On the use of the operator product expansion
in finite-energy sum rules for light-quark correlators

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Abstract

Tau-based finite-energy sum-rule (FESR) analyses often assume that scales $s_0 \sim m_\tau^2$ are large enough that (i) integrated duality violations (DVs) can be neglected, and (ii) contributions from non-perturbative OPE condensates of dimension D scale as $(\Lambda_{\text{QCD}}/m_\tau)^D$, allowing the OPE series to be truncated at low dimension. The latter assumption is not necessarily valid since the OPE series is not convergent, while the former is open to question given experimental results for the electromagnetic, $I = 1$ vector (V), $I = 1$ axial vector (A) and $I = 1$ $V+A$ current spectral functions, which show DV oscillations with amplitudes comparable in size to the corresponding α_s-dependent perturbative contributions at $s \sim 2-3$ GeV^2. Here, we discuss recently introduced new tools for assessing the numerical relevance of omitted higher-D OPE contributions. Applying these to the "truncated OPE" strategy used in Refs. [1, 2] and earlier work by the same authors, we find that this strategy fails to yield reliable results for the strong coupling from hadronic τ decays.

1 Introduction

The determination of the strong coupling α_s from hadronic τ decays is interesting for two important reasons: (i) In principle, a high precision can be reached, if we "normalize" a determination at the τ mass scale by evolving the coupling to the Z mass, and (ii) because of the

low scale set by the τ mass, it provides a direct test of the running of the coupling predicted by QCD. However, at the same time, the strong coupling at the τ mass, $\alpha_s(m_\tau)$, is rather large (of order 0.3 in the $\overline{\text{MS}}$ scheme), and non-perturbative effects threaten to contaminate the extraction of $\alpha_s(m_\tau)$ from experimental data. It is thus important to introduce methods to quantify such contamination, and to test the resulting strategies for their reliability. In this talk, we discuss the "truncated OPE" (tOPE) strategy [3], which has most recently been used in Refs. [1, 2]. We will demonstrate that this strategy leads to results with unquantifiable systematic errors at the scale of the current desired level of precision, and hence should no longer be used.

The determination of $\alpha_s(m_\tau)$ starts by considering finite-energy sum rules (FESRs) of the form[1]

$$\int_0^\infty ds\, s^n\, \rho(s) = -\frac{1}{2\pi i} \oint_{|z|=s_0} dz\, z^n\, \Pi(z)\,, \tag{1}$$

where $\Pi(z)$ is the (scalar) vacuum polarization obtained from the $V + A$ non-strange $I = 1$ or the electro-magnetic (EM) spectral function $\rho(s)$ (with $s = q^2 > 0$), and the contour $|z| = s_0$ is a circle in the complex $q^2 = z$ plane around the origin with radius s_0, which, if $\rho(s)$ is obtained from hadronic τ decays, is bounded from above by the τ mass, $s_0 \leq m_\tau^2$.

Equation (1) is exact. To proceed, we then approximate

$$\Pi(z) = \Pi_{\text{pert.}}(z) + \Pi_{\text{OPE}}(z) + \Pi_{\text{DV}}(z)\,, \tag{2}$$

where $\Pi_{\text{pert.}}(z)$ is the perturbative part, for which a five-loop expression exists [5], $\Pi_{\text{OPE}}(z)$ is the operator-product-expansion (OPE) part,[2]

$$\Pi_{\text{OPE}}(q^2) = \frac{C_4}{(q^2)^2} - \frac{C_6}{(q^2)^3} + \frac{C_8}{(q^2)^4} + \dots\,, \tag{3}$$

and $\Pi_{\text{DV}}(z)$ is the part violating quark-hadron duality, which is not captured by the OPE [7].

The existence of a duality-violating (DV) part is closely related to the fact that the OPE is not convergent; rather it is (at best) an asymptotic expansion. According to expectations, the DV part, which physically represents the oscillations about perturbative expectations associated with the presence of resonances visible in the spectral function, decreases exponentially with q^2, i.e., non-perturbatively in the OPE expansion parameter $1/q^2$. We note that the weight z^n in Eq. (1) picks out the term proportional to $1/q^{2(n+1)}$ in the OPE—this will be important in what follows.

