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A general approach to noncommutative spaces from Poisson homogeneous spaces: Applications to (A)dS and Poincaré

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Abstract

In this contribution we present a general procedure that allows the construction of noncommutative spaces with quantum group invariance as the quantization of their associated coisotropic Poisson homogeneous spaces coming from a coboundary Lie bialgebra structure. The approach is illustrated by obtaining in an explicit form several noncommutative spaces from (3+1)D (A)dS and Poincaré coisotropic Lie bialgebras. In particular, we review the construction of the κ-Minkowski and κ-(A)dS spacetimes in terms of the cosmological constant Λ. Furthermore, we present all noncommutative Minkowski and (A)dS spacetimes that preserve a quantum Lorentz subgroup. Finally, it is also shown that the same setting can be used to construct the three possible 6D κ-Poincaré spaces of time-like worldlines. Some open problems are also addressed.

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1 Introduction

The aim of this contribution is twofold. Firstly, we present a systematic "six-step" procedure that allows the construction of different noncommutative spaces with a common underlying homogeneous space G/H where G is a Lie group and H is the isotropy Lie subgroup. The approach requires starting with a coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$ such that $\mathfrak{g}$ is the Lie algebra of G and δ is the cocommutator obtained from a classical r-matrix r [1,2]. The main requirement for our development is that δ must satisfy the coisotropic condition $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$ with respect to the isotropy Lie algebra $\mathfrak{h}$ of H [3–5]. Since coboundary Lie bialgebras are the tangent counterpart of Poisson–Lie groups (G, Π) with a Poisson structure Π, the latter just comes from the so-called Sklyanin bracket in this quantum group setting. Therefore, this leads to coisotropic Poisson homogeneous spaces $(G/H, \pi)$ where the Poisson structure π on G/H is obtained via canonical projection of the Poisson–Lie structure Π on the Lie group G. The quantization of $(G/H, \pi)$ gives rise to the corresponding noncommutative space.

Secondly, we illustrate this approach by reviewing, from this general perspective, several very recent noncommutative spaces that could be of interest in a quantum gravity framework [6]. In particular, throughout the paper we will focus on the (3+1)D (Anti-)de Sitter (in short (A)dS)) and Poincaré Lie groups and their associated (3+1)D homogeneous spacetimes together with the 6D Poincaré homogeneous space of time-like geodesics.

The structure of the paper is as follows. In the next section we recall the main necessary mathematical notions and geometric structures. And, as the main result, we present the six-step approach to noncommutative spaces from coisotropic Poisson homogeneous spaces. In Section 3 we apply this procedure in order to recover the well-known κ-Minkowski spacetime [7] as well as the (3+1)D κ-(A)dS spacetimes [8]. In Section 4, we present other noncommutative (3+1)D Minkowski and (A)dS spacetimes, which are quite different from the usual κ-spacetimes ones, by requiring to preserve a quantum Lorentz subalgebra [9].

Now, we stress that in many proposals to quantum gravity theories from quantum groups their cornerstone is usually focused on the (3+1)D noncommutative spacetimes (in general, the κ-Minkowski spacetime), forgetting the role that 6D quantum spaces of geodesics could be played. In fact, in our opinion, any consistent theory should consider, simultaneously, both a (3+1)D noncommutative spacetime and a 6D noncommutative space of worldlines. With this idea and by taking into account the very same six-step procedure of Section 2, we construct the 6D κ-Poincaré quantum space of time-like geodesics [10] in Section 5. Furthermore, there exist two other types of κ-Poincaré deformations beyond the usual "time-like" one; namely, the "space-like" and the "light-like" deformations (see [11,12] and references therein). Thus, we also present in Section 5 these two remaining and very recently obtained 6D noncommutative Poincaré spaces of geodesics [12].

Finally, some remarks and open problems are addressed in the last section.

2 Noncommutative spaces from Poisson homogeneous spaces

In this section, we firstly review the basic mathematical tools necessary for the paper and, secondly, we present a general approach that allows one to construct noncommutative spaces from coisotropic Poisson homogeneous spaces.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ of dimension d. We consider a decomposition of $\mathfrak{g}$, as a vector space, given by the sum of two subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}, \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}. \tag{1}$$

A generic ℓ-dimensional (ℓD) homogeneous space is defined as the left coset space

$$M^\ell = G/H\,, \tag{2}$$

where H is the $(d-\ell)$D isotropy subgroup with Lie algebra $\mathfrak{h}$ (1). Hence we can identify the tangent space at every point $m = gH \in M^\ell$, $g \in G$, with the subspace $\mathfrak{t}$:

$$T_m(M^\ell) = T_{gH}(G/H) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{t} = \operatorname{span}\{T_1, \ldots, T_\ell\}\,. \tag{3}$$

The generators of the isotropy subalgebra $\mathfrak{h}$ keep a point on M^ℓ invariant, the origin O, playing the role of rotations around O, while the ℓ generators belonging to $\mathfrak{t}$ move O along ℓ basic directions, thus behaving as translations on M^ℓ. The local coordinates $(t^1, \ldots, t^\ell)$ associated with the translation generators of $\mathfrak{t}$ (3) give rise to ℓ coordinates on M^ℓ.

A Poisson–Lie (PL) group is a pair (G, Π) where G is a Lie group and Π is a Poisson structure such that the Lie group multiplication $\mu : G \times G \to G$ is a Poisson map with respect to Π on G and the product Poisson structure $\Pi_{G\times G} = \Pi \oplus \Pi$ on $G \times G$. The relation between the Poisson bivector field and the Poisson bracket is given by

$$(\mathrm{d}f_1 \otimes \mathrm{d}f_2)\Pi = \{f_1, f_2\}_\Pi\,. \tag{4}$$

A Poisson manifold (M, π) is a manifold M endowed with a Poisson structure π on M. A Poisson homogeneous space (PHS) for a PL group (G, Π) is a Poisson manifold (M, π) which is endowed with a transitive group action $\alpha : (G \times M, \Pi \oplus \pi) \to (M, \pi)$ which is a Poisson map. Throughout this paper we shall consider that the manifold is a homogeneous space $M \equiv M^\ell = G/H$ (2). Moreover, we restrict to the case when the Poisson structure π on M^ℓ can be obtained by canonical projection of the PL structure Π on G.

