Generalized Heisenberg-Weyl groups and Hermite functions

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Abstract

A generalisation of Euclidean and pseudo-Euclidean groups is presented, where the Weyl-Heisenberg groups, well known in quantum mechanics, are involved. A new family of groups is obtained including all the above-mentioned groups as subgroups. Symmetries, like self-similarity and invariance with respect to the orientation of the axes, are properly included in the structure of this new family of groups. Generalized Hermite functions on multidimensional spaces, which serve as orthogonal bases of Hilbert spaces supporting unitary irreducible representations of these new groups, are introduced.

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1 Introduction

It is well-known the interest of the Heisenberg-Weyl (HW) group in physics, mainly in Quantum Mechanics (QM). The indetermination principle, fundamental in QM, is closely linked to this group and the Fourier transform (FT) [1, 2]. It is also related with the Gabor formalism [3] on the theory of wavelets, where an uncertainty principle for time-frequency operators appears [4]. On the other hand, (the affine spaces) Euclidean, $\mathbb{R}^n$, or pseudo-Euclidean spaces, $\mathbb{R}^{p,q}$ $(p+q=n)$, are the arena of the physical events, where their invariance properties are described by the Euclidean type groups $E_n = \mathbb{R}^n \odot SO(n)$ or $E_{p,q} = \mathbb{R}^{p,q} \odot SO(p,q)$, respectively. The HW and Euclidean groups are involved in relevant invariance properties used in the study of the physical systems. Thus, we can mention, first of all, the pairs of sets of conjugate variables, connected through the HW group, that allows us to get equivalent physical descriptions either in the position or in the momentum representations. The freedom of the choice of the origin in each coordinate system (either position or momenta) that it is know as “homogeneity” and it is related to both kind of groups. The freedom to choose the unit of length or “self-similarity”, that can be implemented via dilations. And finally the freedom to select the orientation of the unit vectors for the orthogonal bases of the physical space ("invariance from orientation"). In these last two cases the Euclidean groups are involved in. However, all these invariances are not completely independent because the FT, which matches coordinate and momentum representations [5], does not allow to fix independently self-similarity and orientation. Both family of groups are independent although some times they appear together in the implementation of the invariances above mentioned, that we consider as a whole.

Recently in [6] we have studied the case related with $\mathbb{R}$, It has been the point of departure for a generalization of our analysis to $\mathbb{R}^n$ and $\mathbb{R}^{p,q}$ realized in [7]. Here the Euclidean-like groups E_n and $E_{p,q}$ and the HW groups H_n and $H_{p,q}$ (where $\mathbb{R}^n \subset H_n$ has been replaced by $\mathbb{R}^{p,q}$) have been enlarged to the groups K_n and $K_{p,q}$ that contain the Euclidean groups and the HW groups as subgroups. Tentatives in this direction has been done but with different motivations and only considering the cases with positively defined metric [8–10].

Here, we present the lower dimensional cases (1D and 2D) in Sections 2 and 3, respectively. The representations of the groups here studied are supported by square integrable functions. The fact that the Hermite functions (HF) constitute a (discrete) basis of $L^2(\mathbb{R})$ (see Subsection 2.2) allows us to introduce in Subsection 3.4 a generalization of the HF in order to describe the above mentioned invariance in 2D. We end with a Section 4 devoted to conclusions.

2 Heisenberg-Weyl groups in the real line $\mathbb{R}$

2.1 The Heisenberg-Weyl group H_1

The HW group in 1D can be realized on the coordinate space $\mathbb{R}$ providing the basic commutation relations of QM as $[x,p] \equiv [x, -i\hbar\frac{\partial}{\partial x}] = i\hbar$. A matrix representation of H_1 in terms of real 3×3 upper triangular matrices of the group $M_3(\mathbb{R})$ [11] is given by

$$H_1[a,b,c] = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,c \in \mathbb{R}\,. \tag{1}$$

Self-similarity and orientation are included by extending H_1 to a new group K_1 realized as

$$K_1[a,b,c,k] = \begin{bmatrix} 1 & a & c \\ 0 & k & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,c \in \mathbb{R},\;\; k \in \mathbb{R}^*\,. \tag{2}$$

Obviously, the group laws in both cases are obtained through matrix multiplication.

