SciPost Phys. Proc. 14, 014 (2023)

Comments on the negative grade KdV hierarchy

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34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The construction of negative grade KdV hierarchy is proposed in terms of a Miura-gauge transformation. Such gauge transformation is employed within the zero curvature representation and maps the Lax operator of the mKdV into its couterpart within the KdV setting. Each odd negative KdV flow is obtained from an odd and its subsequent even negative mKdV flows. The negative KdV flows are shown to inherit the two different vacua structure that characterizes the associated mKdV flows.

Received 17-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.014

1 Introduction

Integrable models have been focus of considerable attention in the past few years. These are very peculiar two dimensional field theories admitting an infinite number of conservation laws and soliton solutions. The algebraic construction of integrable models has provided a series of important achievements allowing their construction and classification in terms of the decomposition of the affine algebra into graded subspaces. Structural connection and the derivation of many properties such as the construction of conservation laws and soliton solutions, can be set from the zero curvature representation [1], [2]. In particular the mKdV hierarchy, based on the affine $sl(2)$ algebra, provides the simplest example of systematic construction of a series of evolution equations associated to a universal object called Lax operator. For the mKdV case the relevant decomposition occurs according to the principal gradation. Explicit constructions for positive and negative graded sub-hierarchies have been obtained. The positive flows are known to be labelled by odd numbers whilst there are no restriction for the negative flows [3].

An interesting relation between the KdV and mKdV hierarchies can be realised by the Miura transformation which maps one hierarchy into the other. In ref. [4], [5] we have related the two hierarchies by a gauge transformation that maps one Lax operator into the other. Such *Miura-gauge transformation* acting upon the zero curvature maps the flows from one hierarchy into the other. For the positive sub-hierarchy the mapping is one to one, i.e., each flow equation of mKdV is mapped into its counterpart within the KdV hierarchy. However this is not true for the negative KdV sub-hierarchy. In sec. 3 we argue that only odd flows are consistent for the KdV hierarchy and since there are even and odd flows within the negative mKdV side, there should be a mapping of a pair of mKdV flows into a single KdV flow. This is indeed true, in sect. 4 we construct these mappings and show that an odd and its subsequent even mKdV flows can be mapped into a single KdV flow. An interesting point to mention is that odd mKdV flows admit only *zero vacuum* whilst the even admit strictly *non-zero vacuum* solutions and the associated KdV flow ends up inheriting both types of structure.

2 mKdV negative hierarchy

In this section let us review the construction of mKdV hierarchy within the algebraic formalism. Consider the affine $\mathcal{G} = \hat{sl}(2)$ centerless Kac-Moody algebra generated by

$$h^{(m)} = \lambda^m h^{(0)}, \quad E_{\pm\alpha}^{(m)} = \lambda^m E_{\pm\alpha}^{(0)}, \quad \text{with} \quad \lambda \in \mathrm{C}, \quad \text{and} \quad n \in \mathrm{Z}, \tag{1}$$

satisfying the following algebra

$$\left[h^{(m)}, E_{\pm\alpha}^{(n)}\right] = \pm 2E_{\pm\alpha}^{(m+n)}, \quad \left[E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}\right] = h^{(m+n)}. \tag{2}$$

Introduce the principal grading operator

$$Q_p = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h, \tag{3}$$

that decomposes the affine algebra into graded subspaces, i.e., $\mathcal{G} = \bigoplus_i \mathcal{G}_i$ with

$$\left[Q_p, \mathcal{G}_a\right] = a\mathcal{G}_a, \qquad [\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b}, \quad a, b \in Z, \tag{4}$$

where, for $\mathcal{G} = \hat{sl}(2)$,

$$\mathcal{G}_{2n} = \left\{h^{(n)} = \lambda^n h\right\}, \qquad \mathcal{G}_{2n+1} = \left\{\lambda^n \left(E_\alpha + \lambda E_{-\alpha}\right), \lambda^n \left(E_\alpha - \lambda E_{-\alpha}\right)\right\}. \tag{5}$$

