Hilbert space structure and classical limit of the low energy sector of U(N) quantum Hall ferromagnets

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Abstract

Using the Lieb–Mattis ordering theorem of electronic energy levels, we identify and construct the Hilbert space of the low energy sector of U(N) quantum Hall/Heisenberg ferromagnets at filling factor M for L Landau/lattice sites. The carrier Hilbert space of irreducible representations of U(N) is described by rectangular Young tableaux of M rows and L columns, and associated with Grassmannian phase spaces U(N)/U(M)×U($N-M$). Replacing U(N)-spin operators by their expectation values in a Grassmannian coherent state allows for a semi-classical treatment of the low energy U(N)-spin-wave coherent excitations (skyrmions) of U(N) quantum Hall ferromagnets in terms of Grasmannian nonlinear sigma models.

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1 Introduction

The magnetic interaction between adjacent $\langle\alpha,\beta\rangle$ dipoles is described by the U(2) (two-component electrons) Quantum Heisenberg Model Hamiltonian

$$H=-\frac{1}{2}\sum_{\langle\alpha,\beta\rangle}\mathcal{J}_x\sigma_x(\alpha)\sigma_x(\beta)+\mathcal{J}_y\sigma_y(\alpha)\sigma_y(\beta)+\mathcal{J}_z\sigma_z(\alpha)\sigma_z(\beta)\,, \tag{1}$$

with $\sigma_{x,y,z}(\alpha)$ Pauli matrices at site α and $\mathcal{J}_{x,y,z}$ coupling constants. For positive $\mathcal{J}$, the dominant coupling between two dipoles may cause nearest-neighbors $\langle\alpha,\beta\rangle$ to have lowest energy when they are aligned (ferromagnetic case). The generalization of this model to N-component electrons arises in, for example, the two-body exchange interaction for N-component planar electrons in a perpendicular magnetic field [1], which adopts the form of a U(N) Quantum Hall Ferromagnet (QHF) Hamiltonian on a square lattice

$$H = -\mathcal{J} \sum_{\langle\alpha,\beta\rangle} \sum_{i,j=1}^{N} S_{ij}(\alpha) S_{ji}(\beta), \tag{2}$$

written in terms of U(N)-spin operators

$$S_{ij}(\alpha) = c_i^\dagger(\alpha) c_j(\alpha), \qquad \left[S_{ij}(\alpha), S_{kl}(\beta)\right] = \delta_{\alpha\beta}\left(\delta_{jk} S_{il}(\beta) - \delta_{il} S_{kj}(\beta)\right), \tag{3}$$

realized in terms of creation $c_i^\dagger(\alpha)$ and annihilation $c_i(\alpha)$ operators of an electron with component $i, j \in \{1, \ldots, N\}$ in a given Landau/lattice site $\alpha \in \{1, \ldots, L\}$ of a given Landau level (namely, the lowest one). The sum over $\langle\alpha, \beta\rangle$ extends over all near-neighbor Landau/lattice sites, and $\mathcal{J}$ is the exchange coupling constant (the spin stiffness for the XY model).

In particular, the electrons become multicomponent when, for example, in addition to the usual two spin components $\uparrow$ and $\downarrow$, they acquire extra “pseudospin” internal components associated with: (a) layer (for multilayer arrangements), (b) valley (like in graphene and other 2D Dirac materials), (c) sub-lattice, etc. In the case of a bilayer quantum Hall system in the lowest Landau level, one Landau site can accommodate $N = 4$ internal states/components $|i\rangle, i = 1, 2, 3, 4$ (“flavors”)

$$|1\rangle = |\uparrow t\rangle, \quad |2\rangle = |\uparrow b\rangle, \quad |3\rangle = |\downarrow t\rangle, \quad |4\rangle = |\downarrow b\rangle, \tag{4}$$

where t and b make reference to the “top” and “bottom” layers, respectively. Since the electron field has $N = 4$ degenerate components, the bilayer system possesses an underlying U(4) symmetry. Likewise, the ℓ-layer case carries a U(2ℓ) symmetry.