The group K_1 has two connected components: the connected component of the identity (K_1^o) characterized by $k>0$; and a 2^{nd} component with $k<0$ (K_1^1).

The parameters a,b,c of H_1 (and K_1) are in correspondence to the three generators X,P,I of the Lie algebra of H_1 (and K_1), Lie$[H_1]$ (Lie$[K_1]$), respectively; and the generator D associated to k only belongs to Lie$[K_1]$. The explicit form of these generators in (1) and (2) is

$$\begin{aligned} X &= \frac{\partial K_1[\dots]}{\partial a}\Big|_{Id} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P = \frac{\partial K_1[\dots]}{\partial b}\Big|_{Id} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ I &= \frac{\partial K_1[\dots]}{\partial c}\Big|_{Id} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D = \frac{\partial K_1[\dots]}{\partial k}\Big|_{Id} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{3}$$

with Id the identity element. The commutation relations for both Lie algebras are

$$[X,P]=I\,,\quad [D,X]=-X\,,\quad [D,P]=P\,,\quad [I,\bullet]=0\,. \tag{4}$$

The real line $\mathbb{R}$ is a metric space that supports two continuous conjugate (in the sense of position-momentum conjugation) bases for $L^2(\mathbb{R})$: $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$ obtained by means of the generalized eigenvectors of the operators X and P, i.e., $X\,|x\rangle = x\,|x\rangle\,, P\,|p\rangle = p\,|p\rangle\,.$ The basis elements of $\{|x\rangle\}_{x\in\mathbb{R}}$ satisfy (and similarly for $\{|p\rangle\}_{p\in\mathbb{R}}$)

$$\langle x|x'\rangle = \sqrt{2\pi}\,\delta(x-x')\,,\qquad \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,|x\rangle\langle x| = \mathbb{I}\,. \tag{5}$$

As we mention before these generalized bases are well defined on certain extensions of the Hilbert space (the Gel'fand triplets or the rigged Hilbert spaces) [12].

As is well known the Fourier transform (FT) and its inverse (IFT) connect both bases [5]

$$FT[|x\rangle,x,p] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\, e^{ipx}|x\rangle = |p\rangle\,,\quad IFT[|p\rangle,p,x] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dp\, e^{-ipx}|p\rangle = |x\rangle\,. \tag{6}$$

There exists a representation of H_1 by unbounded operators on $L^2(\mathbb{R})$, where P and X may be represented by

$$[Pf](x) = -i\,\frac{d}{dx}f(x)\,,\qquad [Xf](x) = x\,f(x)\,,\qquad f(x)\in L^2(\mathbb{R})\,, \tag{7}$$

satisfying $[X,P]=I$. We also may choose another representation of P and X on an abstract infinite dimensional separable Hilbert space $\mathcal{H}$. Since there is always a unitary map $U:\mathcal{H}\to L^2(\mathbb{R})$, the commutation relation between P and X on $L^2(\mathbb{R})$ is translated to $\mathcal{H}$. In order to simplify the notation we also denote the operators on $\mathcal{H}$ by P and X.

The relationship between the elements $|f\rangle\in\mathcal{H}$ and $f(x)\in L^2(\mathbb{R})$ is given by [5]

$$|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\, f(x)\,|x\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dp\, \hat{f}(-p)\,|p\rangle\,, \tag{8}$$

with $f(x)=\langle x|f\rangle$, $\hat{f}(p)=FT[f(x);x,p]$ and $\hat{f}(-p)=\langle p|f\rangle$. Remember that only the vectors $|f\rangle$ belonging to a dense space in $\mathcal{H}$ (i.e., the space of test vectors) can be written as (8).

The action of the group elements e^{-iPb} and e^{-iXa} on the continuous bases is given by

$$e^{-iPb}\,|x\rangle = |x+b\rangle\,,\qquad e^{-iXa}\,|p\rangle = |p-a\rangle\,,\qquad \forall a,b\in\mathbb{R}\,. \tag{9}$$

From these relations we conclude that $\{|x\rangle\}$ ($\{|p\rangle\}$) is equivalent to $\{|x+b\rangle\}$ ($\{|p-a\rangle\}$).

