$ BFs to evaluate the π and K pole contributions, one obtains $|V_{us}| = 0.2212(23)$ using the FOPT and 0.2211(23) using the CIPT prescription. The latter results lie 0.0017 higher than the conventional implementation results obtained using the identical input data [3].

attempt to find alternate determinations in which contributions from the high-s part of the us distribution play a reduced role. The only easy way to accomplish this reduction within the FB FESR framework would be to consider weights containing additional powers of $1-y$, which would further suppress contributions from the upper part of the spectrum, near $y = s/s_0 = 1$. However, adding additional powers of $1-y$ increases the degree of the weights, and forces one to fit additional higher D OPE condensates. The current us data is insufficiently precise to allow progress to be made in reducing the experimental errors on $|V_{us}|$ via this strategy.

An alternate method for reducing the relative role of residual-mode contributions is provided by the lattice-based approach of Ref. [5]. The basic idea is to consider generalized dispersion relations for the us V+A HVP combination, $\tilde{\Pi}_{us;V+A}(Q^2)$, with $Q^2 = -s$, whose spectral function is the linear combination, $\tilde{\rho}_{us;V+A}(s)$, defined in Eq. (2). Explicitly,

$$\tilde{\Pi}_{us;V+A}(Q^2) \equiv \left(1-2\frac{Q^2}{m_\tau^2}\right)\Pi^{(1)}_{us;V+A}(s) + \Pi^{(0)}_{us;V+A}(s)\,. \tag{10}$$

The product $|V_{us}|^2\tilde{\rho}_{us;V+A}(s)$ is directly determinable from $dR_{us;V+A}/ds$ and grows as $\sim s$ as $s \to \infty$. Defining weights

$$\omega_N(s) \equiv \frac{1}{\prod_{k=1}^{N}\left(s+Q_k^2\right)}\,, \tag{11}$$

with $0 < Q_k^2 < Q_{k+1}^2$, which have, by construction, poles at spacelike $s = -Q_k^2$, it follows that, for $N \geq 3$, $\tilde{\Pi}_{us;V+A}(Q^2)$ satisfies the convergent dispersion relation

$$\begin{aligned}\int_0^\infty \tilde{\rho}_{us;V+A}(s)\omega_N(s)ds &= \sum_{k=1}^{N}\underset{s=-Q_k^2}{\mathrm{Res}}\left[\tilde{\Pi}_{us;V+A}(-s)\omega_N(s)\right]\\ &= \sum_{k=1}^{N}\frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j\neq k}\left(Q_j^2-Q_k^2\right)} \equiv \tilde{F}_{\omega_N}\,.\end{aligned} \tag{12}$$

Choosing a uniform spacing Δ of the pole locations, the weights ω_N can be characterized by the spacing, Δ, the number of poles, N, and the pole-interval midpoint, $C = (Q_1^2 + Q_N^2)/2$. With results for $|V_{us}|$ found to be insensitive to modest changes of Δ, we choose to employ $\Delta = 0.2/(N-1)$ GeV^2, ensuring ω_N with different N but the same C have poles spanning the same Q^2 range. C and N were varied to minimize the error on $|V_{us}|$.

For large enough N, and keeping all Q_k^2 below ~ 1 GeV^2, spectral integral contributions from the higher-s, larger-error part of the experimental distribution as well as from $s > m_\tau^2$ can be strongly suppressed. Increasing N lowers the error of the LHS in Eq. 12, but, through increased cancellation, increases the relative error on the RHS. Discretization errors also grow with increasing C. For $N = 4$, for example, they become the largest source of lattice error beginning at $C \sim 0.8$ GeV^2 [5].