A second important ingredient is the choice of a constant grade one element $E^{(1)} \in \mathcal{G}_1$

$$E^{(1)} = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}, \tag{6}$$

such that it decomposes the affine algebra as $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$, where $\mathcal{K}$ is the *Kernel* of $E^{(1)}$:

$$\mathcal{K}_E = \left\{y \in \mathcal{K}, [y, E^{(1)}] = 0\right\} = \left\{E^{(2n+1)} \equiv E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}\right\} \in \mathcal{G}_{2n+1}, \tag{7}$$

and $\mathcal{M}$ is its complement subspace. We now define the spatial Lax operator to be an universal algebraic object within the whole hierarchy to be

$$A_x^{\mathrm{mKdV}}(\phi) \;=\; E^{(1)} + A^{(0)}(\phi) = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)} + \partial_x \phi\, h^{(0)} = \begin{pmatrix} \partial_x \phi & 1 \\ \lambda & -\partial_x \phi \end{pmatrix}, \tag{8}$$

where $v(x, t_{-N}) = \partial_x \phi$ is the field of the theory. We are interested in the negative time flows generated by the temporal Lax operator component of the form [3]

$$A_{t_{-N}}^{\mathrm{mKdV}} = D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}, \qquad N = 1, 2, \cdots, \tag{9}$$

where $D^{(i)} \in \mathcal{G}_i$. Thus, for a given integer N, the zero curvature equation

$$\left[\partial_x + E^{(1)} + A^{(0)},\, \partial_{t_{-N}} + D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}\right] = 0, \tag{10}$$

decomposes according to the grading structure, i.e.,

$$\left[A^{(0)}, D^{(-N)}\right] + \partial_x D^{(-N)} = 0, \tag{11}$$

$$\left[A^{(0)}, D^{(-N+1)}\right] + \left[E^{(1)}, D^{(-N)}\right] + \partial_x D^{(-N+1)} = 0, \tag{12}$$

$$\vdots \qquad \vdots$$

$$\left[E^{(1)}, D^{(-1)}\right] - \partial_{t_N} A^{(0)} = 0. \tag{13}$$

These eqns. can be solved grade by grade in order to determine $D^{(i)}$ and the evolution equation for $A^{(0)}(\phi)$ according to time t_{-N} is given by (13).

The simplest case is found by taking $N = 1$, leading to

$$A_{t_{-1}}^{\text{mKdV}} = e^{-2\phi} E_{\alpha}^{(-1)} + e^{2\phi} E_{-\alpha}^{(0)} = \begin{pmatrix} 0 & \lambda^{-1} e^{-2\phi} \\ e^{2\phi} & 0 \end{pmatrix}, \tag{14}$$

associated with the well known sinh-Gordon equation,

$$\phi_{x,t_{-1}} = e^{2\phi} - e^{-2\phi}. \tag{15}$$

Notice that $v = \partial_x \phi = v_0 = const.$ is the vacuum solution of (15) only if $v_0 = 0 \to \phi = 0$. It therefore follows that the sinh-Gordon equation only admits zero vacuum solution.

Considering now $N = 2$, we find

$$A_{t_{-2}}^{\text{mKdV}} = h^{(-1)} + \left(2e^{-2\phi} d^{-1}(e^{2\phi})\right) E_{\alpha}^{(-1)} - 2e^{2\phi} d^{-1}(e^{-2\phi}) E_{-\alpha}^{(0)}$$

$$= \begin{pmatrix} \lambda^{-1} & \lambda^{-1}\left(2e^{-2\phi} d^{-1}(e^{2\phi})\right) \\ -2e^{2\phi} d^{-1}(e^{-2\phi}) & -\lambda^{-1} \end{pmatrix}, \tag{16}$$

where we have denoted $d^{-1} f = \int_0^x f\, dx'$. It leads to the following nonlocal equation of motion

$$\phi_{x,t_{-2}} = -2\left(e^{-2\phi} d^{-1}(e^{2\phi}) + e^{2\phi} d^{-1}(e^{-2\phi})\right). \tag{17}$$