While the DV part is exponentially suppressed, the data are limited to $s \leq m_\tau^2$, and we are forced to consider the possibility that they may not be negligible. In this respect, it is instructive to consider Fig. 1. The oscillations in the spectral function (red data points) represent the presence of resonances, and are not captured by the OPE (black dashed curve). It is clear that DVs, which represent the oscillations of the data around the dashed curve, are not a small part of the dynamical QCD contribution to the spectral function in this region. Note that in order to make this comparison, one should subtract the parton-model contribution to the spectral function, as it is independent of α_s and thus not part of the dynamics produced by QCD.

2 The truncated-OPE strategy

Two different strategies have been developed to deal with the non-perturbative contamination, i.e., the $D > 0$ terms in the OPE and DVs: the tOPE strategy, and the "DV-model" strategy.[3]

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[1]For a more detailed description of our use of FESRs, see Ref. [4] and references therein.

[2]For the non-strange channel, the $D = 2$ term proportional to the square of the light quark masses can be safely neglected. We have also checked that the logarithmic dependence of the C_D on q^2 can be neglected [6].

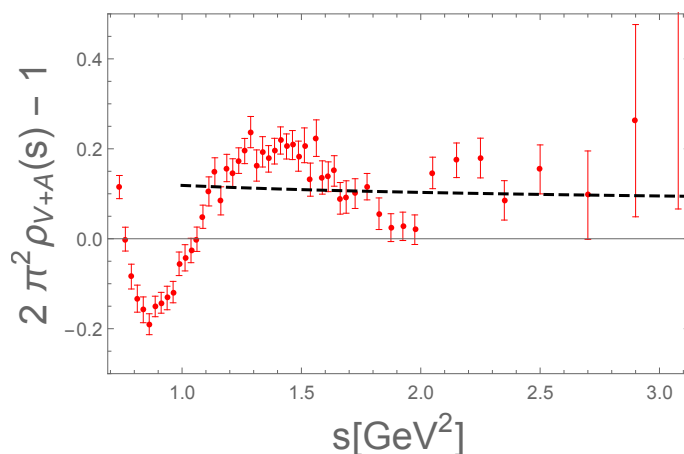[3]A variant of the tOPE strategy can be found in Ref. [8].

Figure 1: Blow-up of the ALEPH data (red data points) [1] in the large-s region of the $V+A$ non-strange spectral function. What is shown is $2\pi^2\rho_{V+A}-1$, *i.e.*, the dynamical QCD contribution to the spectral distribution. Black dashed line: perturbation theory, also with the parton-model contribution subtracted. Figure from Ref. [4].

For the latter, we refer to Boito's talk at this workshop [9]; the most recent application of the DV-model strategy can be found in Ref. [10].

The assumptions underlying the tOPE strategy are the following. First, DVs are neglected, but the dangerous region, where the circular contour on the right-hand side of Eq. (1) crosses the positive real axis, is suppressed by combining the weights of Eq. (1) into polynomials with multiple zeroes at $s = s_0$. These multiple zeroes are thus introduced to suppress DVs, at the intersection of the contour on the right-hand side of Eq. (1) with the positive real axis. Furthermore, s_0 is typically chosen equal to m_τ^2, in order to keep values of $\alpha_s(s_0)$ appearing in the fit as small as possible.

In implementations of the tOPE strategy in the literature, these polynomial have degrees varying between 3 and 7. This implies that OPE terms up to order $D = 16$ contribute to the right-hand side of Eq. (1), and one thus would have to fit $\alpha_s(m_\tau)$ as well as $C_{4,6,8,10,12,14,16}$, eight parameters in total. However, the number of independent polynomials of maximal degree 7 with at least two zeroes at $s = s_0$ is smaller than eight, and thus, necessarily, a number of parameters have to be set equal to zero by hand.