Next, a Lie bialgebra is a pair $(\mathfrak{g}, \delta)$ where $\mathfrak{g}$ is a Lie algebra and $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a linear map called the cocommutator satisfying the following two conditions [2]:
(i) δ is a 1-cocycle:

$$\delta\big([X_i, X_j]\big) = [\delta(X_i), X_j \otimes 1 + 1 \otimes X_j] + \big[X_i \otimes 1 + 1 \otimes X_i\,,\, \delta(X_j)\big]\,, \qquad \forall\, X_i, X_j \in \mathfrak{g}\,. \tag{5}$$

(ii) The dual map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on the dual Lie algebra $\mathfrak{g}^*$ of $\mathfrak{g}$.

Coboundary Lie bialgebras [1,2] are those provided by a skewsymmetric classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ in the form

$$\delta(X_i) = [X_i \otimes 1 + 1 \otimes X_i,\, r]\,, \qquad \forall\, X_i \in \mathfrak{g}\,, \tag{6}$$

such that r must be a solution of the modified classical Yang–Baxter equation (mCYBE)

$$[X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i, [[r, r]]\,] = 0\,, \qquad \forall\, X_i \in \mathfrak{g}\,, \tag{7}$$

where $[[r, r]]$ is the Schouten bracket defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\,, \tag{8}$$

such that

$$r_{12} = r^{ij} X_i \otimes X_j \otimes 1\,, \qquad r_{13} = r^{ij} X_i \otimes 1 \otimes X_j\,, \qquad r_{23} = r^{ij}\, 1 \otimes X_i \otimes X_j\,, \tag{9}$$

and hereafter sum over repeated indices will be understood unless otherwise stated. If the Schouten bracket (8) does not vanish the Lie algebra $\mathfrak{g}$ is said to be endowed with a quasitriangular or standard Lie bialgebra structure $(\mathfrak{g}, \delta(r))$. The vanishing of the Schouten bracket (8) leads to the classical Yang–Baxter equation (CYBE) $[[r, r]] = 0$ and $(\mathfrak{g}, \delta(r))$ is called a triangular or nonstandard Lie bialgebra.

The main point now is that coboundary Lie bialgebras $(\mathfrak{g}, \delta(r))$ are the tangent counterpart of coboundary PL groups (G, Π) [2], where the Poisson structure Π on G is given by the Sklyanin bracket

$$\{f_1, f_2\} = r^{ij}\left(X_i^L f_1 X_j^L f_2 - X_i^R f_1 X_j^R f_2\right), \qquad f_1, f_2 \in \mathcal{C}(G), \tag{10}$$

such that X_i^L and X_i^R are left- and right-invariant vector fields defined by

$$X_i^L f(g) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\left(g\,\mathrm{e}^{tY_i}\right), \qquad X_i^R f(g) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\left(\mathrm{e}^{tY_i} g\right), \tag{11}$$

where $f \in \mathcal{C}(G)$, $g \in G$ and $Y_i \in \mathfrak{g}$. The quantization (as a Hopf algebra) of a PL group (G, Π) is just the corresponding quantum group.

Given a PHS $(M^\ell = G/H, \pi)$ with an underlying coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$ of (G, Π), the Poisson structure π on M^ℓ, coming from canonical projection of the PL structure Π on G, is only ensured to be well-defined whenever the so-called coisotropy condition for the cocommutator δ with respect to the isotropy subalgebra $\mathfrak{h}$ of H is fulfilled [3–5], namely

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \tag{12}$$

This condition means that the commutation relations that define the noncommutative space M_z^ℓ, with underlying classical space M^ℓ (2) and quantum deformation parameter $q = \mathrm{e}^z$, at the first-order in all the quantum coordinates $(\hat{t}^1, \dots, \hat{t}^\ell)$ close on a Lie subalgebra which is just the annihilator $\mathfrak{h}^\perp$ of $\mathfrak{h}$ on the dual Lie algebra $\mathfrak{g}^*$:

$$\mathfrak{h}^\perp \equiv M_z^\ell. \tag{13}$$

The duality between the generators of $\mathfrak{t}$ (3) and the quantum coordinates $(\hat{t}^1, \dots, \hat{t}^\ell)$ spanning M_z^ℓ is determined by a canonical pairing given by the bilinear form

$$\langle \hat{t}^j, T_k \rangle = \delta_k^j, \quad \forall\, j, k. \tag{14}$$

Noncommutative spaces can finally be obtained as quantizations of coisotropic PHS in all orders in the quantum coordinates $(\hat{t}^1, \dots, \hat{t}^\ell)$, so completing the initial quantum space M_z^ℓ (13) which just determines the Lie-algebraic (linear) contribution.

A general approach in order to construct any noncommutative space from any coisotropic PHS $(M^\ell = G/H, \pi)$ with coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$, so fulfilling (12), is summarized in six steps (see [9, 12] and references therein) as follows:

1. Consider a faithful representation ρ of the Lie algebra $\mathfrak{g}$.

2. Compute, by exponentiation, an element of the Lie group G according to the left coset space $M^\ell = G/H$ (2) in the form

$$G_{M^\ell} = \exp\bigl(t^1 \rho(T_1)\bigr) \cdots \exp\bigl(t^\ell \rho(T_\ell)\bigr) H, \tag{15}$$

   where $(T_1, \dots, T_\ell)$ are the translation generators on M^ℓ, H is the $(d-\ell)$D isotropy subgroup, and $(t^1, \dots, t^\ell)$ are local coordinates associated with the above translation generators of $\mathfrak{t}$ (3). Note that these coordinates are independent of the representation chosen in the previous step, provided that it is faithful.

3. Calculate the corresponding left- and right-invariant vector fields (11) from G_{M^ℓ} (15).

4. Consider a classical r-matrix (7) so determining a coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$ (either of quasitriangular or triangular type), which is the tangent counterpart of the corresponding coboundary PL group (G, Π).

5. Obtain the Poisson brackets among the local translation coordinates $(t^1, \ldots, t^\ell)$ via the Sklyanin bracket (10) from the chosen classical r-matrix. The resulting expressions define the coisotropic PHS.

6. Finally, quantize the PHS thus obtaining the noncommutative space in terms of the quantum coordinates $(\hat{t}^1, \ldots, \hat{t}^\ell)$.