These bases support each an infinite dimensional unitary irreducible representation (UIR) of H_1, $U_h(g)$, $h\in\mathbb{R}^*$ [6,13],

$$U_h(g)\equiv U_h(c,a,b) := e^{ihcI}\,e^{ih(aX-bP)} = e^{ih(c-ab/2)I}\,e^{ihaX}\,e^{-ihbP}\,. \tag{10}$$

For instance, in the cases of $\{|x\rangle\}$ as well as $L^2(\mathbb{R})$ the action is given by

$$U_h(g)\,|x\rangle = e^{ihc}\,e^{iha(x+b/2)}\,|x+b\rangle\,,\qquad (\mathcal{U}_h(g)f)(x) = e^{ihc}\,e^{iha(x-b/2)}\,f(x-b)\,. \tag{11}$$

We mentioned before that H_1 does not exhaust the invariances of the real line if we add the hypothesis of self-similarity and orientation and we have to considerer K_1. Since $\{|x\rangle\}$ ($\{|p\rangle\}$) is equivalent to $\{|k\,x\rangle\}$ $\big(\{|k'\,p\rangle\}\big)$ and from (6) we find that $k'=k^{-1}\in\mathbb{R}^*$. In other words, $\mathbb{R}$

supports a UIR, $U_{h,\mathcal{C}}$, of K_1. For the connected component K_1^o of K_1 and for the dilations we use the formula (53) of [6] obtaining that $e^{idD}\,|x\rangle = e^{d/2}\,|e^d\,x\rangle$. Therefore,

$$U_{h,\mathcal{C}}(\tilde{g})\,|x\rangle = e^{d/2}\,e^{ih(c+\mathcal{C})}\,e^{iha(e^d\,x+b/2)}\,|e^d\,x+b\rangle\,,\quad \tilde{g}=(a,b,c,d)\in K_1^o\,, \tag{12}$$

where $\mathcal{C}\in\mathbb{R}$ denotes the eigenvalues of the quadratic Casimir of K_1^o, $\mathcal{C}=X\,P-I\,D$. When we consider also the dilations with $k<0$ we introduce the (unitary) parity operator $\mathcal{P}$ ($x\to -x$) and we obtain in a unified manner that

$$\mathcal{U}_{h,\mathcal{C}}(\tilde{g},\alpha)\,|x\rangle = \mathcal{U}_{h,\mathcal{C}}(\tilde{g})\,|x^\alpha\rangle = e^{d/2}\,e^{ih(c+\mathcal{C})}\,e^{iha(e^d\,x^a+b/2)}\,|e^d\,x^\alpha+b\rangle\,, \tag{13}$$

where α stands either for the identity ($x^{\mathcal{I}}=x$) and $(\tilde{g},\mathcal{I})\in K_1^o$ or the parity ($x^{\mathcal{P}}=-x$) and $(\tilde{g},\mathcal{P})\in K_1^1$. We can rewrite (13) in terms of $k\in R^*$ with $|k|=e^d$ and $d\in\mathbb{R}$ as

$$\mathcal{U}_{h,\mathcal{C}}(c,a,b,k)\,|x\rangle = \sqrt{|k|}\,e^{ih(c+\mathcal{C})}\,e^{iha(k\,x+b/2)}\,|k\,x+b\rangle\,. \tag{14}$$

The corresponding action on the functions of $L^2(\mathbb{R})$ is given by

$$\big(\mathcal{U}_{h,C}(\tilde{g},\alpha)f\big)(x) = \frac{1}{\sqrt{|k|}}\,e^{ih(c+C)}\,e^{i\,h\,a(x-b/2)}\,f\left(k^{-1}\,(x-b)\right)\,. \tag{15}$$

2.2 The Hermite functions appear on the scene

It is well known that the FT of the Hermite Functions $\{\psi_m(x)\}_{m\in\mathbb{N}}$ are also HF, i.e.

$$FT[\psi_m(x),x,p] = i^m\,\psi_m(p)\,,\quad IFT[\psi_m(p),p,x] = (-i)^m\,\psi_m(x)\,. \tag{16}$$

Hence, both are complete orthonormal bases in $L^2(\mathbb{R})$ [14].