One such set of polynomials, used in Ref. [2] and referred to as "optimal" there, is the set

$$w_{2n} = 1 - (n+2)x^{n+1} + (n+1)x^{n+2}, \quad n = 1, \ldots, 5, \tag{4}$$

with $x = s/s_0$. These weights have a double zero at $s = s_0$ (they are "doubly pinched"), and probe $\alpha_s(s_0)$ as well as $C_{6,8,10,12,14,16}$. With five weights, and taking $s_0 = m_\tau^2$, one has five data points, which leads to the choice to set $C_{12} = C_{14} = C_{16} = 0$, so that one has four parameters for a fit to five data points,[4] Different, but similar, sets of weights have been considered in Ref. [2] as well as in our analysis of this strategy [4]. Here we will only discuss the set (4), as the conclusions from our more extensive study of other sets of weights is the same. Finally, in the tOPE strategy, one primarily considers the $V+A$ channel, as DV and OPE-truncation effects are argued to be less severe in $V+A$ than in the V or A channels separately.

In a first look at the tOPE strategy, let us compare two choices for the OPE coefficients C_{12}, C_{14}, and C_{16}, which are not part of the fit, and for which thus *a priori* values have to be chosen. We compare the choice of Refs. [1,2], which effectively sets $C_{12} = C_{14} = C_{16} = 0$ (choice 1),

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[4]The weights of Eq. (4) do not project on the $D = 4$ term in the OPE.

with choice 2:

$$C_{12} = 0.161 \, \text{GeV}^{12}, \qquad C_{14} = -0.17 \, \text{GeV}^{14}, \qquad C_{16} = -0.55 \, \text{GeV}^{16}. \qquad (5)$$

This choice is equally arbitrary, but equally reasonable. The results of applying the tOPE strategy with either of these two choices are:

	$\alpha_s(m_\tau)$	C_6 (GeV^6)	C_8 (GeV^8)	C_{10} (GeV^{10})	χ^2/dof
choice 1	0.317(3)	0.0014(4)	−0.0010(5)	0.0004(3)	1.26/1
choice 2	0.295(4)	−0.0130(4)	0.0356(5)	−0.0836(3)	1.09/1

$$(6)$$

The fits were done with FOPT [4], and only the fit errors are shown. The OPE coefficients in Eqs. (5) and (6) are very reasonable, increasing in absolute value with the order in the OPE, but consistent with it being an asymptotic expansion.

Clearly, the choice-1 and choice-2 fits are inconsistent, and lead to values of $\alpha_s(m_\tau)$ which are about 7% apart; this is about double the total error quoted in Ref. [2]. Moreover, there is no way to tell which of these two fits is closer to the truth; in fact, both fits may be wrong.

3 Tests of the truncated-OPE strategy on data for $e^+e^- \to$ hadrons

In order to probe this unsatisfactory state of affairs in more detail, we will apply the tOPE strategy next to R-ratio data obtained from $e^+e^- \to$ hadrons. The key observation is, of course, that if the tOPE strategy works at $s_0 = m_\tau^2$, as necessitated with data from τ decays, it should certainly work at $s_0 > m_\tau^2$, where R-ratio data are available.

There are, of course, differences with the τ-based approach. First, $e^+e^- \to$ hadrons only gives access to V channel data, whereas it is advocated to apply the tOPE strategy to $V + A$. However, Refs. [1,2] find that the tOPE strategy applied to the V channel τ-decay data yields results consistent with those from $V + A$, with equally good fit qualities. Another difference is that R-ratio data contain an $I = 0$ component, in addition to the $I = 1$ component related to the τ-based V spectral function. However, the $I = 0$ component is an $SU(3)$-flavor partner of the $I = 1$ component, and it is known that the strange quark mass that breaks $SU(3)$ has a very small effect on α_s [11]. We thus conclude that lessons learned from applying the tOPE strategy to R-ratio data are relevant for the application to τ-based analyses at as well. Below we will use R-ratio data from Refs. [12,13].

