Group theoretical derivation of consistent particle theories

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Abstract

Current quantum theories of an elementary free particle assume unitary space inversion and anti-unitary time reversal operators. In so doing robust classes of possible theories are discarded. The present work shows that consistent theories can be derived through a strictly deductive development from the principle of relativistic invariance and position covariance, also with anti-unitary space inversion and unitary time reversal operators. In doing so the class of possible consistent theories is extended for positive but also zero mass particles. In particular, consistent theories for a Klein-Gordon particle are derived and the non-localizability theorem for a non zero helicity massless particle is extended.

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1 Introduction

Relativistic quantum theories of single free particle can be deductively derived from the principles of *relativistic invariance* and *covariance* [1] - [4]; the first principle implies that the Hilbert space of the quantum theory of a free particle must admit a *transformer triplet* $(U, \mathcal{S}, \mathcal{T})$ formed by a unitary representation U of the universal covering group $\tilde{\mathcal{P}}_+^\uparrow$ of the proper orthochronus Poincaré group $\mathcal{P}_+^\uparrow$ and by the operators $\mathcal{S}$ and $\mathcal{T}$, which realize the quantum transformations implied by the transformations of $\mathcal{P}_+^\uparrow$, by space inversion $\mathfrak{s}$ and by time reversal $\mathfrak{t}$, respectively. Yet the literature, except some works [5] [6] with specific aims different from the present one, excludes transformer triplets with $\mathcal{S}$ anti-unitary or with $\mathcal{T}$ unitary, from the pionering works of Wigner, Bargmann [1] - [3], to subsequent investigations [4] - [9] . In so doing robust classes of triplets, and hence of possible theories, are lost. For instance, there is no such a triplet for a consistent theory of Klein-Gordon particles.[1]

The motivation for the exclusion of $\mathcal{T}$ unitary or $\mathcal{S}$ anti-unitary was their implication of negative spectral values for the hamiltonian operator P_0, values deemed inconsistent because

[1]Klein-Gordon theory, indeed, was obtained through canonical quantization [10], [11], but it predicts inconsistencies, such as negative probabilities [12].

P_0 was identified with the *positive* relativistic kinetic energy operator $E_{kin} = \mu(1-\dot{\mathbf{Q}}^2)^{-1/2}$, where $\dot{\mathbf{Q}}$ is the "velocity" operator. But remark 3.1 shall show that the hamiltonian operator P_0 does not always coincide with E_{kin}, so that a unitary T or an anti-unitary S can be consistent.

In the present article we show how a strictly deductive development of consistent quantum theories of elementary free particle can be successfully carried out without *apriori* preclusions about the unitary or anti-unitary character of S or T. As results, classes of consistent possible theories for a positive mass particle are expicitly identified, which meaningfully extend the class of the current theories; in particular, consistent theories of Klein-Gordon particle are derived. Also in the case of a massless particle the approach extends the class of possible theories. Furthermore, the non-localizability theorem for non zero helicity massless particles is extended to the new theories with T unitary or S anti-unitary.

Section 2 shows how the relativistic invariance principle implies that every theory of elementary free particle admits a transformer triplet. In section 3 the class of possible consistent theories for a positive mass particle is identified; this class contains consistent theories with S anti-unitary, e.g. consistent theories of Klein-Gordon particle. Section 4 identifies the class of consistent theories for a zero mass elementary free particle; once again, besides the current theories, it contains theories with S anti-unitary or T unitary. A more accurate and more general argument is presented, which denies localizability of non zero helicity mass zero particles

2 General implications of Poincaré invariance

2.1 Prerequisites and notation

First of all, it is worth to fix the notaion for any quantum theory based on a Hilbert space $\mathcal{H}$:

- $\Omega(\mathcal{H})$ denotes the set of all self-adjoint operators representing observables;

- $\mathcal{S}(\mathcal{H})$ denotes the set of all density operators ρ identified with quantum states;

- $\mathcal{U}(\mathcal{H})$ denotes the group of all unitary unitary operators;

- $\mathcal{V}(\mathcal{H})$ is the larger group of all unitary or anti-unitary operators.

