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NLO finite system size corrections to $2 \to 2$ scattering in ϕ^4 theory using newly derived sum of sinc functions

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Abstract

Previously an equation of state for the relativistic hydrodynamics encountered in heavy-ion collisions at the LHC and RHIC has been calculated using lattice gauge theory methods. This leads to a prediction of very low viscosity, due to the calculated trace anomaly. Finite system corrections to this trace anomaly could challenge this calculation, since the lattice calculation was done in an effectively infinite system. In order to verify this trace anomaly it is sensible to add phenomenologically relevant finite system corrections. We investigate massive ϕ^4 theory with periodic boundary conditions on n of the 3 spatial dimensions. $2 \to 2$ NLO scattering is then computed. Using a newly derived formula for an arbitrary dimensional sum of sinc functions, we show that the NLO finite size corrections preserve unitarity.

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1 Introduction

There is an apparent formation of Quark Gluon Plasma (QGP) in heavy ion collisions [1–3], where the correlations between the outgoing low-momentum particles appear to be well described by nearly inviscid relativistic hydrodynamics. This calculation uses an Equation of State (EoS) provided by a lattice QCD calculation that is extrapolated to infinite system size [4].

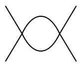
It is currently unclear what happens in QCD just above the transition temperature $T = 180$ MeV. There is strong evidence of a second order phase transition, but the nature of the new phase is largely unknown. It is therefore necessary to understand how reliably the experimental behaviour found in the finite systems (such as heavy ion or parton collisions) can be extrapolated to effectively infinite systems, such as the QGP found in the ~ 0.000001 seconds after Big Bang.

A possibly significant assumption to be investigated is that heavy ion collisions can be well approximated as infinite sized systems [5]. Indeed quenched lattice QCD calculations have shown significant possible corrections dependent on the size of the system [6]. An analytic derivation of the finite size effects on the equation of state (or equivalently the trace anomaly) is therefore sought. This work is a step in that direction, with the intention to develop and understand the mathematical techniques necessary for a full treatment necessary for finite temperature finite sized QCD.

2 Finite Sized ϕ^4 Theory

Let us consider the ϕ^4 Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 , \tag{1}$$

in a system with periodic boundary conditions. If we consider n compact spatial dimensions, with the ith dimension being parameterized by $[-\pi L_i, \pi L_i]$ with periodic boundary conditions. This discretizes the possible spatial momenta to $\vec{p} = \left(\frac{k_1}{L_1}, \frac{k_2}{L_2}, \ldots, \frac{k_n}{L_n}, \right)$ where $\vec{k} \in \mathbb{Z}^n$. In analogy with [7] we can define $-i\lambda^2 V(p^2) \equiv$  with p being the total incoming momentum. One then finds [8] in $n = 3$ spatial dimensions that, up to NLO, one gets the renormalized

$$\overline{V}(p^2, \{L_i\}) = -\frac{1}{2(4\pi)^2}\int_0^1 dx \Bigg\{ \log\left(\frac{\mu^2}{\Delta^2}\right)$$

$$+ 2\sum_{\vec{m}\in\mathbb{Z}^3}{}' \cos(2\pi x \sum m_i p^i L_i) K_0\left(2\pi\sqrt{\Delta^2}\sqrt{\sum (m_i L_i)^2}\right)\Bigg\} . \tag{2}$$

Here we recognize the log term as corresponding to the standard result in infinite ϕ^4 systems [7]. As one would then expect the second term in the integral vanishes in the limit as all $L_i \to \infty$ since $\lim_{x\to\infty} K_0(x) = 0$. We can see that we could reduce the effective number of finite dimensions by quite simply taking the corresponding $L_i \to \infty$, since only terms in the sum with the corresponding $m_i = 0$ will survive the limit. We then find that, as in the infinite system case,

$$\mathcal{M} = \lambda\left[1 + \lambda\left(\overline{V}(s) + \overline{V}(t) + \overline{V}(u)\right)\right] , \tag{3}$$

up to NLO, with s, t and u being the usual Mandelstam variables.