Notice that for $\phi = \phi_0 = v_0 x$ the following identity

$$e^{-2v_0 x} d^{-1}(e^{2v_0 x}) + e^{2v_0 x} d^{-1}(e^{-2v_0 x}) = 0, \tag{18}$$

holds only for $v_0 \neq 0$ and $v = v_0$ is the vacuum solution of (17), only if $v_0 \neq 0$. In fact, it can be shown that all models associated to negative even values of N only admit non-zero vacuum solutions [3]. Let us consider the zero curvature equation in the vacuum regime, i.e.,

$$\left[A_x^{vac} = E^{(1)} + v_0 h^{(0)},\ A_{t_{-N}}^{vac} = D_{vac}^{(-N)} + D_{vac}^{(-N+1)} + \cdots + D_{vac}^{(-1)}\right] = 0. \tag{19}$$

The lowest grade equation is

$$\left[v_0 h^{(0)},\ D_{vac}^{(-N)}\right] = 0. \tag{20}$$

Thus, if $v_0 \neq 0$ $D_{vac}^{(-N)}$ must commute with $h^{(0)}$ and therefore $D_{vac}^{(-N)} \in \mathcal{G}_{-2n}$ and $N = 2n$. Conversely if $v_0 = 0$ the lowest grade eqn. becomes

$$\left[E^{(1)},\ D_{vac}^{(-N)}\right] = 0, \tag{21}$$

implying $D_{vac}^{(-N)} \in \mathcal{K}_E$ and N is odd. Thus, the negative mKdV hierarchy splits in two sub-hierarchies: one even admitting strictly non-zero vacuum ($v_0 \neq 0$) and one odd admiting, only zero vacuum ($v_0 = 0$) solutions. The systematic construction of soliton solutions for the negative mKdV hierarchies was previously studied and can be written as follows (see [3]). For the odd sub-hierarchy the one soliton solution was constructed from dressing the zero vacuum solution ($A_x^{vac} = E^{(1)}$) leading to

$$v(x, t_{-2n+1}) = \partial_x \ln\left(\frac{1-\beta e^{2kx+\omega_{-2n+1}t_{-2n+1}}}{1+\beta e^{2kx+\omega_{-2n+1}t_{-2n+1}}}\right), \quad \text{with} \quad \omega_{-2n+1} = 2k^{-2n+1}. \tag{22}$$

For the even sub-hierarchy the constant value of the vacuum, v_0 introduces a deformation in the Lax operator, $A_x^{vac} = E^{(1)} + v_0 h^{(0)}$ and hence upon the dressing method. In [3] the solutions were constructed employing deformed vertex operators yielding for the one soliton solution,

$$v(x, t_{-2n}) = v_0 + \partial_x \ln\left(\frac{1+\beta(v_0-k)e^{2kx+\omega_{-2n}t_{-2n}}}{1+\beta(v_0+k)e^{2kx+\omega_{-2n}t_{-2n}}}\right), \quad \text{with} \quad \omega_{-2n} = \frac{2k}{v_0(k^2-v_0^2)^n}, \tag{23}$$

where in both cases, β is a free parameter.