The Poincaré group $\mathcal{P}$ is a very important mathematical structure for the present work, because it is the group of symmetry transformations for a free particle. $\mathcal{P}$ is the group generated by $\mathcal{P}_+^\uparrow \cup \{\mathfrak{t}, \mathfrak{s}\}$, where $\mathcal{P}_+^\uparrow$ is the proper orthochronus Poincaré group, $\mathfrak{t}$ and $\mathfrak{s}$ are the time reversal and space inversion transformations. The proper orthochronus group $\mathcal{P}_+^\uparrow$ is a connected group generated by 10 one-parameter subgroups, namely the subgroup $\mathcal{T}_0$ of time translations, the three subgroups $\mathcal{T}_j$ $(j = 1,2,3)$ of spatial translations, the three subgroups $\mathcal{R}_j$ of spatial rotations, the three subgroups $\mathcal{B}_j$ of Lorentz boosts, relative to the three spatial axes x_j. Time reversal $\mathfrak{t}$ and space inversion $\mathfrak{s}$ are not connected with the identity transformation $e \in \mathcal{P}$. Given any vector $\underline{x} = (x_0, \mathbf{x}) \in \mathbb{R}^4$, where x_0 is called the *time component* of $\underline{x}$ and $\mathbf{x} = (x_1, x_2, x_3)$ is called the *spatial component* of $\underline{x}$, time reversal $\mathfrak{t}$ transforms $\underline{x} = (x_0, \mathbf{x})$ into $(-x_0, \mathbf{x})$ and space inversion $\mathfrak{s}$ transforms $\underline{x} = (x_0, \mathbf{x})$ into $(x_0, -\mathbf{x})$.

The universal covering group of $\mathcal{P}_+^\uparrow$ is the semidirect product $\tilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \circledS SL(2,\mathbb{C})$ of the time-space translation group $\mathbb{R}^4$ and the group $SL(2,\mathbb{C}) = \{\underline{\Lambda} \in GL(2,\mathbb{C}) \mid \det \underline{\Lambda} = 1\}$. Accordingly, $\tilde{\mathcal{P}}_+^\uparrow$ is simply connected and there is a canonical homomorphism $\mathsf{h} : \tilde{\mathcal{P}}_+^\uparrow \to \mathcal{P}_+^\uparrow$, $\tilde{g} \to \mathsf{h}(\tilde{g}) \in \mathcal{P}_+^\uparrow$, which restricts to an isomorphism within a small enough neighborhood of the identity $(0, \mathbb{I}_{\mathbb{C}^2})$ of $\tilde{\mathcal{P}}_+^\uparrow$. By $\tilde{\mathcal{T}}_0$, $\tilde{\mathcal{T}}_j$, $\tilde{\mathcal{R}}_j$, $\tilde{\mathcal{B}}_j$, $\tilde{\mathcal{L}}_+^\uparrow$ we denote the subgroups of $\tilde{\mathcal{P}}_+^\uparrow$ which correspond to the subgoroups $\mathcal{T}_0$, $\mathcal{T}_j$, $\mathcal{R}_j$, $\mathcal{B}_j$, $\mathcal{L}_+^\uparrow$ of $\mathcal{P}_+^\uparrow$, through the homomorphism h.

2.2 Quantum theoretical implications for an elementary free particle

Since a free particle is a particular kind of isolated system, we begin by showing the derivation of the general structure of the quantum theory of an isolated system. By $\mathcal{F}$ we denote the class of the (inertial) reference frames that move uniformly with respect to each other. A physical system is an *isolated system* if the following *invariance principle* holds.

$\mathcal{IP}$ *The theory of an isolated system is invariant with respect to changes of frames within* $\mathcal{F}$.