In the next sections we illustrate the above procedure by applying it to several (A)dS and Poincaré quantum deformations giving rise to noncommutative spaces that could be relevant in a quantum gravity framework [6].

3 κ-Minkowski and κ-(A)dS noncommutative spacetimes

Let us consider the (3+1)D Poincaré and (A)dS Lie algebras expressed as a one-parametric family of Lie algebras denoted by $\mathfrak{g}_\Lambda$ depending on the cosmological constant Λ. In a kinematical basis spanned by the generators of time translations P_0, spatial translations $\mathbf{P} = (P_1, P_2, P_3)$, boost transformations $\mathbf{K} = (K_1, K_2, K_3)$ and rotations $\mathbf{J} = (J_1, J_2, J_3)$, the commutation relations of $\mathfrak{g}_\Lambda$ are given by

$$\begin{array}{lll} [J_a, J_b] = \epsilon_{abc} J_c \,, & [J_a, P_b] = \epsilon_{abc} P_c \,, & [J_a, K_b] = \epsilon_{abc} K_c \,, \\ [K_a, P_0] = P_a \,, & [K_a, P_b] = \delta_{ab} P_0 \,, & [K_a, K_b] = -\epsilon_{abc} J_c \,, \\ [P_0, P_a] = -\Lambda K_a , & [P_a, P_b] = \Lambda \epsilon_{abc} J_c \,, & [J_a, P_0] = 0 \,. \end{array} \tag{16}$$

From now on, Latin indices run as $a, b, c = 1, 2, 3$ while Greek ones run as $\mu = 0, 1, 2, 3$. The Lie algebra $\mathfrak{g}_\Lambda$ comprises the dS algebra $\mathfrak{so}(4, 1)$ for $\Lambda > 0$, the AdS algebra $\mathfrak{so}(3, 2)$ for $\Lambda < 0$ and the Poincaré one $\mathfrak{iso}(3, 1)$ when $\Lambda = 0$.

The first step in our approach is to consider a faithful representation $\rho : \mathfrak{g}_\Lambda \to \mathrm{End}(\mathbb{R}^5)$ for $X \in \mathfrak{g}_\Lambda$, that reads

$$\rho(X) = x^\mu \rho(P_\mu) + \xi^a \rho(K_a) + \theta^a \rho(J_a) = \begin{pmatrix} 0 & \Lambda x^0 & -\Lambda x^1 & -\Lambda x^2 & -\Lambda x^3 \\ x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\ x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \xi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix} . \tag{17}$$

By exponentiation we obtain a one-parametric family of Lie groups, G_Λ, that covers the dS $\mathrm{SO}(4, 1)$ for $\Lambda > 0$, the AdS $\mathrm{SO}(3, 2)$ for $\Lambda < 0$, and the Poincaré $\mathrm{ISO}(3, 1)$ for $\Lambda = 0$. The (3+1)D Minkowski and (A)dS homogeneous spacetimes (2), M^{3+1}_Λ, are defined by

$$M^{3+1}_\Lambda = G_\Lambda / H \,, \qquad H = \mathrm{SO}(3, 1) = \langle \mathbf{K}, \mathbf{J} \rangle \,, \tag{18}$$

where the Lie algebra $\mathfrak{h}$ of H is the Lorentz subalgebra and $\mathfrak{t} = \mathrm{span}\,\{P_\mu\}$ (1). Observe that the constant sectional curvature of M^{3+1}_Λ is $\omega = -\Lambda$.

Our aim now is to construct the κ-noncommutative counterpart of M^{3+1}_Λ (18). According to (15) (step 2 in Section 2) we compute G_Λ in terms of local coordinates (x^μ, ξ^a, θ^a) as

$$G_\Lambda = \exp\!\big(x^0 \rho(P_0)\big) \exp\!\big(x^1 \rho(P_1)\big) \exp\!\big(x^2 \rho(P_2)\big) \exp\!\big(x^3 \rho(P_3)\big) H \,, \tag{19}$$

where the Lorentz subgroup $H = \mathrm{SO}(3,1)$ is parametrized by

$$H = \exp\big(\xi^1 \rho(K_1)\big) \exp\big(\xi^2 \rho(K_2)\big) \exp\big(\xi^3 \rho(K_3)\big) \exp\big(\theta^1 \rho(J_1)\big) \exp\big(\theta^2 \rho(J_2)\big) \exp\big(\theta^3 \rho(J_3)\big) . \quad (20)$$

Notice that here the index $\ell = 4$ in (2) and the generic local coordinates (t^1, t^2, t^3, t^4) in (15) corresponds to the spacetime coordinates (x^0, x^1, x^2, x^3).

Following the step 3 in Section 2 we compute the left- and right-invariant vector fields (11) from G_Λ. In the step 4 we have to consider a classical r-matrix and we distinguish two cases between κ-Poincaré with $\Lambda = 0$ and κ-(A)dS with $\Lambda \neq 0$.

The κ-Poincaré classical r-matrix is a solution of the mCYBE (7) and reads [7, 13]

$$r_0 = \frac{1}{\kappa} (K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) , \quad (21)$$

that satisfies the coisotropy condition (12) with respect to $\mathfrak{h} = \mathrm{span}\{\mathbf{K}, \mathbf{J}\}$ and where the quantum deformation parameter $\kappa = 1/z$. The corresponding Sklyanin bracket (10) leads to linear Poisson brackets for the classical coordinates x^μ which determine the κ-Minkowski PHS. This can therefore be quantized directly in terms of the quantum coordinates $\hat{x}^\mu$. Hence we recover well-known κ-Minkowski spacetime [7] (see also [5, 11, 14, 15] and references therein) which is of Lie-algebraic type:

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa} \hat{x}^a , \qquad [\hat{x}^a, \hat{x}^b] = 0 , \quad (22)$$

completing the final steps 5 and 6 in Section 2.