3 KdV negative hierarchy

For the KdV hierarchy we employ the same algebraic structure of section 3, i.e., principal gradation, Q_p (3) and the constant grade one element $E^{(1)}$ (6). We propose the following Lax operator,

$$A_x^{\text{KdV}}(J) = E^{(1)} + A^{(-1)} = E_\alpha^{(0)} + E_{-\alpha}^{(1)} + J\, E_{-\alpha}^{(0)} = \begin{pmatrix} 0 & 1 \\ \lambda + J & 0 \end{pmatrix}, \tag{24}$$

where $A^{(-1)} = J\, E_{-\alpha}^{(0)} \in \mathcal{G}_{-1}$ and $J = J(x, \tau_N)$ is the field of KdV hierarchy. For the sub-hierarchy that leads to negative time-flow τ_{-N} , the temporal-part Lax operator is given by

$$A_{\tau_{-N}}^{\text{KdV}}(J) = \mathcal{D}^{(-N-2)} + \mathcal{D}^{(-N-1)} + \cdots + \mathcal{D}^{(-1)}, \tag{25}$$

where $\mathcal{D}^{(i)} \in \mathcal{G}_i$. The zero curvature decomposes according to the graded structure as

$$\begin{aligned} \left[A^{(-1)}, \mathcal{D}^{(-N-2)}\right] &= 0, & (26)\\ \partial_x \mathcal{D}^{(-N-2)} + \left[A^{(-1)}, \mathcal{D}^{(-N-1)}\right] &= 0, & (27)\\ \partial_x \mathcal{D}^{(-N-1)} + \left[E^{(1)}, \mathcal{D}^{(-N-2)}\right] + \left[A^{(-1)}, \mathcal{D}^{(-N)}\right] &= 0, & (28)\\ &\vdots & \\ \partial_x \mathcal{D}^{(-1)} + \left[E^{(1)}, D^{(-2)}\right] - \partial_{\tau_{-N}} A^{(-1)} &= 0, & (29)\\ \left[E^{(1)}, \mathcal{D}^{(-1)}\right] &= 0, & (30) \end{aligned}$$

which allows solving for all $\mathcal{D}^{(i)}$ and determines the equation of motion (29) according to τ_{-N}. Notice that the lowest grade equation (26) implies that $\mathcal{D}^{(-N-2)}$ is proportional to $E_{-\alpha}^{(-m)}$ and therefore $N = 2m - 1$. For this reason *all equations of motion for the KdV hierarchy are associated with odd temporal flows*, in contrast to the mKdV case, where there are equations of motion associated to both, even and odd flows.

The equations of motion for KdV hierarchy are more conveniently expresed in terms of non-local field $J(x, \tau_N) = \partial_x \eta(x, \tau_N)$. The first negative flow is obtained from zero curvature with $N = 1$, leads to the following temporal Lax operator,

$$\begin{aligned} A_{\tau_{-1}}^{\text{KdV}} &= \frac{\eta_{\tau_{-1}}}{2}\left(E_\alpha^{(-1)} + E_{-\alpha}^{(0)}\right) + \frac{\eta_{x,\tau_{-1}}}{4} h^{(-1)} + \frac{2\eta_x \eta_{\tau_{-1}} - \eta_{2x,\tau_{-1}}}{4} E_{-\alpha}^{(-1)} \\ &= \begin{pmatrix} \frac{\eta_{x,\tau_{-1}}}{4\lambda} & \frac{\eta_{\tau_{-1}}}{2\lambda} \\ \frac{2\eta_x\eta_{\tau_{-1}} - \eta_{2x,\tau_{-1}}}{4\lambda} + \frac{\eta_{\tau_{-1}}}{2} & -\frac{\eta_{x,\tau_{-1}}}{4\lambda} \end{pmatrix}, \end{aligned} \tag{31}$$

and equation of motion

$$4\eta_x\eta_{x,\tau_{-1}} + 2\eta_{2x}\eta_{\tau_{-1}} - \eta_{3x,\tau_{-1}} = 0\,. \tag{32}$$

This equation was first proposed in [6] using the inverse of recursion operator. Later in [7], its Hamiltonian and soliton solutions were discussed.