If Σ belongs to $\mathcal{F}$, then Σ_g denotes the frame related to Σ by such g, for every $g \in \mathcal{P}$. Given an observable $\mathcal{A}$ represented by the operator $A \in \Omega(\mathcal{H})$, let $\mathcal{M}_A$ be a procedure to measure $\mathcal{A}$; then the invariance principle implies that another measuring procedure $\mathcal{M}'_A$ must exist, which is with respect to Σ_g identical to what is $\mathcal{M}_A$ with respect to Σ, otherwise the principle $\mathcal{IP}$ would be violated. Hence, $\mathcal{IP}$ implies the existence [12] of the so called *quantum transformation associated to* g, i.e., of a mapping

$$S_g : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}), \quad A \to S_g[A],$$

where $S_g[A]$ is the self-adjoint operator that represents the observable measured by $\mathcal{M}'_A$.

To every element $\tilde{g}$ of the covering group $\tilde{\mathcal{P}}_+^\uparrow$ we can associate the quantum transformation $S_{\mathsf{h}(\tilde{g})} \equiv S_{\tilde{g}}$ through the canonical homomorphism h. In [12] it is proved that the properties of quantum transformations, under a continuity condition for $\tilde{g} \to S_{\tilde{g}}$, imply that

Imp.1. a continuous unitary representation U of $\tilde{\mathcal{P}}_+^\uparrow$ exists such that $S_{\tilde{g}}[A] = U_{\tilde{g}} A U_{\tilde{g}}^{-1}$, and

Imp.2. two operators $\mathsf{S}, \mathsf{T} \in \mathcal{V}(\mathcal{H})$ exist such that $S_{\mathsf{s}}[A] = \mathsf{S} A \mathsf{S}^{-1}$ and $S_{\mathsf{t}}[A] = \mathsf{T} A \mathsf{T}^{-1}$.

Thus, the principle $\mathcal{IP}$ has the following fundamental implication.

(FI) *The quantum theory of an isolated system admits a transformer triplet* $(U, \mathsf{S}, \mathsf{T})$ *such that implications Imp.1 and Imp.2 hold.*

Given a transformer triplet $(U, \mathsf{S}, \mathsf{T})$, let $P_0, P_j, J_j, K_j \in \Omega(\mathcal{H})$ be the *selfadjoint generators* of U; so [12], if $\tilde{g} \in \tilde{\mathcal{T}}_0$ (resp., $\tilde{\mathcal{T}}_j$, $\tilde{\mathcal{R}}_j$, $\tilde{\mathcal{B}}_j$) is identified by the parameter t (resp., a, θ, u), then

$$U_{\tilde{g}} = e^{iP_0 t}, \quad (\text{resp.}, U_{\tilde{g}} = e^{iP_j a},\ U_{\tilde{g}} = e^{J_j \theta},\ U_{\tilde{g}} = e^{iK_j \frac{1}{2}\ln\frac{1+u}{1-u}}). \tag{1}$$

The generator P_0 relative to time translations is the *hamiltonian operator*, so that

$$(i) \quad \frac{d}{dt} A_t \equiv \dot{A}_t = i[P_0, A_t], \qquad (ii) \quad \frac{d}{dt}\rho_t \equiv \dot{\rho}_t = -i[P_0, \rho_t]. \tag{2}$$

By *"elementary" free particle* we mean an isolated system whose quantum theory has a *unique* three-operator $\mathbf{Q} \equiv (Q_1, Q_2, Q_3)$ with $Q_j \in \Omega(\mathcal{H})$, called *position* operator, such that $(U(\tilde{\mathcal{P}}_+^\uparrow), \mathsf{S}, \mathsf{T}; \mathbf{Q})$ is an *irreducible* system of operators, and satisfying the following conditions.

(Q.1) $[Q_j, Q_k] = \mathbb{O}$, for all $j, k = 1, 2, 3$; this condition establishes that a measurement of position yields all three values of the coordinates of the same specimen of the system.