When $\Lambda \neq 0$ we consider the κ-(A)dS classical r-matrix, which is also a a solution of the mCYBE (7), given by [8, 16, 17]

$$r_\Lambda = \frac{1}{\kappa} (K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2) , \quad (23)$$

such that the parameter η is related to the cosmological constant Λ and the sectional curvature ω of the (A)dS spacetimes (18) by

$$\omega = \eta^2 = -\Lambda . \quad (24)$$

Thus η is real for AdS and a purely imaginary number for dS. The Sklyanin bracket now gives rise to the (nonlinear) κ-(A)dS PHS in the form [8]

$$\begin{aligned} \{x^0, x^1\} &= -\frac{1}{\kappa} \frac{\tanh(\eta x^1)}{\eta \cosh^2(\eta x^2) \cosh^2(\eta x^3)} , \\ \{x^0, x^2\} &= -\frac{1}{\kappa} \frac{\tanh(\eta x^2)}{\eta \cosh^2(\eta x^3)} , \\ \{x^0, x^3\} &= -\frac{1}{\kappa} \frac{\tanh(\eta x^3)}{\eta} , \end{aligned} \quad (25)$$

$$\begin{aligned} \{x^1, x^2\} &= -\frac{1}{\kappa} \frac{\cosh(\eta x^1) \tanh^2(\eta x^3)}{\eta} , \\ \{x^1, x^3\} &= \frac{1}{\kappa} \frac{\cosh(\eta x^1) \tanh(\eta x^2) \tanh(\eta x^3)}{\eta} , \\ \{x^2, x^3\} &= -\frac{1}{\kappa} \frac{\sinh(\eta x^1) \tanh(\eta x^3)}{\eta} . \end{aligned} \quad (26)$$

Consequently, in contrast to the κ-Minkowski spacetime (22) when $\Lambda \neq 0$ the 3-space (26), determined by x^a, is no longer commutative and ordering ambiguities arise in (25) and (26) which precludes a direct quantization. This problem can be circumvented by introducing five ambient coordinates in the (A)dS spacetimes (18) denoted $(s^4, s^\mu) \in \mathbb{R}^5$ such that they fulfil the pseudosphere constraint

$$\Sigma_\Lambda \equiv (s^4)^2 - \Lambda(s^0)^2 + \Lambda\big((s^1)^2 + (s^2)^2 + (s^3)^2\big) = 1 \,. \tag{27}$$

These read [8,9]

$$\begin{aligned} s^4 &= \cos(\eta x^0)\cosh\left(\eta x^1\right)\cosh\left(\eta x^2\right)\cosh\left(\eta x^3\right), \\ s^0 &= \frac{\sin(\eta x^0)}{\eta}\cosh\left(\eta x^1\right)\cosh(\eta x^2)\cosh\left(\eta x^3\right), \\ s^1 &= \frac{\sinh(\eta x^1)}{\eta}\cosh\left(\eta x^2\right)\cosh\left(\eta x^3\right), \\ s^2 &= \frac{\sinh\left(\eta x^2\right)}{\eta}\cosh\left(\eta x^3\right), \\ s^3 &= \frac{\sinh\left(\eta x^3\right)}{\eta}, \end{aligned} \tag{28}$$

and the spacetime coordinates x^μ are called geodesic parallel coordinates. Notice also that $q^\mu = s^\mu/s^4$ are Beltrami projective coordinates in M_Λ^{3+1} (18) which can be obtained through the projection with pole $(0,0,0,0,0) \in \mathbb{R}^5$ of a point with ambient coordinates (s^4, s^μ) onto the projective hyperplane with $s^4 = +1$ (see [18] for details). Next, if we compute the Poisson brackets among (s^4, s^μ) from (25) and (26), consider the quantum coordinates $(\hat{s}^4, \hat{s}^\mu)$ along with the ordered monomials $(\hat{s}^0)^k (\hat{s}^1)^l (\hat{s}^3)^m (\hat{s}^2)^n (\hat{s}^4)^j$, we finally obtain the κ-(A)dS spacetimes $M_{\Lambda,\kappa}^{3+1}$ expressed as a quadratic algebra [8]

$$\begin{aligned} &[\hat{s}^0, \hat{s}^a] = -\frac{1}{\kappa}\,\hat{s}^a\hat{s}^4, \qquad [\hat{s}^4, \hat{s}^a] = \frac{\eta^2}{\kappa}\,\hat{s}^0\hat{s}^a, \qquad [\hat{s}^0, \hat{s}^4] = -\frac{\eta^2}{\kappa}\,\hat{\mathcal{S}}_{\eta/\kappa}, \\ &[\hat{s}^1, \hat{s}^2] = -\frac{\eta}{\kappa}(\hat{s}^3)^2, \qquad [\hat{s}^1, \hat{s}^3] = \frac{\eta}{\kappa}\,\hat{s}^3\hat{s}^2, \qquad [\hat{s}^2, \hat{s}^3] = -\frac{\eta}{\kappa}\,\hat{s}^1\hat{s}^3, \end{aligned} \tag{29}$$

where the quantum 3-space $\hat{\mathcal{S}}_{\eta/\kappa}$ operator is given by

$$\hat{\mathcal{S}}_{\eta/\kappa} = \left(\hat{s}^1\right)^2 + \left(\hat{s}^2\right)^2 + \left(\hat{s}^3\right)^2 + \frac{\eta}{\kappa}\,\hat{s}^1\hat{s}^2 \,. \tag{30}$$

Obviously, Jacobi identities are satisfied. We remark that $M_{\Lambda,\kappa}^{3+1}$ (29) has a Casimir operator

$$\hat{\Sigma}_{\Lambda,\kappa} = \left(\hat{s}^4\right)^2 - \Lambda\left(\hat{s}^0\right)^2 + \frac{\Lambda}{\kappa}\,\hat{s}^0\hat{s}^4 + \Lambda\,\hat{\mathcal{S}}_{\eta/\kappa}\,, \tag{31}$$

which is the quantum analogue of the pseudosphere (27) (recall that $\Lambda = -\eta^2$ (24)).

As expected, under the flat limit $\eta \to 0$ (i.e., $\Lambda \to 0$), the ambient coordinates (s^4, s^μ) (28) provide the usual Cartesian ones $(1, x^\mu)$ in the Minkowski spacetime and the κ-(A)dS spacetimes (29) reduce to the κ-Minkowski spacetime (22).

4 Noncommutative (A)dS and Minkowski spacetimes with quantum Lorentz subgroups

In this section we present very recent results concerning (3+1)D noncommutative (A)dS and Minkowski spacetimes that preserve a quantum Lorentz subgroup which were obtained in [9] by following the same six-step procedure described in Section 2. We advance that these are quite different from the κ-Minkowski (22) and κ-(A)dS (29) spacetimes reviewed in the previous section. Hence, we keep the same notation as in Section 3, in particular we shall make use of the expressions (16)–(20), (24), (27) and (28).