The RHS of Eq. (12) is determined by the values of the HVP combination $\tilde{\Pi}_{us;V+A}$ at the Euclidean locations $Q^2 = Q_k^2$, and can be measured on the lattice, obviating the need for the OPE approximation, while $dR_{us;V+A}/ds$ can used to fix, up to the unknown factor $|V_{us}|^2$, the spectral integral contributions to the LHS for s up to $s = m_\tau^2$. With $s > m_\tau^2$ contributions approximated using pQCD, one has,

$$|V_{us}| = \sqrt{\tilde{R}_{us;w_N}\Big/\left(\tilde{F}_{\omega_N} - \int_{m_\tau^2}^{\infty}\tilde{\rho}_{us}^{\,\mathrm{pQCD}}(s)\omega_N(s)ds\right)}\,, \tag{13}$$

where $\tilde{R}_{w_N} \equiv \frac{m_\tau^2}{12\pi^2 S_{EW}}\int_0^{m_\tau^2}\frac{1}{(1-y_\tau)^2}\frac{dR_{us;V+A}(s)}{ds}\omega_N(s)ds$.

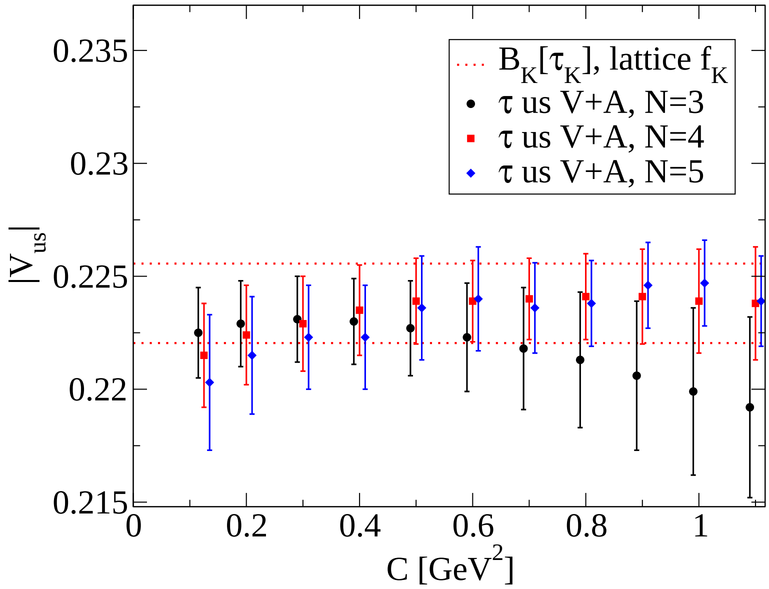

Figure 3: *Lattice-based results for* $|V_{us}|$ *as a function of* C *for* $N = 3$, 4 *and* 5. *The dashed lines show, for comparison, the bounds on* $|V_{us}|$ *obtained using the measured single-prong* τ *BF* B_K *together with the RBC/UKQCD lattice result for* f_K.

The *us* V+A HVPs which determine $\tilde{F}_{\omega_N}$ in Eq. (13) have been evaluated using results from the two near-physical quark-mass, $n_f = 2+1$, $48^3 \times 96$ and $64^3 \times 128$ Möbius domain wall fermion ensembles of Ref. [28]. Slight mistunings of the u, d and s quark masses were corrected by measuring HVPs with partially quenched physical valence masses. All-mode averaging [29] was used throughout to reduce costs.

From studies of finite volume (FV) effects in the lattice determination of the analogous electromagnetic current HVP entering the lattice calculation of the light-quark contribution to the anomalous magnetic moment of the muon [30,31], one expects FV effects to be very small in the present analysis so long as Q_1^2 is kept greater than ~ 0.2 GeV^2. A further suppression of FV effects is also expected here since the mass of the $K\pi$ state expected to dominate FV effects for the *us* V+A HVP is much larger than that of the 2π state which dominates FV effects for the electromagnetic HVP. The impact of $K\pi$-induced FV effects on $|V_{us}|$ is estimated using ChPT [5], and found to be $\sim 0.1\%$ for the optimized version of the lattice analysis.