(Q.2) For every $g \in \mathcal{P}$, the position operator $\mathbf{Q}$ and the transformed position operator $S_g[\mathbf{Q}]$ satisfy the transformation properties of position with respect to g.

As proved in [12], the transformer triplet $(U, \mathsf{S}, \mathsf{T})$ of the quantum theory of an elementary free particle must be *irreducible*. Thus, the identification of all possible theories of an elementary free particle can be carried out in two steps: first by identifying all irreducible transformer triplets $(U, \mathcal{S}, \mathcal{T})$, and then selecting those triplets for which a unique position operator $\mathbf{Q}$ exists.

The mathematical group structural properties of $\mathcal{P}$ imply [12], [16] that each irreducible triplet $(U, \mathcal{S}, \mathcal{T})$ is characterized by a number $\mu \in \mathbb{C}$, called *mass*, with $\mu^2 \in \mathbb{R}$, such that $P_0^2 - \mathbf{P}^2 = \mu^2 \mathbb{I}$.

3 Quantum theories of positive mass elementary free particle

To identify the positive mass possible theories, we shall identify the irreducible triplets with $\mu > 0$; then, the triplets admitting a three-operator $\mathbf{Q}$ satisfying $(Q.1)$, $(Q.2)$ are singled out.

3.1 Positive mass irreducible triplets

Following [12], for any pair (μ, s), where $\mu > 0$ and s is an integral or half-integral number $s \in \frac{1}{2}\mathbb{N}$ called *spin*, there is at least one irreducible triplet. Conversely, every irreducible triplet is characterized by one such a pair. The following theorem yields a first classification.

Theorem 3.1. *If $(U, \mathcal{S}, \mathcal{T})$ is an irreducible triplet with non-negative mass $\mu \geq 0$, then*
i) $\sigma(P_0) = (-\infty, -\mu]$ or $\sigma(P_0) = [\mu, \infty)$ or $\sigma(P_0) = (-\infty, -\mu] \cup [\mu, \infty)$, where $\sigma(P_0)$ is the spectrum of P_0.

Moreover, $\sigma(P_0) = (-\infty, -\mu] \cup [\mu, \infty)$ if and only if $\mathcal{T}$ is unitary or $\mathcal{S}$ is anti-unitary.
ii) Each class $\mathcal{I}(\mu, s)$ of all irreducible triplets with positive mass $\mu > 0$ decomposes as

$$\mathcal{I}(\mu, s) = \mathcal{I}^-(\mu, s) \cup \mathcal{I}^+(\mu, s) \cup \mathcal{I}^{-+}(\mu, s), \tag{3}$$

where $\mathcal{I}^-(\mu, s)$, $\mathcal{I}^+(\mu, s)$ and $\mathcal{I}^{-+}(\mu, s)$ are respectively the classes of irreducible triplets with $\sigma(P_0) = (-\infty, -\mu]$, $\sigma(P_0) = [\mu, \infty)$ and $\sigma(P_0) = (-\infty, -\mu] \cup [\mu, \infty)$.

The representation U of a triplet in $\mathcal{I}^+(\mu, s)$ or $\mathcal{I}^-(\mu, s)$ can be irreducible or not. We refer to [12] for a complete identification of the irreducible triplets of $\mathcal{I}^\pm(\mu, s)$ with U irreducible. Therein also instances of triplet in $\mathcal{I}^+(\mu, s)$ and $\mathcal{I}^+(\mu, s)$ with U reducible are explicitly shown.

The representation U of a triplet in $\mathcal{I}^{-+}(\mu, s)$ is always reducible [12], namely $U = U^+ \oplus U^-$ where $U^\pm$ belongs to a triplet in $\mathcal{I}^\pm(\mu, s)$. Moreover, U^+ is reducible if and only if U^- is reducible.