For N-component electrons, the Pauli exclusion principle allows $M \le N$ electrons per Landau/lattice site (the filling factor). Selecting a ground state ($|0\rangle_{\mathrm{F}}$ denotes de Fock vacuum)

$$|\Phi_0\rangle = \Pi_{\alpha=1}^{L} \Pi_{i=1}^{M} c_i^\dagger(\alpha) |0\rangle_{\mathrm{F}}, \tag{5}$$

which fills all L lattice sites with the first M internal levels $i = 1, \ldots, M \le N$, spontaneously breaks the U(N) symmetry (SSB) since a general unitary transformation mixes the first M “spontaneously chosen” occupied internal levels with the $N - M$ unoccupied ones. The ground state $|\Phi_0\rangle$ is still invariant under the stability subgroup U(M) × U($N - M$) of transformations among the M occupied levels and the $N - M$ unoccupied levels, respectively. Therefore, the transformations that do not leave $|\Phi_0\rangle$ invariant are parametrized by the Grassmannian coset $\mathbb{G}_M^N = \mathrm{U}(N)/\mathrm{U}(M) \times \mathrm{U}(N-M)$, which reduces to the well known Bloch sphere $\mathbb{S}^2 = \mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$ for $N = 2$ spin components and $M = 1$ electron per Landau site (“symmetric multi-qubits” [2]).

In this article, we aim to describe the carrier Hilbert space associated with these U(N) representations, their coherent states [3], and the classical limit. The structure of the Hilbert space for a U(N) QHF with L Landau/lattice sites and filling factor M is sketched in Section 2. U(N) irreducible representations (IRs) are classified with Young diagrams. Lieb-Mattis ordering of electronic energy levels (based on the pouring principle for Young diagrams) identifies rectangular Young diagrams of L columns and M rows as the carrier Hilbert space of the lower energy sector. We also provide a Fock (boson and fermion) representation of basis states alternative to the Young tableau representation.

In the classical/continuum limit $L \to \infty$ (large U(N)-spin representations or large number of lattice sites), the U(N)-spin operators S_{ij} become c-numbers, and the low energy U(N)-spin-wave coherent excitations are named “skyrmions” [4–6]. These coherent excitations turn out to be governed by a ferromagnetic order parameter associated with this SSB and labeled by $(N-M)\times M$ complex matrices Z parametrizing the complex Grassmannian manifold $\mathbb{G}_M^N$ in Section 3. In fact, Grassmannian nonlinear sigma models (NLσM) describe the classical dynamics associated with these SU(N) quantum spin chains [7–12], generalizing the SU(2) NLσM for the continuum dynamics of Heisenberg (anti)ferromagnets [13–15]. In references such as [9, 10], N represents the number of fermion "flavors", whereas L is referred to as the number of "colours" n_c.

2 Lieb-Mattis theorem and low energy U(N) ferromagnetism

Given the Fourier transform

$$\mathcal{S}_{ij}(q)=\sum_{\alpha=1}^{L}e^{\mathrm{i}q\alpha}S_{ij}(\alpha)\,, \tag{6}$$

the long-wavelength (low momentum $q\simeq 0$) ground state excitations of QHFs are described by the collective operators

$$\mathcal{S}_{ij}(0)=\sum_{\alpha=1}^{L}S_{ij}(\alpha)\,, \tag{7}$$

which are invariant under site permutations $\alpha\longleftrightarrow\alpha'$. The kind of IRs of U($N$) related to translation invariance are those described by rectangular Young diagrams of M rows and L columns

$$[L^M]=M\left\{\overbrace{\begin{array}{|c|c|c|}\hline & \cdots & \\ \hline \vdots & \vdots & \vdots \\ \hline & \cdots & \\ \hline\end{array}}^{L}\right.\,. \tag{8}$$

This means that physical states are symmetric (bosonic) under permutations of the L lattice sites and antisymmetric (fermionic) under permutation of the M electrons (the filling factor) at each lattice site. This reasoning gives an introductory and heuristic proof of the main Proposition 2.

As an interesting comment, in the quantum Hall effect approach, each electron occupies on average a surface area of $2\pi\ell_B^2$ (a Landau site, with ℓ_B the magnetic length) that is pierced by one magnetic flux quantum $\phi_0=2\pi\hbar/e$. This image allows for a dual bosonic Schwinger realization of collective U(N)-spin operators

$$\mathcal{S}_{ij}=\sum_{\mu=1}^{M}a_{i\mu}^{\dagger}a_{j\mu}\,,\qquad i,j=1,\ldots,N\,, \tag{9}$$

this time in terms of creation $a_{i\mu}^{\dagger}$ and annihilation $a_{j\mu}$ *boson* operators of magnetic flux quanta attached to the electron $\mu=1,\ldots,M$ with component $i=1,\ldots,N$. From the usual bosonic commutation relations $[a_{i\mu},a_{j\nu}^{\dagger}]=\delta_{ij}\delta_{\mu\nu}$ we recover the U(N)-spin commutation relations (3). We shall not further pursue this bosonic picture here. For more information, we address the reader to the Reference [16].