Invariance properties of K_1 are implemented to a generalization of the HF obtained using the UIR's of K_1 (13) in position coordinates x (and similarly for p) as follows

$$\chi_m(x,a,b,k) := |k|^{1/2}\;e^{-ia(k\,x+b/2)}\,\psi_m(k\,x+b)\,,\qquad a,b\in\mathbb{R}\,,\;k\in\mathbb{R}^*\,. \tag{17}$$

In this way we obtain two families of functions depending on 3 real parameters (a,b,k)

$$\{\chi_m(x,a,b,k)\}\,,\quad \{\chi_m(p,a,b,k)\}\,,\quad \forall\;k\neq 0,a,b\in\mathbb{R}\,. \tag{18}$$

Orthonormal and completeness relations of the HF induce similar relations for these families of generalized HF, so they are also orthonormal bases in $L^2(\mathbb{R})$. However, these generalized HF are not eigenfunctions of the FT and its inverse, contrarily to the ordinary HF (16)

$$\begin{aligned} FT[\chi_m(x,a,b,k),x,p] &= i^m\;\chi_m(p,b,-a,k^{-1})\,,\\ IFT[\chi_m(p,a,b,k),p,x] &= (-i)^m\;\chi_m(x,-b,a,k^{-1})\,. \end{aligned} \tag{19}$$

3 Euclidean and pseudo-Euclidean plane cases

In this Section we will consider the 2D configuration spaces: the Euclidean plane ($\mathbb{R}^2$) and the pseudo-Euclidean plane ($\mathbb{R}^{1,1}$) with metrics of signature $(+,+)$ and $(+,-)$, respectively.

3.1 The groups H_2 and K_2 on the plane

The HW group on 2D, H_2, admits a finite representation by real 4×4 matrices as

$$H_2[\mathbf{a},\mathbf{b},c]=\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}\equiv\begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_1,a_2,b_1,b_2,c\in\mathbb{R}. \tag{20}$$

This group can be enlarged by adding the group of proper rotations $SO(2)$ and the dilations on the plane, $\mathbb{R}^*$, so as to obtain the group K_2

$$K_2[\mathbf{a},\mathbf{b},c,k,R(\theta)]=\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & kR(\theta) & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad R(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\in SO(2), \tag{21}$$

with $\theta\in[0,2\pi)$ and $k\in\mathbb{R}^*$. The group law is obtained by matrix multiplication, as usual,

$$K_2[\mathbf{a},\mathbf{b},c,k,R]\cdot K_2[\mathbf{a}',\mathbf{b}',c',k',R']=K_2[\mathbf{a}'+k'R'^T\mathbf{a},\ \mathbf{b}+kR\,\mathbf{b}',\ c+c'+\mathbf{a}\cdot\mathbf{b}',\ kk',\ RR'], \tag{22}$$

where $RR'\equiv R(\theta)R(\theta')=R(\theta+\theta')$.

3.2 The groups $H_{1,1}$ and $K_{1,1}$ on the pseudo-plane

A new generalization of H_2, $H_{1,1}$, can be obtain by replacing $\mathbb{R}^2$ by $\mathbb{R}^{(1,1)}$. It formally is like H_2 (20) but replacing $SO(2)$ by $SO_0(1,1)$), the connected component of the identity of $SO(1,1)$. The group $K_{1,1}$ comes from $H_{1,1}$ by adding $\mathbb{R}^*$,

$$K_{1,1}[\mathbf{a},\mathbf{b},c,k,\Lambda(\eta)]=\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\Lambda(\eta) & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad \Lambda(\eta)=\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\in SO_0(1,1), \tag{23}$$

with $\eta\in\mathbb{R}$ and $k\in\mathbb{R}^*$. The group law for $K_{1,1}$ is similar to that of K_2 (22), provided R is replaced by Λ. Note that K_2 has only a connected component while $K_{1,1}$ has two.

3.3 The Lie algebras of K_2 and $K_{1,1}$

Both algebras are 7D with infinitesimal generators X_1, X_2, P_1, P_2, I, D and, moreover, J for Lie$[K_2]$ and K for Lie$[K_{1,1}]$. A 4×4 matrix realization of the generators is

$$\begin{aligned} X_\alpha&=\frac{\partial K_-}{\partial a^\alpha}\Big|_{Id}=\begin{pmatrix} 0 & \boldsymbol{\alpha}^T & 0 \\ \mathbf{0} & \mathbb{O}_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad P_\alpha=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{O}_2 & \boldsymbol{\alpha} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \\ I&=\frac{\partial K_-}{\partial c}\Big|_{Id}=\begin{pmatrix} 0 & \mathbf{0}^T & c \\ \mathbf{0} & \mathbb{O}_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad D=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \end{aligned} \tag{24}$$