If we now take $N = 3$ in (25) and find for the associated temporal Lax operator,

$$\begin{aligned} A^{\text{KdV}}_{\tau_{-3}} &= \frac{\eta_{\tau_{-3}}}{2}\left(E^{(-1)}_{\alpha} + E^{(0)}_{-\alpha}\right) + \frac{\eta_{x,\tau_{-3}}}{4}\, h^{(-1)} - \frac{\mathcal{B}}{8}\left(E^{(-2)}_{\alpha} + E^{(-1)}_{-\alpha}\right) \\ &\quad + \frac{2\eta_{\tau_{-3}}\eta_x - \eta_{2x,\tau_{-3}}}{8}\, E^{(-1)}_{-\alpha} - \frac{\mathcal{B}_x}{16}\, h^{(-2)} + \frac{\mathcal{B}_{2x} - \eta_x\,\mathcal{B}}{8}\, E^{(-2)}_{-\alpha} \\ &= \begin{pmatrix} \frac{\eta_{x,\tau_{-3}}}{4\lambda} - \frac{\mathcal{B}_x}{16\lambda^2} & \frac{\eta_{\tau_{-3}}}{2\lambda} - \frac{\mathcal{B}}{8\lambda^2} \\ \frac{1}{2}\eta_{\tau_{-3}} + \frac{2\eta_{\tau_{-3}}\,\eta_x - \eta_{2x,\tau_{-3}} - \mathcal{B}}{8\lambda} + \frac{\mathcal{B}_{2x} - \eta_x\,\mathcal{B}}{8\lambda^2} & -\frac{\eta_{x,\tau_{-3}}}{4\lambda} + \frac{\mathcal{B}_x}{16\lambda^2} \end{pmatrix}, \end{aligned} \tag{33}$$

where

$$\mathcal{B} = d^{-1}(4\eta_x\eta_{x,\tau_{-3}} + 2\eta_{2x}\eta_{\tau_{-3}} - \eta_{3x,\tau_{-3}})\,. \tag{34}$$

The corresponding equation of motion is given by

$$\begin{aligned} &-\frac{1}{2}\eta_{5x,\tau_{-3}} + 4\eta_x\left(-2\eta_{x,\tau_{-3}}\eta_x + \eta_{3x,\tau_{-3}} - \eta_{2x}\eta_{\tau_{-3}}\right) + 5\eta_{2x}\eta_{2x,\tau_{-3}} \\ &\quad + 4\eta_{x,\tau_{-3}}\eta_{3x} + \eta_{4x}\eta_{\tau_{-3}} + \eta_{2x}\, d^{-1}\left(4\eta_x\eta_{x,\tau_{-3}} + 2\eta_{2x}\eta_{\tau_{-3}} - \eta_{3x,\tau_{-3}}\right) = 0\,. \end{aligned} \tag{35}$$

Notice that vacuum solution $\eta = \eta_0 =$ constant, either zero or non-zero, satisfy both equations of motion (32) and (35). Such behavior differs from the mKdV hierarchy where the equations of motion associated with odd-time flows are satisfied with zero vacuum and the even-time flows with non-zero vacuum (constant). This coalescence in vacuum solution presented in KdV hierarchy can be explained more generally from zero curvature projected around vacuum, i.e,

$$\left[A^{\text{KdV}}_x\big|_{\text{vac}},\, A^{\text{KdV}}_{\tau_{-N}}\big|_{\text{vac}}\right] = \left[E^{(1)} + \eta_0\, E^{(0)}_{-\alpha}, \mathcal{D}^{(-N-2)}_{\text{vac}} + \mathcal{D}^{(-N-1)}_{\text{vac}} + \cdots + \mathcal{D}^{(-1)}_{\text{vac}}\right] = 0\,. \tag{36}$$

Its lowest grade component leads to

$$\left[\eta_0\, E^{(0)}_{-\alpha}, \mathcal{D}^{(-N-2)}_{\text{vac}}\right] = \left[\eta_0\, E^{(0)}_{-\alpha},\, a_{-N-2}\, E^{(-1/2(N+1))}_{-\alpha}\right] = 0\,, \tag{37}$$

which is automatically satisfied no matter whether η_0 is zero or non-zero if $N = 2n - 1$. It therefore follows that the negative KdV hierarchy are associated to odd flows, $\tau_{-N} = \tau_{-2n-1}$ and admit both, zero and non-zero vacuum solutions.