The class of all irreducible triplets of $\mathcal{I}^{-+}(\mu, s)$ with U^+ irreducible can be found in [12], where also triplets of $\mathcal{I}^{-+}(\mu, s)$ with U^+ reducible are concretely shown.

3.2 Theories of elementary free particle with positive mass

To determine the possible theories of positive mass elemetary free particle, we have to select irreducible triplets of $\mathcal{I}(\mu, s)$ identified in [12] for which a position $\mathbf{Q}$ satisfying $(Q.1)$ and $(Q.2)$ exists. Condition $(Q.2)$ can be only partially imposed. In fact, while the covariance properties with respect to translations, rotations, time reversal and space inversion are known and explicitly expressed by the following relations [12]

$$(i)\ [Q_j, P_k] = i\delta_{jk}, \quad (ii)\ [J_j, Q_k] = i\epsilon_{jkl} Q_l, \quad (iii)\ \mathcal{T}\mathbf{Q} = \mathbf{Q}\mathcal{T}, \quad (iv)\ \mathcal{S}\mathbf{Q} = -\mathbf{Q}\mathcal{T}, \tag{4}$$

the explicit relations that establish the transformation properties of position with respect to boosts are not available, yet [12]. However, conditions (4) are sufficient to uniquely identify $\mathbf{Q}$ for some subclasses of irreducible triplets, according to the following theorem [12].

Theorem 3.2. *Given a triplet in $\mathcal{I}^+(\mu,0)$ with U irreducible there is a unique three-operator* $\mathbf{Q}$*, satisfying (Q.1) and (4). Modulo unitary isomorphism, the resulting theory has Hilbert space* $\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$*, where* $d\nu(\mathbf{p}) = \frac{dp_1 dp_2 dp_3}{p_0}$ *with* $p_0 = \sqrt{\mu^2 + \mathbf{p}^2}$*,*

– *generators defined by*

$$(P_j\psi)(\mathbf{p}) = p_j\psi(\mathbf{p}), \quad (P_0\psi)(\underline{p}) = p_0\psi(\underline{p}), \quad J_k = J_k^{(0)}, \quad K_j = K_j^{(0)}, \tag{5}$$

where $J_k^{(0)} = -i\left(p_l\frac{\partial}{\partial p_j} - p_j\frac{\partial}{\partial p_l}\right)$, $K_j^{(0)} = ip_0\frac{\partial}{\partial p_j}$;

– $\mathcal{S} = \Upsilon$, $\mathcal{T} = \mathcal{K}\Upsilon$*, where* $\mathcal{K}$ *and* Υ *are defined by* $\mathcal{K}\psi(\mathbf{p}) = \overline{\psi(\mathbf{p})}$*,* $(\Upsilon\psi)(\mathbf{p}) = \psi(-\mathbf{p})$*.*

– *The position operator is* $\mathbf{Q} = \mathbf{F}$*, where* $\mathbf{F}$ *is the Newton-Wigner [13] operator defined by*

$$F_j = i\frac{\partial}{\partial p_j} - \frac{i}{2p_0^2}p_j\,. \tag{6}$$

Analogously, there is only one theory based on a triplet in $\mathcal{I}^-(\mu,0)$ *with* U *irreducible. It differs from that in* $\mathcal{I}^+(\mu,0)$ *by* $P_0 = -p_0$ *and* $K_j = -K_j^{(0)}$*. There are only two theories based on triplets of* $\mathcal{I}^{-+}(\mu,0)$ *with* U^+ *irreducible. They share the Hilbert space and generators:*[2]
$\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu) \oplus L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$

$$P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix}, \quad P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad J_k = \begin{bmatrix} J_k^{(0)} & 0 \\ 0 & J_k^{(0)} \end{bmatrix}, \quad K_j = \begin{bmatrix} K_j^{(0)} & 0 \\ 0 & -K_j^{(0)} \end{bmatrix}. \tag{7}$$