where $\boldsymbol{\alpha}$ is either the column vector $(1,0)^T$ for $\alpha=1$ or $(0,1)^T$ for $\alpha=2$ and $\mathbb{O}_2$ is the 2×2 zero matrix. The generators J and K are represented as

$$J=\frac{\partial K_-}{\partial\theta}\Big|_{Id}=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & -i\sigma_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad K=\frac{\partial K_-}{\partial\eta}\Big|_{Id}=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \sigma_1 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \tag{25}$$

where σ_i are Pauli matrices. The non-vanishing commutation relations are

$$[X_\alpha, P_\beta] = \delta_{\alpha\beta} I\,, \quad [D, X_\alpha] = -X_\alpha\,, \quad [D, P_\alpha] = +P_\alpha\,, \tag{26}$$

together with these for Lie$[K(2)]$

$$[J, X_\alpha] = \epsilon_{\alpha\beta}\, X_\beta\,, \quad [J, P_\alpha] = \epsilon_{\alpha\beta}\, P_\beta\,, \tag{27}$$

where $\epsilon_{\alpha\beta}$ is the skew-symmetric tensor, and these ones for Lie$[K(1,1)]$

$$[K, X_\alpha] = (-1)^\alpha\, \epsilon_{\alpha\beta}\, X_\beta\,, \quad [K, P_\alpha] = (-1)^{\alpha+1}\, \epsilon_{\alpha\beta}\, P_\beta\,. \tag{28}$$

3.4 Bases on the plane and the hyperplane

Now we will consider together the 2D real affine space $\mathbb{X}$ associated to either the vector space $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ and the Hilbert space $L^2(\mathbb{X})$ on which we define the position operator $\mathbf{X} \equiv (X_1, X_2)$ and their conjugate momentum operator $\mathbf{P} \equiv (P_1, P_2)$. These operators act on the eigenvectors $|\mathbf{x}\rangle\,(\equiv |x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle)$ and $|\mathbf{p}\rangle$, respectively, as $X_\alpha\,|\mathbf{x}\rangle = x_\alpha|\mathbf{x}\rangle$ and $P_\alpha\,|\mathbf{p}\rangle = p_\alpha|\mathbf{p}\rangle\,, \alpha = 1, 2\,.$ These eigenvectors are transformed into each other by means of Fourier type transformations (6) but in 2D

$$|\mathbf{p}\rangle = \frac{1}{2\pi}\int_{\mathbb{X}} d\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{x}\rangle\,, \qquad |\mathbf{x}\rangle = \frac{1}{2\pi}\int_{\mathbb{X}} d\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{p}\rangle\,. \tag{29}$$

As for the 1D case (9) we have similar relations: $e^{-i\mathbf{b}\cdot\mathbf{P}}\,|\mathbf{x}\rangle = |\mathbf{x}+\mathbf{b}\rangle$ and $e^{-i\mathbf{a}\cdot\mathbf{X}}\,|\mathbf{p}\rangle = |\mathbf{p}-\mathbf{a}\rangle$ $(\mathbf{a}, \mathbf{b} \in \mathbb{X})$. Hence the basis $\{|\mathbf{x}\rangle\}$ is equivalent to $\{|\mathbf{x}+\mathbf{b}\rangle\}$ and the same for $\{|\mathbf{p}\rangle\}$ and $\{|\mathbf{p}-\mathbf{a}\rangle\}$.

The use of the 2D FT serves us to realize that the five operators given by $\mathbf{X}$, $\mathbf{P}$ and I determine a UIR representation of H_2 or $H_{1,1}$ by exponentiation.