Further details of the evaluation of the lattice HVP combinations $\tilde{F}_{\omega_N}$, together with a detailed error budget, may be found in Ref. [5]. Here we provide only an update of that analysis, employing the recently updated HFLAV *us* BF input reported by A. Lusiani at this meeting [3]. Currently, statistical errors dominate at lower C and discretization errors at larger C. Using, as in Ref. [5], the HFLAV B_K result to evaluate the K pole contribution, the optimized version of this update, obtained for $N = 4$ and $C \simeq 0.7$ GeV^2, is

$$|V_{us}| = 0.2240(13)_{exp}\,(13)_{latt}\,. \tag{14}$$

Employing $K_{\mu 2}$+SM expectation for B_K yields instead $0.2254(10)_{exp}\,(13)_{latt}$. The lattice error is comparable to the current experimental error, and straightforwardly improvable should future experimental improvements warrant it.

In Fig. 3, we present results for $|V_{us}|$, now as a function of C, for $N = 3$, 4, 5. We include some low C points, where estimated FV effects reach $\sim 0.2\%$, and some points with C near 1 GeV^2, where, for $N = 3$, residual-mode and pQCD contributions are less strongly suppressed.

The dashed horizontal lines show, for comparison, the band corresponding to the result for $|V_{us}|$ obtained using the same B_K together with the RBC/UKQCD determination of f_K.

The results of the inclusive lattice analysis for different N are in excellent agreement for intermediate C, and, moreover, show very good C independence in this region. The falloff at higher C for $N = 3$ is exactly what one would expect, given the less effective suppression of contributions from the large-s residual-mode and pQCD regions, were there to be missing strength in the old low-statistics, 1999 ALEPH results in the high-multiplicity region. Such missing strength would similarly lower the results obtained from FB FESR analyses. The $N = 3$ lattice results make it very likely this is the case. The results also show that the more efficient suppression of contributions from the high-multiplicity region obtained at higher C for $N = 4$ and 5 can be accomplished without blowing up lattice errors. This represents a major advantage of the lattice approach.

5 Conclusions

We have shown that, even after taking into account the most recent updates of the results for exclusive strange BFs, the assumptions for the effective $D = 6$ and 8 OPE condensates underlying the conventional implementation of the FB FESR framework still produce results for $|V_{us}|$ displaying unphysical s_0- and weight-dependence. The conventional implementation assumptions must therefore be abandoned going forward. The alternate FB FESR implementation [4], in which the s_0 dependence of weighted spectral integrals is used to fit not just $|V_{us}|$ but also the relevant $D > 4$ OPE condensates in the same analysis was shown to cure these problems and produce results for $|V_{us}|$ which, as when earlier versions of the BF input were used, are once more ~ 0.0020 higher than those obtained from the same data using conventional implementation assumptions.

We have also outlined an alternate lattice-based dispersive approach to determining $|V_{us}|$ which requires as input only strange hadronic τ decay data. This approach employs precision lattice data, rather than the OPE approximation, as theoretical input. We have shown that weights exist which strongly reduce the relative role of contributions to the weighted spectral integrals from the regions where data errors are large without at the same time blowing up the associated lattice statistical and systematic errors. The lattice approach yields experimental errors on $|V_{us}|$ considerably smaller than those obtained from the alternate FB FESR framework, and has theory errors that can be straightforwardly reduced through higher statistics and larger volume lattice studies once improved experimental data warrants these improvements. The lattice-based results for $|V_{us}|$ are in good agreement with those of determinations from other sources, including the expectations of 3-family unitarity.

Acknowledgements

We thank the RBC/UKQCD Collaboration for fruitful discussion and support and the Centro de Ciencias de Benasque Pedro Pascual for hospitality at the workshop “High-precision QCD at low energy”, where this project was initiated.

Funding information Research leading to the results reported here was supported by funding from the European Research Council under the European Union’s Seventh Framework Programme (Grant No. FP7/2007-2013)/ERC Grants No. 279757 and No. STFC ST/P000711/1. Computing time granted through the STFC-funded DiRAC facility (Grants No. ST/K005790/1, No. ST/K005804/1, No. ST/K000411/1, and No. ST/H008845/1) is also gratefully acknowledged. Software used includes the CPS QCD code, supported in part by the U.S. DOE SciDAC program, and the BAGEL assembler kernel generator for high-performance optimized kernels and fermion solvers [32]. This work was also supported by resources provided by the Scientific Data and Computing Center (SDCC) at Brookhaven National Laboratory, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy and by JSPS KAKENHI Grants No. 17H02906 and No. 17K14309. HO is supported in part by RIKEN Special Postdoctoral Researcher program, Nara Women's University Intramural Grant for Project Research and RJH, RL, and KM by grants from the Natural Science and Engineering Research Council of Canada. CL acknowledges support through a DOE Office of Science Early Career Award and by U.S. DOE Contract No. DESC0012704 (BNL). AP is supported in part by UK STFC Grants No. ST/L000458/1 and No. ST/P000630/1. The work of JZ was supported by Australian Research Council grants FT100100005 and DPI140103067.