We consider the family of the (3+1)D Poincaré and (A)dS Lie algebras $\mathfrak{g}_\Lambda$ (16) and search for classical r-matrices (7) that keep the Lorentz subalgebra $\mathfrak{h} = \text{span}\{\mathbf{K},\mathbf{J}\} = \mathfrak{so}(3,1)$ as a sub-Lie bialgebra, that is,

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}\,, \tag{32}$$

which is a more restrictive version of the coisotropy condition (12). This restriction implies that the corresponding PHS is constructed through the Lorentz isotropy subgroup $H = \mathrm{SO}(3,1)$ such that $(H, \Pi|_H)$ is a PL subgroup of (G_Λ, Π) and it is called a PHS of Poisson subgroup type.

Then we start with the most general element $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$. Since the dimension of $\mathfrak{g}_\Lambda$ is $d = 10$, r depends on 45 initial deformation parameters. From it, we directly compute the cocommutator δ (6) such that $(\mathfrak{g}_\Lambda, \delta(r))$ defines a Lie bialgebra if and only if r is a solution of the mCYBE (7). Moreover, we have to impose the condition (32).

The simplest case is to require that $\delta(\mathfrak{h}) = 0$ which means that the Lorentz subgroup remains underformed. The final result is summarized as [9]:

Proposition 1. *The only PL group (G_Λ, Π) such that $\Pi|_H = 0$ is the trivial one.*

Therefore the only PHS $(M^{3+1}_\Lambda = G_\Lambda/H, \pi)$ of Poisson Lorentz subgroup type such that $\Pi|_H = 0$ is the trivial one. In other words, there does not exist any quantum deformation of the (3+1)D Poincaré and (A)dS Lie algebras preserving the Lorentz subalgebra $\mathfrak{h}$ underformed.

Now the main question is whether there exists a quantum deformation of $\mathfrak{g}_\Lambda$ preserving a non-trivial quantum Lorentz subalgebra, that is, $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h} \neq 0$. The answer is positive. By taking into account previous results concerning quantum Poincaré groups [19, 20] and quantum deformations of the Lorentz algebra $\mathfrak{h} = \mathfrak{so}(3,1)$ [21], it can be proven that the classification of the quantum deformations of $\mathfrak{g}_\Lambda$ keeping a quantum Lorentz subalgebra can be casted into three types as follows [9]:

Proposition 2. *There exist three classes of PHS $(M^{3+1}_\Lambda = G_\Lambda/H, \pi)$ for each of the maximally symmetric relativistic spacetimes of constant curvature (Minkowski and (A)dS) (18) such that the isotropy Lorentz subgroup H is a PL subgroup of (G_Λ, Π). All of them are obtained from coboundary PL structures on their respective isometry group G_Λ which are determined, up to $\mathfrak{g}_\Lambda$-isomorphisms, by the classical r-matrices*

$$\begin{aligned} r_{\mathrm{I}} &= z\,(K_1 \wedge K_2 + K_1 \wedge J_3 - K_3 \wedge J_1 - J_1 \wedge J_2) - z'\,(K_2 \wedge K_3 - K_2 \wedge J_2 - K_3 \wedge J_3 + J_2 \wedge J_3)\,,\\ r_{\mathrm{II}} &= z\,K_1 \wedge J_1\,,\\ r_{\mathrm{III}} &= z\,(K_1 \wedge K_2 + K_1 \wedge J_3)\,, \end{aligned} \tag{33}$$

where z and z' are free quantum deformation parameters. These three classical r-matrices are solutions of the CYBE $[[r,r]] = 0$.

Hence the three classes correspond to triangular or nonstandard deformations. Types II and III would provide one-parametric deformations, while type I would lead to a two-parametric one with arbitrary deformation parameters z and z'. Recall that the κ-$\mathfrak{g}_\Lambda$ deformations described in the previous section have a quasitriangular or standard character.

Next we apply the approach presented in Section 2 in order to construct the corresponding PHS from the above classical r-matrices in terms of the local coordinates x^μ through the Sklyanin bracket (10). However, the resulting expressions are rather cumbersome and strong ordering ambiguities appear, so there is no a direct quantization for any class. In order to solve

Table 1: The three types of (A)dS and Minkowski noncommutive spacetimes with quantum Lorentz subgroups determined by Proposition 2. These are expressed in quantum ambient spacetime coordinates $\hat{s}^\mu$ (28) or in ($\hat{s}^\pm = \hat{s}^0 \pm \hat{s}^1, \hat{s}^2, \hat{s}^3$). The quantum coordinate $\hat{s}^4$ always commutes with $\hat{s}^\mu$.

Type I $\quad r_{\mathrm{I}} = z(K_1 \wedge K_2 + K_1 \wedge J_3 - K_3 \wedge J_1 - J_1 \wedge J_2)$
$\qquad\qquad -z'(K_2 \wedge K_3 - K_2 \wedge J_2 - K_3 \wedge J_3 + J_2 \wedge J_3)$

• Subfamily with $z = 0$

$[\hat{s}^-, \hat{s}^2] = -2z'\hat{s}^+\hat{s}^3 \qquad [\hat{s}^-, \hat{s}^3] = 2z'\hat{s}^+\hat{s}^2 \qquad [\hat{s}^2, \hat{s}^3] = z'(\hat{s}^+)^2 \qquad [\hat{s}^+, \cdot\,] = 0$

• Subfamily with $z' = 0$

$[\hat{s}^-, \hat{s}^+] = 2z\hat{s}^+\hat{s}^2 \qquad [\hat{s}^-, \hat{s}^2] = z\hat{s}^-\hat{s}^+ - 2z(\hat{s}^3)^2 \qquad [\hat{s}^-, \hat{s}^3] = 2z\hat{s}^3\hat{s}^2$
$[\hat{s}^2, \hat{s}^3] = z\hat{s}^+\hat{s}^3 \qquad [\hat{s}^+, \hat{s}^2] = -z(\hat{s}^+)^2 \qquad [\hat{s}^+, \hat{s}^3] = 0$