Let $\mathcal{H}$ be an abstract infinite-D separable Hilbert space and $S : \mathcal{H} \to L^2(\mathbb{X})$ a unitary map. If $|f\rangle \in \mathcal{H}$ and $S\,|f\rangle = f(x)$ we have the following relation in a suitable dense subspace of $\mathcal{H}$

$$|f\rangle = \int_{\mathbb{X}} d\mathbf{x}\, f(\mathbf{x})\,|\mathbf{x}\rangle\,, \qquad f(\mathbf{x}) = \langle \mathbf{x}|f\rangle\,. \tag{30}$$

The action of an element of K_2 (or $K_{1,1}$) on $\mathbb{X}$ implies that $|\mathbf{x}\rangle$ transforms as

$$|\mathbf{x}\rangle \to |\mathbf{x}'\rangle = |k|\, e^{ih(c+\mathcal{C}+\mathbf{a}\cdot\mathbf{b}/2)}\, e^{ih\mathbf{a}\cdot(k\Lambda\mathbf{x}+\mathbf{b})}\,|k\,\Lambda\mathbf{x}+\mathbf{b}\rangle\,, \tag{31}$$

see (10), (13) and (14). This action allows to calculate the action of a UIR on $L^2(\mathbb{X})$

$$(U(g)f)(\mathbf{x}) = |k|^{-1}\, e^{ic}\, e^{-ik^{-1}\mathbf{a}\cdot\Lambda^{-1}(\mathbf{x}-\mathbf{b})}\, f(k^{-1}\Lambda^{-1}(\mathbf{x}-\mathbf{b}))\,. \tag{32}$$

The interested reader can easily compute similar expressions for $|\mathbf{p}\rangle$ and $f(\mathbf{p})$.

3.5 Bases on $L^2(\mathbb{X})$

The HFs $\psi_\alpha(x_\alpha)$ determine an orthonormal basis on $L^2(\mathbb{R})$ (Subsection 2.2). So the functions

$$\Psi_{\mathbf{m}}(\mathbf{x}) := \psi_{m_1}(x^1)\,\psi_{m_2}(x^2)\,, \qquad \mathbf{m} = (m_1, m_2) \in \mathbb{N}^2\,, \tag{33}$$

constitute an orthonormal basis on $L^2(\mathbb{X})$, i.e. for any $f(x) \in L^2(\mathbb{X})$ we have that

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{N}^2}^{\infty} c^{\mathbf{m}}\,\Psi_{\mathbf{m}}(\mathbf{x}) \equiv \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} c^{m_1,m_2}\;\psi_{m_1}(x^1)\,\psi_{m_2}(x^2)\,, \quad c^{m_1,m_2} \in \mathbb{C}\,. \tag{34}$$

The double HF or the 2D HF functions $\Psi_{\mathbf{m}}(\mathbf{x})$ verify the following relations

$$\begin{aligned} \int_{\mathbb{R}^2} d\mathbf{x}\,[\Psi_{\mathbf{m}'}(\mathbf{x})]^*\,\Psi_{\mathbf{m}}(\mathbf{x}) &= \delta_{\mathbf{m},\mathbf{m}'} \equiv \delta_{m_1,m_1'}\,\delta_{m_2,m_2'}\,,\\ \sum_{\mathbf{m}\in\mathbb{N}^2} [\Psi_{\mathbf{m}}(\mathbf{x})]^*\,\Psi_{\mathbf{m}}(\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}) \equiv \delta(x^1-y^1)\,\delta(x^2-y^2)\,. \end{aligned} \tag{35}$$

They are real functions and eigenfunctions of the FT and of its inverse, i.e.,

$$FT\,[\Psi_{\mathbf{m}}(\mathbf{x});\mathbf{x};\mathbf{p}] = i^{\tilde{\mathbf{m}}}\,\Psi_{\mathbf{m}}(\mathbf{p})\,, \qquad IFT\,[\Psi_{\mathbf{m}}(\mathbf{p});\mathbf{p};\mathbf{x}] = (-i)^{\tilde{\mathbf{m}}}\,\Psi_{\mathbf{m}}(\mathbf{x})\,, \quad \tilde{\mathbf{m}} := \sum_{\alpha} m_\alpha\,. \tag{36}$$

In this 2D case we can profit from the invariance properties of 2D HF to construct a representation of the groups K_2 (or $K_{1,1}$) supported on a set of *generalized* HF. We start by defining

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda) := |k|\,e^{-i\,\mathbf{a}(k\Lambda\mathbf{x}+\mathbf{b}/2)}\,\Psi_{\mathbf{m}}(k\Lambda\mathbf{x}+\mathbf{b})\,. \tag{37}$$

Now we are able to obtain an explicit form of the 2D generalized HF in terms of the 1D generalized HF, (17) and (18), as

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda) = \chi_{m_1}((\Lambda\mathbf{x})^1,a^1,b^1,k)\,\chi_{m_2}((\Lambda\mathbf{x})^2,a^2,b^2,k)\,, \tag{38}$$

where $(\Lambda\mathbf{x})^\alpha$ denotes la α-th contravariant component of the vector $\Lambda\mathbf{x}$.