The two theories differ for the different pairs $(\mathcal{S}_1, \mathcal{T}_1)$*,* $(\mathcal{S}_2, \mathcal{T}_2)$ *of space inversion and time reversal operators; indeed* $\mathcal{S}_1 = \mathcal{S}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathcal{K}$ *while* $\mathcal{T}_1 = \mathcal{K}\Upsilon\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\mathcal{T}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*.*
For both theories the position operator is $\mathbf{Q} = \begin{bmatrix} \mathbf{F} & 0 \\ 0 & \mathbf{F} \end{bmatrix}$*.*

For all triplets with $s > 0$ (Q.1) and (4) are not sufficient [12] to completely identify $\mathbf{Q}$.

Remark 3.1. In both theories based on $\mathcal{I}^{-+}(\mu,0)$ the hamiltonian operator P_0 has also negative spectral values. But since the "velocity" is $\dot{\mathbf{Q}} = \frac{d}{dt}\mathbf{Q} = i[P_0, \mathbf{Q}] = \begin{bmatrix} \frac{\mathbf{p}}{p_0} & 0 \\ 0 & -\frac{\mathbf{p}}{p_0} \end{bmatrix}$, we compute that $E_{kin} = \mu(1 - \dot{\mathbf{Q}}^2)^{-1/2} = p_0 > \mathbb{0}$, i.e. the theories are consistent.

3.3 Conclusions for the positive mass case

According to section 3.2, four classes of possible consistent theories are completely determined by following the present approach, with U or U^+ irreducible. However, the class of theories based on $\mathcal{I}^{\pm}(\mu,0)$ with U reducible and the class of theories based on $\mathcal{I}^{-+}(\mu,0)$ with U^+ reducible are not empty; concrete examples are given in [12]. They are *new species theories*, i.e. they correspond to none of the known theories. Hence, our approach extends the class of consistent theories of positive mass elementary spin 0 free particle.

Moreover, it provides consistent theories for Klein-Gordon particles. Indeed, by means of a unitary transformation, operated by the operator $Z = Z_1 Z_2$, where $Z_2 = \frac{1}{\sqrt{p_0}}\mathbb{I}$ and Z_1 is the

[2] If $\psi \in L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu) \oplus L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$, we write $\psi \equiv \psi_1 \oplus \psi_2 \equiv \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$, $\psi_1, \psi_2 \in L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$).

inverse of the *Fourier-Plancherel* operator, the theories based on $\mathcal{I}^{-+}(\mu,0)$ turn out to be equivalent to two theories with Hilbert space $\hat{\mathcal{H}} = Z\left(L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu)\right) \equiv L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3)$, with the self-adjoint generators

$$\hat{P}_j = \begin{bmatrix} -i\frac{\partial}{\partial x_j} & 0 \\ 0 & -i\frac{\partial}{\partial x_j} \end{bmatrix}, \quad \hat{P}_0 = \sqrt{\mu^2 - \nabla^2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\hat{J}_j = -i\left(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{K}_j = \frac{1}{2}\left(x_j\sqrt{\mu^2-\nabla^2} + \sqrt{\mu^2-\nabla^2}x_j\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

while $\hat{\mathrm{T}}_1 = \mathcal{K}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{\mathrm{S}}_1 = \mathcal{K}\Upsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\hat{\mathrm{T}}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\hat{\mathrm{S}}_2 = \mathcal{K}\Upsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The position operator is $\hat{Q}_j = \begin{bmatrix} x_j & 0 \\ 0 & x_j \end{bmatrix}$.

These $\hat{P}_j$, $\hat{J}_j$, $\hat{K}_j$, $\hat{\mathcal{H}}$ are generators and Hilbert space of Klein-Gordon theory of spin-0 particle [10] [11]. However, since the position operator is the multiplication operator, the position probability density must be $\rho(t,\mathbf{x}) = |\hat{\psi}_1(t,\mathbf{x})|^2 + |\hat{\psi}_2(t,\mathbf{x})|^2$, hence non-negative. Thus, our extended class includes consistent theories for Klein-Gordon particle free from the inconsistent negative probabilities of the early theory.