First, we repeat the fit with weights (4) at $s_0 \approx m_\tau^2$, now using the R-ratio data. We find[5]

$$\chi^2 \text{ fit}: \qquad \alpha_s(m_\tau) = 0.308(4), \qquad p\text{-value} = 2 \times 10^{-15}, \qquad (7)$$
$$\text{diagonal fit}: \qquad \alpha_s(m_\tau) = 0.245(10).$$

This is clearly a disaster. We note that a diagonal fit uses a diagonal fit quality, but takes the full data covariance matrix into account for error propagation; for more detail on this, see Refs. [6,14]. Next, let us take s_0 larger: if we take $s_0 = 3.6 \, \text{GeV}^2$, the p-value of the χ^2 fit becomes larger than 10%. In fact, at $s_0 = 3.6 \, \text{GeV}^2$, we find

$$\chi^2 \text{ fit}: \qquad \alpha_s(m_\tau) = 0.264(5), \qquad p\text{-value} = 0.41, \qquad (8)$$
$$\text{diagonal fit}: \qquad \alpha_s(m_\tau) = 0.256(12).$$

This is a clear improvement, but it yields a very low value for α_s: at the Z mass, this would translate into $\alpha_s(m_Z) = 0.110$! Other sets of weights considered in Refs. [1,2] lead to very similar results. For more details, we refer to Ref. [14].

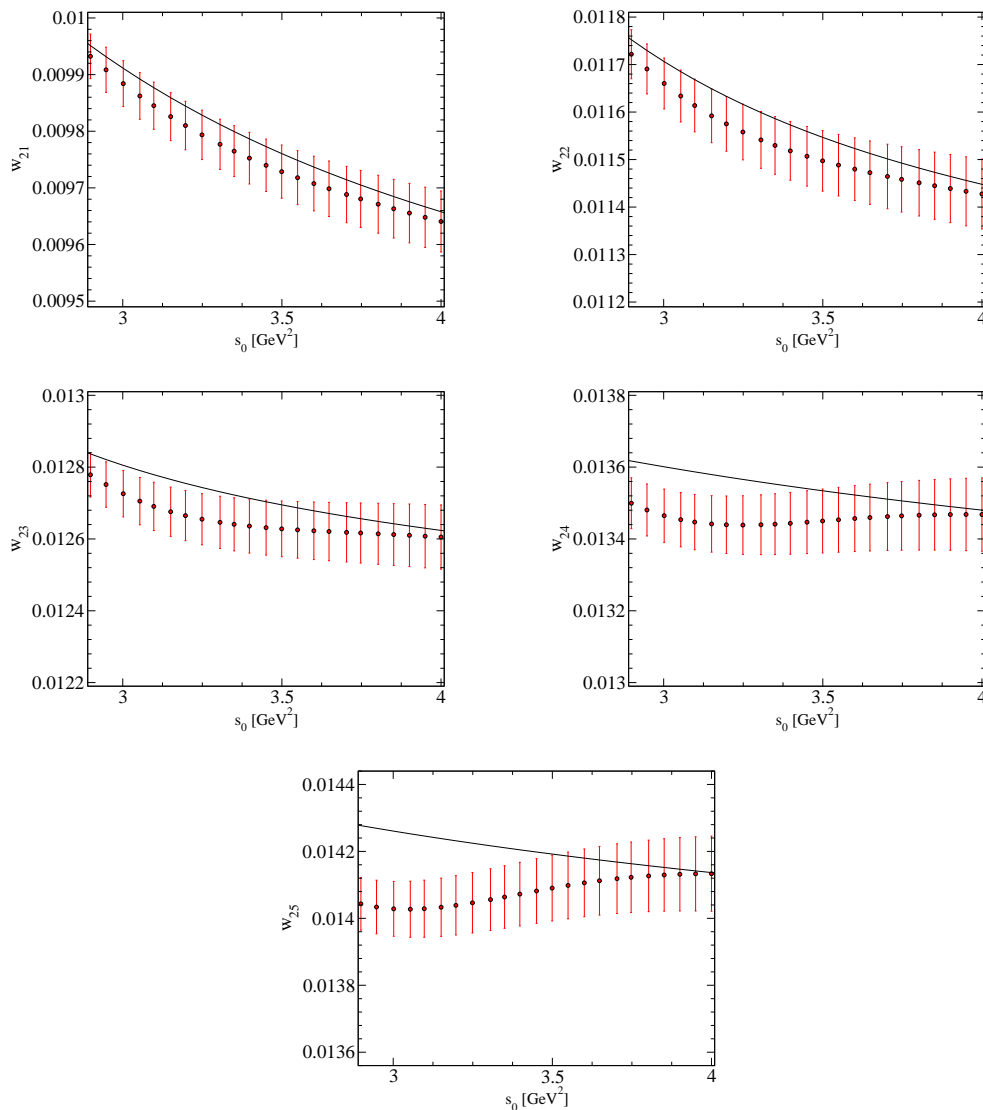[5]We omit results for the OPE coefficients $C_{D>0}$ for this discussion.