Type II $\quad r_{\mathrm{II}} = zK_1 \wedge J_1$

$[\hat{s}^0, \hat{s}^1] = 0 \qquad [\hat{s}^0, \hat{s}^2] = z\hat{s}^1\hat{s}^3 \qquad [\hat{s}^0, \hat{s}^3] = -z\hat{s}^1\hat{s}^2$
$[\hat{s}^2, \hat{s}^3] = 0 \qquad [\hat{s}^1, \hat{s}^2] = z\hat{s}^0\hat{s}^3 \qquad [\hat{s}^1, \hat{s}^3] = -z\hat{s}^0\hat{s}^2$

Type III $\quad r_{\mathrm{III}} = z(K_1 \wedge K_2 + K_1 \wedge J_3)$

$[\hat{s}^2, \hat{s}^+] = z(\hat{s}^+)^2 \qquad [\hat{s}^2, \hat{s}^-] = -z\hat{s}^-\hat{s}^+ \qquad [\hat{s}^-, \hat{s}^+] = 2z\hat{s}^+\hat{s}^2 \qquad [\hat{s}^3, \hat{s}^\mu] = 0$

this problem we proceed similarly to the κ-(A)dS spacetimes (29). We again consider the ambient coordinates (s^4, s^μ) (28) (subjected to the pseudosphere constraint (27)), compute their PL brackets from those initially given in terms of x^μ, and finally obtain the corresponding noncommutive spacetimes by choosing an appropriate order in the quantum coordinates $(\hat{s}^4, \hat{s}^\mu)$ (so satisfying the Jacobi identities).

As a final result, we display in Table 1 all the (3+1)D Minkowski and (A)dS noncommutive spacetimes that preserve a non-trivial quantum Lorentz subgroup [9].

Now some remarks are in order.

- The ambient quantum coordinate $\hat{s}^4$ is always a central element for all the three types of noncommutive spacetimes, $[\hat{s}^4, \hat{s}^\mu] = 0$, so that these are just defined by the (3+1) quantum variables $\hat{s}^\mu$.

- In this respect, we remark that the corresponding noncommutive Minkowski spacetimes can directly be obtained through the flat limit $\Lambda \to 0$ (or $\eta \to 0$), in such a manner that the quantum coordinates $(\hat{s}^4, \hat{s}^\mu)$ reduce to the usual quantum Cartesian ones $(1, \hat{x}^\mu)$. Since $\hat{s}^4$ is absent in all the expressions presented in Table 1, the noncommutive Minkowski spacetimes adopt the very same formal expressions in the quantum Cartesian coordinates $\hat{x}^\mu$.

- For types I and III it is found that the explicit noncommutive spacetimes are naturally adapted to a null-plane basis [9,22] and for this reason we have considered the quantum coordinates $(\hat{s}^\pm = \hat{s}^0 \pm \hat{s}^1, \hat{s}^2, \hat{s}^3)$ instead of $\hat{s}^\mu$. Thus they lead to $(\hat{x}^\pm = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2, \hat{x}^3)$ for the Minkowski cases.

- In type I we have distinguished two subfamilies with either z or z' equal to zero in order to clarify the presentation of the results. Nevertheless, observe that the general noncommutive spacetimes of type I is just the superposition (the sum) of both subfamilies.

- We remark that the type II noncommutative spacetime has already been obtained for the quadratic Minkowski case in [23] (set $\hat{s}^\mu \equiv \hat{x}^\mu$) by following a different approach from ours; that is, from a twisted quantum Poincaré group and then applying the FRT procedure. Notice that, in fact, the classical r-matrix $r_{\rm II}$ (33) is just a Reshetikhin twist.

- Finally, the type III noncommutative spacetimes can be regarded as (2+1)D quantum spaces since the quantum coordinate $\hat{s}^3$ is a central operator, $[\hat{s}^3, \cdot\,] = 0$. We recall that when this structure is, again, only applied to the Minkowski case ($\hat{x}^\pm = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2$), it was already obtained from a Drinfel'd double structure of the (2+1)D Poincaré group in [24]. In addition, we stress that the corresponding quantum algebra for $\mathfrak{g}_\Lambda$ comes from the lower dimensional Lorentz subalgebra $\mathfrak{so}(2,1)$ spanned by $\{J_3, K_1, K_2\}$ which is just the well-known nonstandard (or Jordanian) quantum deformation of $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1)$ (see [25–28]). For higher-dimensional quantum (A)dS algebras keeping such a nonstandard quantum $\mathfrak{sl}(2,\mathbb{R})$ Hopf subalgebra we refer to [29].

5 κ-Poincaré space of time-like worldlines and beyond

So far we have constructed several (3+1)D Minkowski and (A)dS noncommutive spacetimes by applying the approach given in Section 2. However, we stress that such a procedure is rather general and can be applied to any homogeneous space. Hence in this section we shall consider the 6D homogeneous space of time-like Poincaré geodesics and obtain its κ-noncommutative version [10].

With this aim we consider the following Cartan decomposition of the Poincaré algebra $\mathfrak{g}_\Lambda \equiv \mathfrak{g}$ and $G_\Lambda \equiv G$ with commutation relations (16) with $\Lambda = 0$ (see (1)):

$$\mathfrak{g} = \mathfrak{t}_{\rm tl} \oplus \mathfrak{h}_{\rm tl}\,, \qquad \mathfrak{t}_{\rm tl} = \text{span}\{\mathbf{P}, \mathbf{K}\}\,, \qquad \mathfrak{h}_{\rm tl} = \text{span}\{P_0, \mathbf{J}\} = \mathbb{R} \oplus \mathfrak{so}(3)\,. \tag{34}$$

The homogeneous space of time-like geodesics is of dimension six and is defined by

$$\mathcal{W}_{\rm tl} = G/H_{\rm tl}\,, \tag{35}$$

where the isotropy subgroup $H_{\rm tl} = \mathbb{R} \otimes \mathrm{SO}(3)$ comes from the Lie subalgebra $\mathfrak{h}_{\rm tl}$ (34).