The Hilbert space of a U(N) QHF with L Landau/lattice sites at integer filling factor M is the $\binom{N}{M}^L$-dimensional L-fold tensor product space $\mathcal{H}_N^{\otimes L}[1^M]=\bigotimes_{\alpha=1}^{L}\mathcal{H}_N^{\alpha}[1^M]$. In Young diagram notation

$$M\left\{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\right.\otimes\overset{L\text{ times}}{\cdots}\otimes\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\quad\longleftrightarrow\quad[1^M]^{\otimes L}=[1^M]\otimes\overset{L}{\cdots}\otimes[1^M]\,. \tag{10}$$

Basis vectors of $\mathcal{H}_N^\alpha[1^M]$ are the M-particle Slater determinants (for $M=1$ we have "quNits", as a N-ary quantum-digit generalization of qubits) written in Fock and Young tableau notation as

$$\Pi_{\mu=1}^M c_{i_\mu}^\dagger(\alpha)|0\rangle_{\mathrm{F}} = \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_M \\ \hline \end{array}\,, \tag{11}$$

obtained by filling out columns of the corresponding Young diagram with components $i_\mu \in \{1,\ldots,N\}$ in strictly increasing order $i_1 < \cdots < i_M$. One can see that there are exactly $\binom{N}{M}$ different arrangements of this kind (the dimension of $\mathcal{H}_N^\alpha[1^M]$). This tensor product representation of U(N) is reducible. For example, the Clebsch-Gordan decomposition of a tensor product of $L=2$ IRs of U(N) of shape $[1^M]$, with filling factor $M=2$ and $N\geq 4$ components, is represented by the following Young diagrams

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = {\color{red}\ydiagram{2,2}} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1} \quad \longleftrightarrow \quad [1^2]\otimes[1^2] = {\color{red}[2^2]} \oplus [2,1^2] \oplus [1^4]\,, \tag{12}$$

where we have highlighted in red the rectangular case $[2^2]$ for later discussion. The $P(=ML)$-particle ground state (5) can be written in Young tableau notation

$$|\Phi_0\rangle = \Pi_{\alpha=1}^L \Pi_{i=1}^M c_i^\dagger(\alpha)|0\rangle_{\mathrm{F}} = \begin{array}{|c|c|c|}\hline 1 & \ldots & 1 \\ \hline \vdots & \vdots & \vdots \\ \hline M & \ldots & M \\ \hline \end{array}\,, \tag{13}$$

and then it belongs to the carrier Hilbert space $\mathcal{H}_N[L^M]$ of the rectangular IR $[L^M]$ with dimension

$$D[L^M] = \frac{\prod_{i=N-M+1}^N \binom{i+L-1}{i-1}}{\prod_{i=2}^M \binom{i+L-1}{i-1}} \overset{M=1}{\longrightarrow} \binom{L+N-1}{L} \overset{N=2}{\longrightarrow} L+1\,. \tag{14}$$

Note that $\mathcal{H}_2[L^1]$ is just the usual $(2j+1)$-dimensional Hilbert space for the angular momentum $j=L/2$ representation of SU(2). We denote Young diagrams of $P=ML$ boxes/particles by (a partition of P)

$$h = [h_1,\ldots,h_N] = \overbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \cline{1-5} & \ldots & \\ \cline{1-3} \end{array}}^{h_1} \qquad \begin{array}{l} h_1 \geq \cdots \geq h_N\,, \\ h_1 + \cdots + h_N = P\,. \end{array} \tag{15}$$

The shorthand $[h, \overset{M}{\ldots}, h, 0, \ldots, 0] = [h^M]$ is often used. Before presenting the central proposition of this work, we should define the concept of "dominance order $\succeq$" of Young diagrams of P particles as: h dominates h' (h is "more symmetric" than h') if

$$[h_1,\ldots,h_N] \succeq [h'_1,\ldots,h'_N] \Leftrightarrow h_1+\cdots+h_k \geq h'_1+\cdots+h'_k \quad \forall\, k\,. \tag{16}$$

Lieb-Mattis' theorem [17] states that, under general conditions on the symmetric Hamiltonian of the system, if $h \succeq h'$ then $E(h) < E(h')$, with $E(h)$ the ground state energy inside each IR h of U(N). Then we can establish the following

Proposition: The rectangular Young diagram of shape $[L^M]$ dominates all Young diagrams arising in the Clebsch-Gordan direct sum decomposition of the L-fold tensor product (10).