It turns out [12] that in all triplets with non zero spin, the position operator $\mathbf{Q}$ is not uniquely determined by (Q.1) and (4). On the other hand, the transformation properties of position with respect to boosts, expressed for instance by a relation for $[K_j, Q_k]$, are not available in order to better identify $\mathbf{Q}$ by imposing them.

To each solution $\mathbf{Q}$ of (Q.1) and (4) there correspond a different $[K_j, Q_k]$, in general. For instance, Dirac theory for spin 1/2 particle [14] [15] is *completely* characterized by the relation $[K_j, Q_k] = -\frac{i}{2}(Q_j\dot{Q}_k + \dot{Q}_k Q_j)$ satisfied by the posistion operator of Dirac theory; however, other solutions $\mathbf{Q}$ yielding other relations for $[K_j, Q_k]$ are theoretically consistent too.

4 Quantum theories of zero mass elementary free particle

Analogously to the positive mass case, the possible quantum theories of zero mass particle are determined first by identifying the class $\mathcal{I}_0$ of the irreducible transformer triplets with $\mu = 0$, and then by selecting those triplets that admit a unique position operator. According to theorem 3.1.i the class $\mathcal{I}_0$ decomposes as $\mathcal{I}_0^+ = \mathcal{I}_0^+ \cup \mathcal{I}_0^- \cup \mathcal{I}_0^{-+}$, where $\mathcal{I}_0^-$ (resp., $\mathcal{I}_0^+$, $\mathcal{I}_0^{-+}$) denotes the class of irreducible triplets with $\sigma(P_0) = (-\infty, 0]$ (resp., $\sigma(P_0) = [0,\infty)$, $\sigma(P_0) = \mathbb{R}$).

4.1 Zero mass irreducible triplets

In [16] the irreducible triplets of $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$ with U irreducible, and of $\mathcal{I}_0^{-+}$ with U^+ irreducible are completely identified. The results are collected by the following statement.

Theorem 4.1. *Modulo unitary isomorphisms, there is only one triplet* $(U, \mathrm{S}, \mathrm{T})$ *in* $\mathcal{I}_0^+$ *and in* $\mathcal{I}_0^-$ *with* U *irreducible, whose Hilbert space is* $\mathcal{H} = L_2(\mathbb{R}^3, d\nu)$*, and*

$$(P_j\psi)(\mathbf{p}) = p_j\psi(\mathbf{p}), \quad P_0\psi(\mathbf{p}) = \pm p_0\psi(\mathbf{p}), \quad J_j = J_j^{(0)}, \quad K_j = \pm K_j^{(0)}, \quad \mathrm{S} = \Upsilon, \quad \mathrm{T} = \mathcal{K}\Upsilon.$$

If $(U, \mathrm{S}, \mathrm{T})$ *is an irreducible triplet of* $\mathcal{I}_0^{-+}$ *with* U^+ *irreducible, then* $m \in \mathbb{Z}$ *exists such* *that* $\mathcal{H} = L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu)$ *and*

$$P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix},$$
$$J_j = \begin{bmatrix} J_j^{(0)} + j_j & 0 \\ 0 & J_j^{(0)} - j_j \end{bmatrix}, \quad K_j = \begin{bmatrix} K_j^{(0)} + k_j & 0 \\ 0 & -K_j^{(0)} + k_j \end{bmatrix},$$

where $j_1 = \frac{m}{2}\frac{p_1 p_0}{p_1^2+p_2^2}$, $j_2 = \frac{m}{2}\frac{p_2 p_0}{p_1^2+p_2^2}$, $j_3 = 0$, $k_1 = -\frac{m}{2}\frac{p_2 p_3}{p_1^2+p_2^2}$, $k_2 = \frac{m}{2}\frac{p_3 p_1}{p_1^2+p_2^2}$, $k_3 = 0$.