By following the procedure presented in Section 2, we first parametrize the Poincaré Lie group from the 5D matrix representation (17) with $\Lambda = 0$ taking into account the order given in (15), that is,

$$\begin{aligned} G_{\mathcal{W}_{\rm tl}} = \exp\big(\eta^1 \rho(K_1)\big) \exp\big(y^1 \rho(P_1)\big) \exp\big(\eta^2 \rho(K_2)\big) \exp\big(y^2 \rho(P_2)\big) \\ \times \exp\big(\eta^3 \rho(K_3)\big) \exp\big(y^3 \rho(P_3)\big) H_{\rm tl}\,, \end{aligned} \tag{36}$$

where $H_{\rm tl}$ is the stabilizer of the worldline corresponding to a massive particle at rest at the origin of the (3+1)D Minkowski spacetime, namely

$$H_{\rm tl} = \exp\big(\phi^1 \rho(J_1)\big) \exp\big(\phi^2 \rho(J_2)\big) \exp\big(\phi^3 \rho(J_3)\big) \exp\big(y^0 \rho(P_0)\big)\,. \tag{37}$$

Therefore the classical coordinates $(t^1, \dots, t^\ell)$ in (15) correspond to (η^a, y^a) in (36) (recall that now $\ell = 6$). Next we consider the κ-Poincaré r-matrix (21) and by projecting the Sklyanin bracket (10) to the homogeneous space (35) we obtain a coisotropic PHS for the classical space of time-like geodesics which can be straightforwardly quantized since no ordering problems appear. In this way, the κ-Poincaré space of time-like geodesics $\mathcal{W}_{\mathrm{tl},\kappa}$ in terms of the six quantum coordinates $(\hat{y}^a, \hat{\eta}^a)$ turns out to be [10]:

$$
\begin{aligned}
[\hat{y}^1, \hat{y}^2] &= \frac{1}{\kappa}\left(\hat{y}^2 \sinh\hat{\eta}^1 - \frac{\hat{y}^1 \tanh\hat{\eta}^2}{\cosh\hat{\eta}^3}\right), \\
[\hat{y}^1, \hat{y}^3] &= \frac{1}{\kappa}\big(\hat{y}^3 \sinh\hat{\eta}^1 - \hat{y}^1 \tanh\hat{\eta}^3\big), \\
[\hat{y}^2, \hat{y}^3] &= \frac{1}{\kappa}\big(\hat{y}^3 \cosh\hat{\eta}^1 \sinh\hat{\eta}^2 - \hat{y}^2 \tanh\hat{\eta}^3\big), \\
[\hat{y}^1, \hat{\eta}^1] &= \frac{1}{\kappa}\frac{\big(\cosh\hat{\eta}^1 \cosh\hat{\eta}^2 \cosh\hat{\eta}^3 - 1\big)}{\cosh\hat{\eta}^2 \cosh\hat{\eta}^3}, \\
[\hat{y}^2, \hat{\eta}^2] &= \frac{1}{\kappa}\frac{\big(\cosh\hat{\eta}^1 \cosh\hat{\eta}^2 \cosh\hat{\eta}^3 - 1\big)}{\cosh\hat{\eta}^3}, \\
[\hat{y}^3, \hat{\eta}^3] &= \frac{1}{\kappa}\big(\cosh\hat{\eta}^1 \cosh\hat{\eta}^2 \cosh\hat{\eta}^3 - 1\big),
\end{aligned}
\tag{38}
$$

together with

$$
[\hat{\eta}^a, \hat{\eta}^b] = 0, \quad \forall\, a, b, \qquad [\hat{y}^a, \hat{\eta}^b] = 0, \quad a \neq b. \tag{39}
$$

The above commutators can also be written in terms of quantum Darboux operators $(\hat{q}^a, \hat{p}^a)$ on a 6D smooth submanifold $(\eta^1, \eta^2, \eta^3) \neq (0,0,0)$; these are defined by

$$
\begin{aligned}
\hat{q}^1 &:= \frac{\cosh\hat{\eta}^2 \cosh\hat{\eta}^3}{\cosh\hat{\eta}^1 \cosh\hat{\eta}^2 \cosh\hat{\eta}^3 - 1}\,\hat{y}^1, \\
\hat{q}^2 &:= \frac{\cosh\hat{\eta}^3}{\cosh\hat{\eta}^1 \cosh\hat{\eta}^2 \cosh\hat{\eta}^3 - 1}\,\hat{y}^2, \\
\hat{q}^3 &:= \frac{1}{\cosh\hat{\eta}^1 \cosh\hat{\eta}^2 \cosh\hat{\eta}^3 - 1}\,\hat{y}^3, \\
\hat{p}^a &:= \hat{\eta}^a,
\end{aligned}
\tag{40}
$$

where the ordering $(\hat{\eta}^a)^m (\hat{y}^a)^n$ has to be preserved. They lead to the canonical commutation relations

$$
\big[\hat{q}^a, \hat{q}^b\big] = \big[\hat{p}^a, \hat{p}^b\big] = 0, \qquad \big[\hat{q}^a, \hat{p}^b\big] = \frac{1}{\kappa}\,\delta_{ab}. \tag{41}
$$

From these expressions we find that the noncommutative space $\mathcal{W}_{\mathrm{tl},\kappa}$ can be regarded as three copies of the usual Heisenberg–Weyl algebra of quantum mechanics where the deformation parameter κ^{-1} replaces the Planck constant $\hbar$. We also recall that a first phenomenological analysis for $\mathcal{W}_{\mathrm{tl},\kappa}$, expressed in the form (38) and (39), was performed in [30].

So far we have constructed the noncommutative space $\mathcal{W}_{\mathrm{tl},\kappa}$ from the usual "time-like" κ-Poincaré deformation with classical r-matrix (21). However we remark that there exist two other possible κ-Poincaré deformations provided by "space-like" and "light-like" classical r-matrices [10, 11]. The quantization procedure described in Section 2 can similarly be applied to these remaining cases in order to construct the quantum counterpart of the 6D homogeneous space $\mathcal{W}_{\mathrm{tl}}$ (35). Therefore we shall keep exactly the expressions (36) and (37) together with the associated invariant vector fields and only change the underlying r-matrix. In what follows we summarize the final results which were recently obtained in [12].