Therefore, the ground state will always belong to the rectangular $[L^M]$ sector. For instance, the rectangular sector $[2^2] \succeq [2,1^2] \succeq [1^4]$ dominates in the Clebsch-Gordan decomposition (12). Intuitively, dominance means that one can go from h to h' by moving a certain number of boxes from upper rows to lower rows, so that h is "more symmetric". Therefore, we shall concentrate on the low-energy carrier Hilbert space $\mathcal{H}_N[L^M]$ of the rectangular IR $[L^M]$ to which the ground state $|\Phi_0\rangle$ in (5) belongs. In particular, we shall construct *coherent* (Skyrmion) ground state excitations. For the role of other mixed permutation symmetry sectors we address the reader to [18].

3 Grassmannian coherent states and nonlinear sigma models

Grassmannian (fermionic) coherent states can be seen as U(N) rotations/excitations over the ground state $|\Phi_0\rangle$

$$|Z\rangle^L = \frac{\exp\left[\sum_{1\leq j\leq M, M+1\leq i\leq N+M} Z_{ij}\mathcal{S}_{ij}\right]|\Phi_0\rangle}{\sqrt{\det(\mathbb{1}_M + Z^\dagger Z)}}, \tag{17}$$

created by appying U(N)-spin collective $\mathcal{S}_{ij}, i > j$, ladder operators. These Grassmannian coherent states are then labeled by $(N-M)\times M$ complex matrices Z. For $N = 2$ spin components, $\uparrow$ and $\downarrow$, and $M = 1$ we recover spin $j = L/2$ (atomic) coherent states

$$|z\rangle^L = \frac{e^{zS_{21}}|\Phi_0\rangle}{\sqrt{1+|z|^2}} = (1+|z|^2)^{-j}\sum_{m=-j}^{j}\sqrt{\binom{2j}{j-m}}z^{j-m}|j,m\rangle\,, \tag{18}$$

where we have spanned in terms of the usual angular momentum (Dicke) states $\{|j,m\rangle, m = -j,\dots,j\}$, with $|\Phi_0\rangle = |j,-j\rangle$ and $z = \tan(\theta/2)e^{i\phi}$ is the sthereographic projection of the Bloch sphere $\mathbb{S}^2$ onto the complex plane $\mathbb{C}$. Actually, atomic coherent states can also be written as a tensor product of qubits

$$|z\rangle^L = \Big[\underbrace{\cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)e^{i\phi}|\downarrow\rangle}_{|z\rangle}\Big]^{\otimes L} = |z\rangle^{\otimes L}\,. \tag{19}$$

For $L = 2 \Rightarrow j = L/2 = 1$, we identify the spin triplet $|j,m\rangle$ states

$$|1,1\rangle = |\uparrow\uparrow\rangle\,,\quad |1,0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\,,\quad |1,-1\rangle = |\downarrow\downarrow\rangle\,. \tag{20}$$

For $N = 4$ and filling factor $M = 1$ we have

$$|Z\rangle^L = \frac{[|1\rangle + z_2|2\rangle + z_3|3\rangle + z_4|4\rangle]^{\otimes L}}{(1+|z_2|^2+|z_3|^2+|z_4|^2)^{L/2}}\,, \tag{21}$$

where $Z = (1, z_2, z_3, z_4)^t$ denotes a point on the complex projective space $\mathbb{C}P^3 = \mathrm{U}(4)/\mathrm{U}(1)\times\mathrm{U}(3)$ or the Grassmannian $\mathbb{G}_1^4$.