With $m = 0$ *there are six triplets, each characterized by a different pair* $(\mathrm{T}_n, \mathcal{S}_n)$, $n = 1, 2, ..., 6$.

For every $m \neq 0$ *in* $\mathcal{I}_0^{-+}$ *there are two triplets with different pairs* $(\mathrm{T}_a, \mathcal{S}_a)$ *and* $(\mathrm{T}_b, \mathcal{S}_b)$.

Remark 4.1. *For the zero mass case the helicity operator* $\hat{\lambda} = \frac{\mathbf{J}\cdot\mathbf{P}}{p_0}$ *plays an important role. Theorem 4.1 and (5) imply [16] that* $\hat{\lambda} = 0$ *for the triplets in* $\mathcal{I}_0^{\pm}$.

Using theorem 4.1 we see that $\hat{\lambda} = -\frac{m}{2}$ *for every triplet of* $\mathcal{I}_0^{-+}$ *with* U^+ *irreducible.*

4.2 The theories of elementary free particle with zero mass

The possible theories of elementary free particle with zero mass can now be identified by selecting the triplets that admit a unique position operator. One conclusion shared by the past approaches states that no position operator exists for massless particles with non-zero helicity. Yet, the theoretical structures where such non-existence is proven [7] [9] are triplets where $\mathcal{S}$ is unitary and T is anti-unitary. The present approach highlights that this is a serious shortcoming, because according to theorem 3.1 these structures must be triplets in $\mathcal{I}_0^+$ or $\mathcal{I}_0^-$. But according to section 4.1 irreducible triplets with non-zero helicity can exist only in $\mathcal{I}_0^{-+}$. Therefore, these proofs do not apply.

In fact our approach proves the following theorems [16].

Theorem 4.2. *If* $\hat{\lambda} \neq 0$, *then in every triplet of* $\mathcal{I}_0$ *there is no three-operator satisfying* (Q.1) *and (4.i), (4.ii).*

Theorem 4.3. *For the triplet of* $\mathcal{I}_0^+$ *or of* $\mathcal{I}_0^-$, *with* U *irreducible, there is only one three-operator satisfying* (Q.1) *and* (4), *namely Newton-Wigner operator* $\mathbf{Q} = \mathbf{F}$.

Since the search for a position operator must be restricted to triplets with $\hat{\lambda} = 0$, in $\mathcal{I}_0^{-+}$ only triplets with $m = 0$ have to be checked.

Theorem 4.4. *The triplets of* $\mathcal{I}_0^{-+}$ *with a three-operator satisfying* (Q.1) *and* (4) *are three of the six triplets with* $m = 0$ *of theorem 4.1, characterized by* $\mathrm{T}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathcal{S}_1 = \mathcal{K}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *by* $\mathrm{T}_2 = \mathrm{T}_1$, $\mathcal{S}_2 = \Upsilon\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *and by* $\mathrm{T}_3 = \mathcal{K}\Upsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\mathcal{S}_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathcal{K}$.

In all the three theories $\mathbf{Q} = \begin{bmatrix} \mathbf{F} & 0 \\ 0 & \mathbf{F} \end{bmatrix}$.

4.3 Conclusions for the zero mass case

The current literature in fact restricts the search for theories of massless elementary free particle to triplets with T anti-unitary and $\mathcal{S}$ unitary, i.e. to triplets of $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$. Our approach proves that consistent theories can be developed also if T is unitary or $\mathcal{S}$ is anti-unitary. As a consequence, the class of possible theories extends to include a subclass of $\mathcal{I}_0^{-+}$.

Furthermore, the non-existence proofs of a position operator for non zero helicity massless particles extends to the larger class of possible theories, because the operators ⊲T and ⊲S play no role in the new theorem 4.2.

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