References

[1] E. Gámiz, J. Prades, M. Jamin, F. Schwab and A. Pich, *Determination of m_s and $|V_{us}|$ from hadronic τ decays*, J. High Energ. Phys. **01**, 060 (2003), doi:10.1088/1126-6708/2003/01/060.

[2] A. Pich, *Theoretical progress on the Vus determination from tau decays*, Proc. Sci. **046-KAON**, 008 (2008), doi:10.22323/1.046.0008.

[3] A. Lusiani, *HFLAV τ branching fractions fit and measurements of V_{us} with τ lepton data* (2018), arXiv:1811.06470.

[4] R. J. Hudspith, R. Lewis, K. Maltman and J. Zanotti, *A resolution of the inclusive flavor-breaking $\tau|V_{us}|$ puzzle*, Phys. Lett. B **781**, 206 (2018), doi:10.1016/j.physletb.2018.03.074.

[5] P. A. Boyle et al., *Novel $|V_{us}|$ determination using inclusive strange τ decay and lattice hadronic vacuum polarization functions*, Phys. Rev. Lett. **121**, 202003 (2018), doi:10.1103/PhysRevLett.121.202003.

[6] Y. Amhis, et al., *Averages of b-hadron, c-hadron, and τ-lepton properties as of summer 2016*, Eur. Phys. J. C **77**, 895 (2017), doi:10.1140/epjc/s10052-017-5058-4.

[7] M. Antonelli, V. Cirigliano, A. Lusiani and E. Passemar, *Predicting the τ strange branching ratios and implications for V_{us}*, J. High Energ. Phys. **10**, 070 (2013), doi:10.1007/JHEP10(2013)070.

[8] Y.-S. Tsai, *Erratum: Decay correlations of heavy leptons in $e^+ + e^- \to l^+ + l^-$*, Phys. Rev. D **13**, 771 (1976), doi:10.1103/PhysRevD.13.771.

[9] J. Erler, *Electroweak radiative corrections to semileptonic τ decays*, Rev. Mex. Fis. **50**, 200 (2004).

[10] M. Jamin, J. Antonio Oller and A. Pich, *S-wave $K\pi$ scattering in chiral perturbation theory with resonances*, Nucl. Phys. B **587**, 331 (2000), doi:10.1016/S0550-3213(00)00479-X.

[11] K. Maltman and J. Kambor, *Decay constants, light quark masses, and quark mass bounds from light quark pseudoscalar sum rules*, Phys. Rev. D **65**, 074013 (2002), doi:10.1103/PhysRevD.65.074013.

[12] D. Epifanov et al. [Belle Collaboration], *Study of $\tau^- \to K_S \pi^- \nu_\tau$ decay at Belle* Phys. Lett. B, **654**, 65 (2007) doi:10.1016/j.physletb.2007.08.045.

[13] See, e.g., T. Lück's presentation at ICHEP 2018.

[14] I.M. Nugent et al. [BaBar Collaboration], *Invariant mass spectra of $\tau^- \to h^- h^- h^+ \nu_\tau$ decays*, Nucl. Phys. B Proc. Suppl., **253-255**, 38 (2014) doi:10.1016/j.nuclphysbps.2014.09.010.

[15] S. Ryu, et al. [Belle Collaboration], *Measurement of the branching fractions and mass spectra for τ lepton decays including K_S^0 at Belle*, Nucl. Phys. B Proc. Suppl., **253-255**, 33 (2014) doi:10.1016/j.nuclphysbps.2014.09.009;

[16] B. Aubert et al., *Exclusive branching-fraction measurements of semileptonic τ decays into three charged hadrons, into $\phi\pi^-\nu_\tau$, and into $\phi K^-\nu_\tau$*, Phys. Rev. Lett. **100**, 011801 (2008), doi:10.1103/PhysRevLett.100.011801.