3 Unitarity

In order to verify that unitarity has stayed intact we will show that the optical theorem holds, no matter how many dimensions m are of finite size. For the optical theorem to hold, we need that

$$2\,\mathrm{Im}[\mathcal{M}] = \sigma_{\text{tot}} . \tag{4}$$

It is a straight-forward calculation to find

$$\sigma_{tot} = \frac{\lambda^2}{16\pi}\frac{\pi^{\frac{1-m}{2}}}{\Gamma\left(\frac{3-m}{2}\right)}\frac{1}{L\sqrt{s}}\sum_{0\le l<R^2}^{*}\frac{r_m(l)}{\sqrt{R^2-l}^{\,m-1}} . \tag{5}$$

By following [8] one gets that

$$2\,\mathrm{Im}[\mathcal{M}] = \frac{\lambda^2}{16\pi}\frac{2R}{L\sqrt{s}}\sum_{\vec{k}\in\mathbb{Z}^m}\mathrm{sinc}\left(2\pi R\|\vec{k}\|\right), \tag{6}$$

up to NLO. We can then use A.7 to get

$$2\,\mathrm{Im}[\mathcal{M}] = \frac{\lambda^2}{16\pi}\frac{2R}{L\sqrt{s}}\frac{1}{2R\pi^{\frac{m-1}{2}}\,\Gamma\left(\frac{3-m}{2}\right)}\sideset{}{^*}\sum_{0\leq l<R^2}\frac{r_m(l)}{\sqrt{R^2-l}^{\,m-1}}\,. \tag{7}$$

The equivalence of Equations 5 and 7 shows that the Optical Theorem, and therefore Unitarity, holds independent of the amount of compact dimensions.

4 Conclusion

By passing all considered self-consistency checks, namely having the correct infinite limit and preserving unitarity, we have shown the viability of the mathematical tools developed and employed. Notably it greatly supports the generalization of the number theoretic formula derived in A, which has potential implications in number theory.

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A Sum of Sinc functions

We will need a generalization of a formula for the multi-dimensional sum of sinc functions, namely sums of the form

$$\sum_{\vec{k}\in\mathbb{Z}^m}\mathrm{sinc}(2\pi R\|\vec{k}\|), \tag{A.1}$$

where we define $\mathrm{sinc}(x) = \begin{cases}\frac{\sin(x)}{x} & x\neq 0\\ 1 & x=0\end{cases}$.

Since the sum only depends on the magnitude of $\vec{k}$, we can simplify the sum using the *Sum of Squares function* $r_d(n)$ which gives the amount of $\vec{k}\in\mathbb{Z}^d$ with $\|\vec{k}\|^2 = n$. Allowing us to use

$$\sum_{l=0}^{\infty} r_d(l)\, f\left(\sqrt{l}\right) = \sum_{l=0}^{\infty} r_d(l)\, \hat{f}\left(\sqrt{l}\right) \tag{A.2}$$

from [9]. To employ their analytically continued version of the Poisson summation formula we can first set $f(r) = \mathrm{sinc}(2\pi R r)$. Then we can calculate $\hat{f}$

$$\hat{f}(p) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}\int_0^\infty \mathrm{sinc}(2\pi R r)\,{}_0F_1\left(\frac{m}{2};-\pi^2p^2r^2\right) r^{m-1}dr \tag{A.3}$$

$$= \frac{1}{2R\pi^{\frac{m-1}{2}}\,\Gamma\left(\frac{3-m}{2}\right)}(R^2-p^2)^{\frac{1-m}{2}}\theta(R^2-p^2)\,. \tag{A.4}$$

Now we can apply A.2

$$\sum_{\vec{k}\in\mathbb{Z}^m}\mathrm{sinc}\left(2\pi R\|\vec{k}\|\right) = \sum_{l=0}^{\infty} r_m(l)\,\mathrm{sinc}(2\pi R\sqrt{r}) \tag{A.5}$$

$$= \sum_{l=0}^{\infty} r_m(l)\frac{(R^2-l)^{\frac{1-m}{2}}}{2R\pi^{\frac{m-1}{2}}\,\Gamma\left(\frac{3-m}{2}\right)}\theta(R^2-l)\,. \tag{A.6}$$

Here we should note that if $r_m(R^2) \neq 0$ we will get a term with $\theta(0) = \frac{1}{2}$ which we note will give a $\frac{1}{0}$ term in the sum for $m > 1$. Therefore

$$\sum_{\vec{k}\in\mathbb{Z}^m}\mathrm{sinc}\left(2\pi R\|\vec{k}\|\right) = \frac{1}{2R\pi^{\frac{m-1}{2}}\,\Gamma\left(\frac{3-m}{2}\right)}\sum_{0\le l<R^2}^{*}\frac{r_m(l)}{\sqrt{R^2-l}^{\,m-1}}\,, \tag{A.7}$$

where $\sum^*_{0\le l<R^2}$ means that if $R^2 \in \mathbb{Z}$, then its term in the sum has weight $\frac{1}{2}$. The $m = 2$ case corresponds directly to a formula of Ramanujan [10], making this result a generalization thereof.

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