4 Miura transformation and soliton solutions

In order to map the mKdV and KdV hierarchies let us consider the *Miura-gauge transformation* generated by (see [4], [5])

$$S_1 = e^{\phi_x E^{(0)}_{-\alpha}} = \begin{pmatrix} 1 & 0 \\ \phi_x & 1 \end{pmatrix}, \tag{38}$$

which maps the two Lax operators, A^{mKdV}_x into A^{KdV}_x of eqns. (8) and (24) respectively, i.e.,

$$A^{\text{KdV}}_x = S_1 A^{\text{mKdV}}_x S_1^{-1} + S_1\partial_x S_1^{-1} = E^{(0)}_{\alpha} + E^{(1)}_{-\alpha} + J E^{(0)}_{-\alpha}\,, \tag{39}$$

where

$$J(x,t) = \partial_x \eta(x,t) = (\phi_x)^2 - \phi_{2x}\,. \tag{40}$$

We now analyse how S_1 acts as a local gauge transformation upon A_t^{mKdV}. Let us consider first its action on an even grade element $D^{(-2n)} = c_{-n} h^{(-n)}$:

$$\begin{aligned} D^{(-2\mathrm{n})} &\rightarrow e^{\phi_x E_{-\alpha}^{(0)}} \left(c_{-n} h^{(-n)} \right) e^{-\phi_x E_{-\alpha}^{(0)}} + e^{\phi_x E_{-\alpha}^{(0)}} \partial_t \left(e^{-\phi_x E_{-\alpha}^{(0)}} \right) \\ &= \underbrace{c_{-n} h^{(-n)}}_{\mathcal{G}_{-2n}} + \underbrace{2c_{-n}\phi_x E_{-\alpha}^{(-n)}}_{\mathcal{G}_{-2n-1}} - \underbrace{\partial_t \phi_x E_{-\alpha}^{(0)}}_{\mathcal{G}_{-1}}\,. \end{aligned} \tag{41}$$

On the other hand, if we consider $D^{(-2n+1)} = a_{-n} E_{\alpha}^{(-n)} + b_{-n} E_{-\alpha}^{(-n+1)}$ under the local gauge generated by (38) we find

$$\begin{aligned} D^{(-2\mathrm{n}+1)} &\rightarrow e^{\phi_x E_{-\alpha}^{(0)}} \left(a_n E_{\alpha}^{(-n)} + b_n E_{-\alpha}^{(-n+1)} \right) e^{-\phi_x E_{-\alpha}^{(0)}} + e^{\phi_x E_{-\alpha}^{(0)}} \partial_t \left(e^{-\phi_x E_{-\alpha}^{(0)}} \right) \\ &= - \underbrace{a_n (\phi_x)^2 E_{-\alpha}^{(-n)}}_{\mathcal{G}_{-2n-1}} - \underbrace{a_n \phi_x h_1^{(-n)}}_{\mathcal{G}_{-2n}} + \underbrace{a_n E_{\alpha}^{(-n)} + b_n E_{-\alpha}^{(-n+1)}}_{\mathcal{G}_{-2n+1}} - \underbrace{\partial_t \phi_x E_{-\alpha}^{(0)}}_{\mathcal{G}_{-1}}\,. \end{aligned} \tag{42}$$