Figure 2: tOPE fits using optimal weights with $s_0 = s_0^* = 3.6$ GeV^2 in Eq. (1). The red data points show the left-hand side of Eq. (1) for each weight; the black curves show the fits of the right-hand side. Figure from Ref. [14].

One might argue that the tests of Eqs. (7,8) are possibly somewhat inconclusive. However, the R-ratio data allow us to subject the tOPE strategy to a more stringent test, by considering the s_0 dependence of tOPE-based fits. Again, the simple observation is that if the tOPE strategy works at values of $s_0 \geq m_\tau^2$, there should be a good match between theory and experiment for *all* values of $s_0 \geq m_\tau^2$. Figure 2 shows the left-hand side and the right-hand side of Eq. (1) obtained in tOPE χ^2 fits employing optimal weights at a fixed $s_0 \equiv s_0^* = 3.6$ GeV^2, as a function of s_0. While superficially, one might conclude that the agreement between experiment (left-hand side of Eq. (1)) and theory (right-hand side Eq. (1)) as a function of s_0 is not unreasonable, this is not actually the case. In fact, it is not easy to judge the level of agreement visually, as there are strong correlations, both between the spectral integrals at different s_0, between the fitted theory integrals at different s_0, and between the theory integrals and the spectral integrals used to fit the parameters of the theory representation. A careful look reveals a possible sign of trouble: clearly the slopes of the fit curves at $s_0 > s_0^*$ are rather different than the corresponding slopes in the data, for the weights w_{23}, w_{24} and w_{25}.
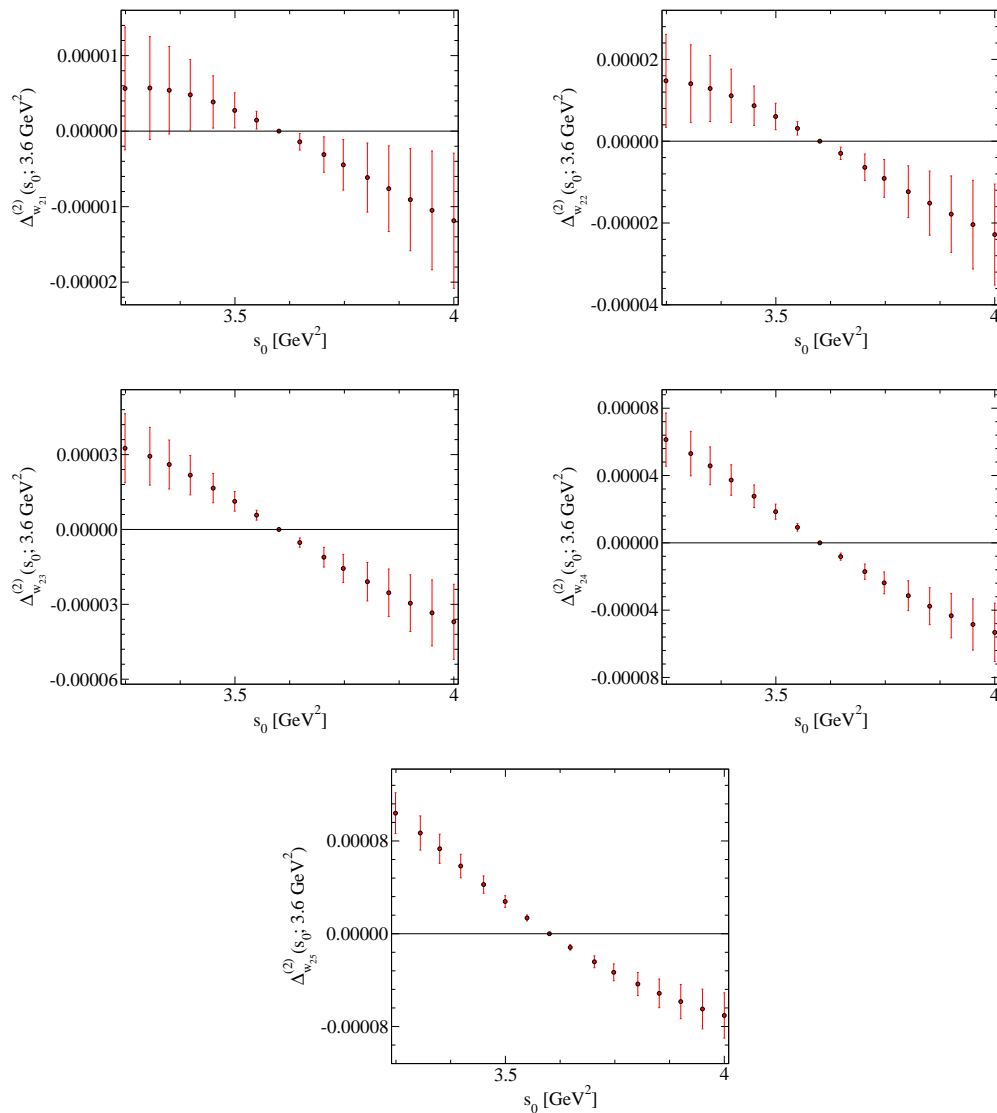
Figure 3: The double differences $\Delta^{(2)}(s_0; s_0^*)$ as a function of s_0, for $s_0^* = 3.6$ GeV^2. See text. Figure from Ref. [14].