We consider the "space-like" r-matrix given by

$$r = \frac{1}{\kappa}\,(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1)\,, \tag{42}$$

which is also a solution of the mCYBE (7), so quasitriangular. The corresponding quantum Poincaré algebra was obtained in [31] (c.f. Type 1. (a) with $z = 1/\kappa$). When computing the PHS it is found that again there are no ordering problems so that this can be quantized directly leading to the commutation relations defining $\mathcal{W}_{\mathrm{tl},\kappa}$ from the "space-like" κ-Poincaré deformation; these are

$$\begin{aligned}
[\hat{y}^1, \hat{y}^2] &= -\frac{1}{\kappa}\,\hat{y}^1 \tanh \hat{\eta}^2 \tanh \hat{\eta}^3\,,\\
[\hat{y}^1, \hat{y}^3] &= \frac{1}{\kappa}\,\frac{\hat{y}^1}{\cosh \hat{\eta}^3}\,,\\
[\hat{y}^2, \hat{y}^3] &= \frac{1}{\kappa}\,\frac{\hat{y}^2}{\cosh \hat{\eta}^3}\,,\\
[\hat{y}^1, \hat{\eta}^1] &= -\frac{1}{\kappa}\,\frac{\tanh \hat{\eta}^3}{\cosh \hat{\eta}^2}\,,\\
[\hat{y}^2, \hat{\eta}^2] &= -\frac{1}{\kappa}\tanh \hat{\eta}^3\,,\\
[\hat{y}^3, \hat{\eta}^3] &= -\frac{1}{\kappa}\sinh \hat{\eta}^3\,,
\end{aligned} \tag{43}$$

with the same vanishing brackets given by (39).

Finally, in the kinematical basis (16) with $\Lambda = 0$ the "light-like" κ-Poincaré deformation is determined by

$$r = \frac{1}{\kappa}\,(K_3 \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + J_1 \wedge P_2 - J_2 \wedge P_1)\,, \tag{44}$$

which is triangular with vanishing Schouten bracket. This element provides the so-called "null-plane" quantum Poincaré algebra introduced in [32, 33] (where $z = 1/\kappa$) in terms of a null-plane basis [22] instead of the kinematical one. Notice that the "light-like" r-matrix (44) is just the sum of the "time-like" r-matrix (21) and the "space-like" one (42). Consequently, as expected, the resulting PHS can directly be quantized giving rise to $\mathcal{W}_{\mathrm{tl},\kappa}$ from the "light-like" κ-Poincaré deformation which turns out to be given by the sum of (38) and (43) (preserving the same vanishing brackets (39)); namely

$$\begin{aligned}
[\hat{y}^1, \hat{y}^2] &= \frac{1}{\kappa}\left(\hat{y}^2 \sinh \hat{\eta}^1 - \frac{\hat{y}^1 \tanh \hat{\eta}^2\big(\sinh \hat{\eta}^3 + 1\big)}{\cosh \hat{\eta}^3}\right),\\
[\hat{y}^1, \hat{y}^3] &= \frac{1}{\kappa}\left(\hat{y}^3 \sinh \hat{\eta}^1 - \frac{\hat{y}^1\big(\sinh \hat{\eta}^3 - 1\big)}{\cosh \hat{\eta}^3}\right),\\
[\hat{y}^2, \hat{y}^3] &= \frac{1}{\kappa}\left(\hat{y}^3 \cosh \hat{\eta}^1 \sinh \hat{\eta}^2 - \frac{\hat{y}^2(\sinh \hat{\eta}^3 - 1)}{\cosh \hat{\eta}^3}\right),\\
[\hat{y}^1, \hat{\eta}^1] &= \frac{1}{\kappa}\left(\frac{\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - \sinh \hat{\eta}^3 - 1}{\cosh \hat{\eta}^2 \cosh \hat{\eta}^3}\right),\\
[\hat{y}^2, \hat{\eta}^2] &= \frac{1}{\kappa}\left(\frac{\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - \sinh \hat{\eta}^3 - 1}{\cosh \hat{\eta}^3}\right),\\
[\hat{y}^3, \hat{\eta}^3] &= \frac{1}{\kappa}\big(\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - \sinh \hat{\eta}^3 - 1\big)\,.
\end{aligned} \tag{45}$$

We remark that quantum Darboux operators $(\hat{q}^a, \hat{p}^a)$ satisfying (41) can also be defined for these latter noncommutative spaces [12].

6 Concluding remarks and open problems

In this "twofold" contribution we have, firstly, presented in Section 2 a general approach to construct noncommutative spaces from coisotropic PHS spaces determined by a coboundary Lie bialgebra structure and, secondly, we have applied it to the physically relevant (3+1)D (A)dS and Poincaré Lie groups. Besides the well-known (3+1)D κ-spacetimes shown in Section 3, we have also presented quite different (i.e. non-equivalent) (3+1)D noncommutative (A)dS and Minkowski spacetimes by requiring to preserve a quantum Lorentz subgroup invariant in Section 4. In addition, we have also considered noncommutative spaces beyond the (3+1)D noncommutative spacetimes, which are the usual models considered in quantum gravity. In this respect, we have presented the only three possible 6D noncommutative spaces of time-like geodesics provided the three types of κ-Poincaré quantum deformations in Section 5. We stress that a classification of all 6D noncommutative spaces of κ-Poincaré geodesics, covering the usual time-like worldlines, already here described, along with the space-like and light-like geodesics can be found in [12].

To conclude, we would like to comment on some open problems. Obviously, the procedure considered here can be applied to any coisotropic PHS space providing new noncommutative spaces. As far as (3+1)D (A)dS and Minkowski noncommutative spacetimes are concerned, we have presented their well-known κ-deformation together with all possible quantum spacetimes preserving a non-trivial quantum Lorentz subgroup. These results constitute the cornerstone of a large number of possibilities for a further development. Nevertheless, we remark that quantum spaces of geodesics have not been considered and studied so deeply. In fact, to the best of our knowledge, only κ-deformations for quantum Poincaré geodesics have been achieved. This fact not only suggests the consideration of other types of quantum Poincaré geodesics but, in our opinion, the relevant open problem is to construct quantum (A)dS spaces of geodescics; there are no results on this problem from a quantum group setting. In fact, for the κ-Poincaré space of time-like worldlines (from the usual κ-Poincaré algebra) its fuzzy properties have been studied in [30] and by following [30, 34] a similar analysis could be faced with the other types of κ-Poincaré geodesics. Consequently, the construction of (A)dS noncommutative spaces of geodesics (covariant under their corresponding (A)dS quantum groups) could be achieved following the same approach here presented, and thus the role of a nonvanishing cosmological constant (or curvature) in this novel noncommutative geometric setting could be further analysed. Work on all these lines is in progress.

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