Thus, any even negative mKdV time flow of the form $A_{t_{-2n}}^{\mathrm{mKdV}} = D^{(-2n)} + D^{(-2n+1)} + \cdots + D^{(-1)}$is mapped into its KdV counterpart with the following graded structure,

$$\begin{aligned} A_{\tau_{-2n+1}}^{\mathrm{KdV}} &= e^{\phi_x E_{-\alpha}^{(0)}} \left(D^{(-2n)} + D^{(-2n+1)} + \cdots + D^{(-1)} \right) e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-2n}} E_{-\alpha}^{(0)} \\ &= \mathcal{D}^{(-2n-1)} + \mathcal{D}^{(-2n)} + \cdots + \mathcal{D}^{(-1)}\,. \end{aligned} \tag{43}$$

For odd negative mKdV time flow of the form $A_{t_{-2n+1}}^{\mathrm{mKdV}} = D^{(-2n+1)} + D^{(-2n+1)} + \cdots + D^{(-1)}$ will be mapped into

$$\begin{aligned} A_{\tau_{-2n+1}}^{\mathrm{KdV}} &= e^{\phi_x E_{-\alpha}^{(0)}} \left(D^{(-2n+1)} + D^{(-2n+2)} + \cdots + D^{(-1)} \right) e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-2n+1}} E_{-\alpha}^{(0)} \\ &= \mathcal{D}^{(-2n-1)} + \mathcal{D}^{(-2n)} + \mathcal{D}^{(-2n+1)} + \cdots + \mathcal{D}^{(-1)}\,. \end{aligned} \tag{44}$$

Since A_x^{KdV} is universal for both, even and odd KdV flows, the zero curvature representation (26) - (30) implies that $A_{t_{-2n+1}}^{\mathrm{mKdV}}$ and $A_{t_{-2n}}^{\mathrm{mKdV}}$ are transformed by the Miura-gauge transformation (38), into a single graded KdV structure $A_{\tau_{-2n+1}}^{\mathrm{KdV}}$ (43)-(44) (associated to flow τ_{-2n+1}). We therefore conclude that both *negative even and negative odd mKdV flows collapse within the same KdV odd flow*, i.e.,

$$t_{-2n+1}^{mKdV}, t_{-2n}^{mKdV} \quad \underset{S_1}{\Longrightarrow} \quad \tau_{-2n+1}^{KdV}\,. \tag{45}$$

Notice that this explains why each KdV negative flow admits both zero and non-zero vacuum solutions. They inherit the zero and the non-zero vacuum information from mKdV negative odd and its subsequent negative even flows respectively. Let us illustrate explicitly for the first two negative mKdV flows, namely, t_{-1} and t_{-2}.

For t_{-1}^{mKdV} the field $\phi = \phi(x, t_{-1})$ satisfies the sinh-Gordon eqn (15). We then have

$$\begin{aligned} A_{\tau_{-1}}^{\mathrm{KdV}} &= S_1 A_{t_{-1}}^{\mathrm{mKdV}} S_1^{-1} + S_1 \partial_{t_{-1}} S_1^{-1} \\ &= e^{\phi_x E_{-\alpha}^{(0)}} \left(e^{-2\phi} E_{\alpha}^{(-1)} + e^{2\phi} E_{-\alpha}^{(0)} \right) e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-1}} E_{-\alpha}^{(0)}\,, \end{aligned} \tag{46}$$

leading to

$$A_{\tau_{-1}}^{\mathrm{KdV}} = e^{-2\phi} \left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)} \right) + \frac{\eta_{x,t_{-1}}}{4} h^{(-1)} - (\phi_x)^2 e^{-2\phi} E_{-\alpha}^{(-1)}\,, \tag{47}$$

where we used the sinh-Gordon equation of motion, $\phi_{x,t_{-1}} = e^{2\phi} - e^{-2\phi}$ and the *Miura* transformation, $\eta_x = (\phi_x)^2 - \phi_{2x}$ to simplify some terms. Note that in terms of zero curvature, we had already constructed $A^{\mathrm{KdV}}_{\tau_{-1}}$ given in (31),

$$A^{\mathrm{KdV}}_{\tau_{-1}} = \frac{\eta_{\tau_{-1}}}{2}\left(E^{(-1)}_{\alpha} + E^{(0)}_{-\alpha}\right) + \frac{\eta_{x,\tau_{-1}}}{4}\, h^{(-1)} + \frac{2\eta_x\eta_{\tau_{-1}} - \eta_{2x,\tau_{-1}}}{4}\, E^{(-1)}_{-\alpha}\,. \tag{48}$$