Whether this difference in slopes is statistically significant or not can be investigated by considering the double differences

$$\Delta^{(2)}(s_0; s_0^*) = \left[I_w^{\text{th}}(s_0) - I_w^{\text{exp}}(s_0) \right] - \left[I_w^{\text{th}}(s_0^*) - I_w^{\text{exp}}(s_0^*) \right], \tag{9}$$

where $I_w^{\text{exp}}(s_0)$ denotes the left-hand side of Eq. (1) for polynomial weight w, and $I_w^{\text{th}}(s_0)$ denotes the right-hand side of Eq. (1). These double differences compare theory (*i.e.*, the fits) with experiment, relative to a reference value s_0^*. Of course, it is important, in computing these double differences, to take all correlations, including those between data and fitted parameters, into account. These double differences should be consistent with zero for the tOPE strategy to pass this type of test.

Figure 3 shows the double differences for the fits shown in Fig. 2. Clearly, for at least the weights w_{23}, w_{24} and w_{25}, they are very far from consistent with zero, for values of s_0 on both sides of s_0^*. (By construction, the double differences vanish exactly at $s_0 = s_0^*$.) From these (and other, see Ref. [14]) R-ratio based tests, we conclude that the tOPE strategy fails.

Because of the close similarity between the EM and τ-based spectral functions, it is clear that also for hadronic τ decays, the tOPE cannot be trusted to yield reliable results.

4 Conclusion

Since the τ mass is relatively light, one has deal with the question of possible non-perturbative contamination in any strategy to determine the strong coupling from hadronic τ-decay data. In order to do this, assumptions are needed, and these assumptions need to be tested. Here we considered the truncated-OPE strategy, in which the main assumption is that higher-order terms in the OPE can be neglected, and thus effectively be set equal to zero. This constitutes an arbitrary choice, and it is particularly dangerous, given the asymptotic nature of the OPE, when working with FESRs involving weights for which unsuppressed contributions of high dimension are in principle present. We carried out several tests of the tOPE strategy, quantitatively probing the validity of the assumption made about the OPE.

We found that indeed this assumption cannot be trusted at scales of order the τ mass: The tOPE strategy does not pass EM-based self-consistency tests, described in Sec. 3. Moreover, in Sec. 2 we also showed that it does not pass $V+A$ τ-based self-consistency tests. Our conclusion is that the tOPE strategy is not reliable if the goal is to obtain $\alpha_s(m_\tau)$ with currently competitive accuracy. Values of $\alpha_s(m_\tau)$ obtained with this approach depend very strongly on arbitrary assumptions made about the OPE; these assumptions are not based on QCD.

For a much more detailed description and discussion of this work, as well as many more references, we refer to Refs. [4, 14].

Acknowledgments

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