In order to study the semi-classical/thermodynamical limit $L\to\infty$ of $U(N)$ QHF, one has to replace U(N)-spin operators S_{ij} by their coherent state expectation values $\langle Z|S_{ij}|Z\rangle$, which play the role of a matrix order parameter

$$\mathcal{S}(Z) \equiv \frac{2}{L}\langle Z|\left(S - \frac{L}{2}\mathbb{1}_N\right)|Z\rangle^L = Q(Z)^\dagger E_M Q(Z)\,, \tag{22}$$

$$E_M = \mathrm{diag}(1, \overset{M}{\dots}, 1, -1, \overset{N-M}{\dots}, -1)\,, \tag{23}$$

with

$$Q(Z) = \left(\begin{array}{c|c} \Delta_1 & -Z^\dagger\Delta_2 \\ \hline Z\Delta_1 & \Delta_2 \end{array}\right), \tag{24}$$

$$\Delta_1 = (\mathbb{1}_M + Z^\dagger Z)^{-1/2}, \qquad \Delta_2 = (\mathbb{1}_{N-M} + ZZ^\dagger)^{-1/2}. \tag{25}$$

The low energy physics of the U(N) QHF [when considering only nearest-neighbor interactions $\mathcal{J}_{\alpha\beta} = \mathcal{J}\delta_{\alpha,\beta\pm1}$ in the exchange Hamiltonian (1)] is described by a NLσM field theory with action in the continuum limit ($L \to \infty$ and lattice constant $\ell \to 0$)

$$A[Z] = \int dx_0 dx_1 dx_2 \left[\mathrm{tr}(E_M Q^\dagger \partial_{x_0} Q) + \mathcal{J}\,\mathrm{tr}(\vec{\nabla}\mathcal{S}\cdot\vec{\nabla}\mathcal{S})\right], \tag{26}$$

where $\partial_{x_0} \equiv \partial_0$ means partial derivative with respect to time $t = x_0$, $\vec{\nabla} = (\partial_{x_1}, \partial_{x_2}) \equiv (\partial_1, \partial_2)$ is the gradient and $\vec{\nabla}\mathcal{S}\cdot\vec{\nabla}\mathcal{S}$ is the scalar product. The first (kinetic) term of the action is the Berry term, provided by the coherent state representation of the path integral quantization. The second term describes the energy cost when the order parameter $\mathcal{S}$ is not uniform (see [7–12] and [16] for more information). The topological current

$$J^\mu = \frac{\mathrm{i}}{16\pi}\varepsilon^{\mu\nu\lambda}\mathrm{tr}(\mathcal{S}\partial_\nu\mathcal{S}\partial_\lambda\mathcal{S}) \tag{27}$$

(ε is the Levi-Civita antisymmetric symbol in 1+2 dimensions), leads to the topological (Pontryagin) charge or Skyrmion number

$$\mathcal{C} = \int dx_1 dx_2 J^0. \tag{28}$$

See e.g. Ref. [12] for more information.

4 Conclusion

We have presented several group-theoretical tools to study interacting N-component fermions on a lattice, like U(N) quantum Hall ferromagnets arising from two-body exchange interactions of N-component fermions. In particular, we have restricted ourselves to the lower energy permutation symmetry sector (according to the Lieb-Mattis theorem) corresponding to fermion mixtures described by rectangular Young diagrams with M rows (the filling factor) and L columns (Landau/lattice sites).

The "spontaneously chosen" ground state $|\Phi_0\rangle$ breaks the original U(N) symmetry and the associated U(N) ferromagnetic order parameter $\mathcal{S}$ [the expectation value of collective U(N)-spin operators S in a Grassmannian coherent state $|Z\rangle$] describes coherent state excitations ("Skyrmions") in the semi-classical $L \to \infty$ limit, whose dynamics is governed by a Grassmannian nonlinear sigma model.

The subject of SU(N) fermions and SU(N) magnetism has been recently further fueled in condensed matter physics with exciting advances in cooling, trapping and manipulating fermionic alkaline-earth atoms trapped in optical lattices (see e.g. [19, 20] for a realization of a SU(N) generalization of the Hubbard model). Multilayer quantum Hall arrangements, bearing larger U(N) symmetries, also display interesting new physics (see [21] for the bilayer case); Such is the case of superconducting properties of twisted bilayer (and trilayer) graphene predicted by [22] and observed by [23]. Furthermore, magnetic Skyrmion materials display a robust topological magnetic structure, being a candidate for the next generation of spintronic memory devices.

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