From the condition for eqns (47) and (48) to agree we find

$$\eta_{\tau_{-1}} = 2\cdot e^{-2\phi(x,t_{-1})}\,. \tag{49}$$

On the other hand, if we now consider t_{-2}^{mKdV} with $\phi = \phi(x,t_{-2})$ satisfiyng (17), we get from the Miura gauge transformation $A^{\mathrm{KdV}}_{\tau_{-1}} = S_1 A^{\mathrm{mKdV}}_{t_{-2}} S_1^{-1} + S_1\partial_{t_{-2}} S_1^{-1}$,

$$A^{\mathrm{KdV}}_{\tau_{-1}} = e^{\phi_x E^{(0)}_{-\alpha}}\left(h^{(-1)} + 2e^{-2\phi}d^{-1}(e^{2\phi})E^{(-1)}_{\alpha} - 2e^{2\phi}d^{-1}(e^{-2\phi})E^{(0)}_{-\alpha}\right)e^{-\phi_x E^{(0)}_{-\alpha}} - \phi_{x,t_{-2}}E^{(0)}_{-\alpha}\,,$$

leading to

$$A^{\mathrm{KdV}}_{\tau_{-1}} = 2e^{-2\phi}d^{-1}(e^{2\phi})\left(E^{(-1)}_{\alpha} + E^{(0)}_{-\alpha}\right) + \frac{\eta_{x,t_{-2}}}{4}h^{(-1)} + 8(\phi_x - \phi_x^2 e^{-2\phi}d_x^{-1}e^{2\phi})E^{(-1)}_{-\alpha}\,, \tag{50}$$

where we used the equation of motion for t_{-2}^{mKdV} (17) and *Miura transformation*. Thus, (50) only agrees with (48) provided

$$\eta_{\tau_{-1}} = 2\cdot 2e^{-2\phi(x,t_{-2})}d^{-1}(e^{2\phi(x,t_{-2})})\,. \tag{51}$$

Notice that the same $A^{\mathrm{KdV}}_{\tau_{-1}}$ is written in two different ways, one in terms of the sinh-Gordon field $\phi(x,t_{-1})$ given by (47)-(49) and another, in terms of solution of eqn. (17) namely $\phi(x,t_{-2})$ in (50)-(51) . This can be checked explicitly with solutions given in (22) and (23) for $n = 1$.

5 Conclusion

We have therefore concluded from the above simple example that solutions of the KdV equation associated to the time flow τ_{-1} inherit different vacuum structures from a pair of mKdV solutions (via Miura transformation) . The first associated to mKdV flow t_{-1}, eqn. (15) (with zero vacuum) satisfying (49) and the second associated to mKdV flow t_{-2}, eqn. (17) (with non-zero vacuum) satisfying (51). The argument can be easily generalized for higher flows, and each KdV flow admits both, zero and non-zero vaccum solutions. They are constructed from pairs of subsequent of mKdV flows each of them admiting different vacuum structures. We expect to report in a future publication the generalization of our construction to the A_r - KdV hierarchy employing the gauge-Miura transformation proposed in [5]. We also expect to discuss the systematic construction of soliton (multisoliton) solutions and their vacuum structure in terms of vertex operators and its deformations along the lines of refs. [3], [4].

Acknowledgements

Funding information JFG and AHZ thank CNPq and Fapesp for support. YFA thanks São Paulo Research Foundation (FAPESP) for financial support under grant #2021/00623−4 and GVL is supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

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