[17] M. J. Lee et al., *Measurement of the branching fractions and the invariant mass distributions for $\tau^- \to h^- h^+ h^- \nu_\tau$ decays*, Phys. Rev. D **81**, 113007 (2010), doi:10.1103/PhysRevD.81.113007.

[18] R. Barate et al., *Study of τ decays involving kaons, spectral functions and determination of the strange quark mass*, Eur. Phys. J. C **11**, 599 (1999), doi:10.1007/s100529900208.

[19] M. Davier, A. Höcker, B. Malaescu, C. Z. Yuan and Z. Zhang, *Update of the ALEPH non-strange spectral functions from hadronic τ decays*, Eur. Phys. J. C **74**, 2803 (2014), doi:10.1140/epjc/s10052-014-2803-9.

[20] J. C. Hardy and I. S. Towner, *Superallowed $0^+ \to 0^+$ nuclear β decays: 2014 critical survey, with precise results for V_{ud} and CKM unitarity*, Phys. Rev. C **91**, 025501 (2015), doi:10.1103/PhysRevC.91.025501.

[21] S. Aoki et al., *Review of lattice results concerning low-energy particle physics*, Eur. Phys. J. C **77**, 112 (2017), doi:10.1140/epjc/s10052-016-4509-7.

[22] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, *Strange-quark mass from tau-lepton decays with $O(\alpha_s^3)$ accuracy*, Phys. Rev. Lett. **95**, 012003 (2005), doi:10.1103/PhysRevLett.95.012003.

[23] K. G. Chetyrkin and A. Kwiatkowski, *Mass corrections to the tau decay rate*, Z. Phys. C - Part. Fields **59**, 525 (1993), doi:10.1007/BF01498634.

[24] M. Tababashi, et al. [Particle Data Group], *Review of particle physics*, Phys. Rev. D **98**, 030001 (2018), doi:10.1103/PhysRevD.98.030001.

[25] C. McNeile, A. Bazavov, C. T. H. Davies, R. J. Dowdall, K. Hornbostel, G. P. Lepage and H. D. Trottier, *Direct determination of the strange and light quark condensates from full lattice QCD*, Phys. Rev. D **87**, 034503 (2013), doi:10.1103/PhysRevD.87.034503.

[26] D. Boito, M. Golterman, M. Jamin, A. Mahdavi, K. Maltman, J. Osborne and S. Peris, *Updated determination of α_s from τ decays*, Phys. Rev. D **85**, 093015 (2012), doi:10.1103/PhysRevD.85.093015.

[27] A. Bazavov et al., *$|V_{us}|$ from $K_{\ell 3}$ decay and four-flavor lattice QCD* (2018), arXiv:1809.02827.

[28] T. Blum et al. [RBC and UKQCD Collaborations], *Domain wall QCD with physical quark masses*, Phys. Rev. D **93**, 074505 (2016), doi:10.1103/PhysRevD.93.074505.

[29] T. Blum, T. Izubuchi and E. Shintani, *New class of variance-reduction techniques using lattice symmetries*, Phys. Rev. D **88**, 094503 (2013), doi:10.1103/PhysRevD.88.094503.

[30] C. Aubin, T. Blum, P. Chau, M. Golterman, S. Peris and C. Tu, *Finite-volume effects in the muon anomalous magnetic moment on the lattice*, Phys. Rev. D **93**, 054508 (2016), doi:10.1103/PhysRevD.93.054508.

[31] J. Bijnens and J. Relefors, *Vector two-point functions in finite volume using partially quenched chiral perturbation theory at two loops*, J. High Energ. Phys. **12**, 114 (2017), doi:10.1007/JHEP12(2017)114.

[32] P. A. Boyle, *The BAGEL assembler generation library*, Comput. Phys. Commun. **180**, 2739 (2009), doi:10.1016/j.cpc.2009.08.010.