The 2D GHF determine an orthonormal basis on $L^2(\mathbb{X})$ since

$$\begin{aligned} \int_{\mathbb{R}^2} d\mathbf{x}\,\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,[\mathfrak{X}_{\mathbf{m}'}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)]^* &= \delta_{\mathbf{m},\mathbf{m}'}\,,\\ \sum_{\mathbf{m}\in\mathbb{N}^2} \mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{y},\mathbf{a},\mathbf{b},k,\Lambda)]^* &= \delta(\mathbf{x}-\mathbf{y})\,. \end{aligned} \tag{39}$$

In addition, for the FT in 2D and its inverse we have the following relations:

$$\begin{aligned} FT\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,;\,\mathbf{x},\mathbf{p}] &= i^{\tilde{\mathbf{m}}}\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{p},\mathbf{b},-\mathbf{a},k^{-1},\Lambda^{-1T})\,,\\ IFT\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{p},\mathbf{a},\mathbf{b},k,\Lambda)\,;\mathbf{p};\mathbf{x}] &= (-i)^{\tilde{\mathbf{m}}}\,\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{b},-\mathbf{a},k^{-1},\Lambda^{-1T})\,. \end{aligned} \tag{40}$$

4 Conclusion

We present a revision of some generalizations of the Euclidean groups [8–10] by considering as an ensemble the equivalence of conjugate variables, and the properties of homogeneity, self-similarity and invariance from orientation that are present in the description of physical systems. The group extensions of the Euclidean-like groups by the HW group give rise to new groups that amalgamate the symmetries associated to both groups together with the invariances that we have just mentioned above. Moreover these groups $K_{p,q}$ (with $q+p=n$) admit a representation in terms of $(n+2)\times(n+2)$ matrices. In particular, we have displayed here the low dimensional cases (1D and 2D). The nD case can be easily implemented from the 2D case [7]. Thus, the elements of the n-D Heisenberg-Weyl group are given (see expression (20)) by

$$H_{p,q}[\mathbf{a},\mathbf{b},c] \equiv \begin{pmatrix} 1 & \mathbf{a}^T & c\\ \mathbf{0} & \mathbb{I}_n & \mathbf{b}\\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \qquad \mathbf{a},\mathbf{b}\in\mathbb{R}^{(p,q)}\,,\ \ c\in\mathbb{R}\,. \tag{41}$$

Now according to (21) and (23) we can write the matrix elements of the new group $K_{p,q}$ as

$$K_{p,q}[\mathbf{a},\mathbf{b},c,k,\Lambda] \equiv \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\Lambda & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \qquad \mathbf{a},\mathbf{b} \in \mathbb{R}^{(p,q)},\ k \in \mathbb{R}^*,\ \Lambda \in SO(p,q). \tag{42}$$

Since the HF are an orthogonal basis of $L^2(\mathbb{R}^1)$ a basis on $L^2(\mathbb{R}^{p,q})$ is obtained in terms of nD HF, which can be easily obtained taking into account formula (33). The function spaces $L^2(\mathbb{R}^{p,q})$ support a UIR of the group $K_{p,q}$, that allows us to define a new set of orthonormal functions, the nD generalized Hermite functions following the expressions (37) and (38).

The existence of both discrete and continuous bases supporting representations of $K_{p,q}$ lead us to introduce a generalization of the Hilbert spaces: the rigged Hilbert spaces (or Gel'fand triplets) [12]. Then the infinitesimal generators of $K_{p,q}$ realized by self-adjoint operators on $L^2(\mathbb{R}^{p,q})$ are, generally, unbounded become bounded (continuous) operators (on two different locally convex topologies) using these rigged Hilbert spaces [7].

The nD Hermite functions appear in many quantum systems with quadratic Hamiltonians [15, 16], hence our results could be of interest, for instance, in Quantum Optics (photon distribution on multimodes mixed states [17]), in multidimensional signals analysis (decomposition of signals in terms of wavelets involves Fourier transform or Gabor transform [3,18,19]) and in vision studies [20–22].

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