Proving the dimension-shift conjecture

Ruth Britto[1,2], Guy R. Jehu[1⋆] and Andrea Orta[1]

1 School of Mathematics and Hamilton Mathematical Institute,
Trinity College Dublin, Ireland
2 Institut de Physique Théorique, Université Paris Saclay, France

⋆ jehu@maths.tcd.ie

15th International Symposium on Radiative Corrections:
Applications of Quantum Field Theory to Phenomenology,
FSU, Tallahasse, FL, USA, 17-21 May 2021
doi:10.21468/SciPostPhysProc.7

Abstract

We prove the conjecture made by Bern, Dixon, Dunbar, and Kosower that describes a simple dimension shifting relationship between the one-loop structure of $\mathcal{N} = 4$ MHV amplitudes and all-plus helicity amplitudes in pure Yang-Mills theory. The proof captures all orders in dimensional regularisation using unitarity cuts, by combining massive spinor-helicity with Coulomb-branch supersymmetry. The form of these amplitudes can be given in terms of pentagon and box integrals using a generalised D-dimensional unitarity technique which captures the full amplitude to all multiplicities.

Received 16-11-2021
Accepted 10-02-2022
Published 20-06-2022
doi:10.21468/ciPostPhysProc.7.009

1 Introduction

On-shell methods have lead to many new perspectives on gauge theories. Not least of these is the gauge/gravity double copy [1,2]. Here we discuss and prove an (up-until-now) conjectural relationship between gauge theories which relate a theory known to be physical, QCD, to a theory of great theoretical and conceptual interest, $\mathcal{N} = 4$ Super-Yang-Mills [3]. This states a simple dimension-shifting relationship between the n-point one-loop gluon amplitudes of each theory A_n^{theory} with differing helicity configurations

$$A_n^{\text{QCD}}\left(1^+, 2^+, \ldots, n^+\right) = -2\epsilon(1-\epsilon)(4\pi)^2 \left[\frac{A_n^{\mathcal{N}=4}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle ij\rangle^4}\right]_{\epsilon\to\epsilon-2}, \tag{1}$$

where ϵ is the usual dimensional-regulator parameter and the $\langle ij\rangle^4$ factor is the standard (Weyl) spinor-helicity contraction which compensates for the spinor-weight (or little-group scaling) between the two amplitudes.

Of particular practical interest is the fact that the relation (1) holds to all orders in ϵ, and thus relates the general-dimensional structure of the integral functions upon which it depends. In [3] the conjecture was verified up to $n=6$ but it has now been proven to all-multiplicities, and to all orders in ϵ [4]. Moreover, the complete all-orders-in-ϵ all-n amplitudes can be computed in both theories.

2 Status of the theories

Since the conjecture of the relationship (1) was first made, there has been great progress in the computation of the relevant scattering amplitudes.

2.1 All-plus QCD

The all-plus QCD amplitude has long been known to vanish at tree-level order, a fact which can most easily be seen using supersymmetric (SUSY) Ward identities [5]

$$A_n^{\text{SUSY; tree}}(1^+,2^+,...,n^+)=0 \quad , \tag{2}$$

and the fact that at tree level

$$A_n^{\text{SUSY; tree}}=A_n^{\text{QCD; tree}} \, . \tag{3}$$

The one-loop order result was an early all-multiplicity result for a gluon amplitude [6,7], and was computed to leading order in ϵ

$$A_n^{\text{QCD,1}-\text{loop}}(1^+,2^+,...,n^+)=\sum_{1\le i_1<i_2<i_3<i_4\le n}\frac{\text{tr}\_(i_1i_2i_3i_4)}{\langle 12...n1\rangle}+\mathcal{O}(\epsilon) \quad . \tag{4}$$

At two-loop order this particular helicity configuration provided the first high-multiplicity ($n>5$) results for the planar sector [8], where thanks to four-dimensional cut-constructibility and on-shell recursion a functional form has been successfully computed up to $n=7$ up to $\mathcal{O}(\epsilon)$ terms. A specific subleading-in-colour contribution to the two-loop all-plus amplitude has recently been computed to two loops, the first partial-amplitude result to be computed to arbitrary multiplicity [9].

2.2 MHV in $\mathcal{N}=4$ SYM

The tree-level amplitude of the MHV configuration in $\mathcal{N}=4$ SYM first deduced by Parke and Taylor [10] has since been expressed in a more general form, which bundles together the states in the supermultiplet [11] into a "superamplitude"

$$A_n^{\text{MHV; tree}}=\frac{\delta^{(8)}(|i\rangle\eta_{iA})}{\langle 12\dots n1\rangle} \, . \tag{5}$$

Then the application of functional derivatives gives the tree amplitude with two negative-helicity gluons

$$A_n^{\mathcal{N}=4;\text{ tree}}(1^+,\dots,i^-,\dots,j^-,\dots,n^+)=\frac{\delta^4}{\delta\eta_i^4}\frac{\delta^4}{\delta\eta_j^4}A_n^{\text{MHV; tree}} \, . \tag{6}$$

At loop-level, the MHV amplitude is one of the most resounding successes of four-dimensional unitarity constructions. The amplitude is easily constrained and fixed by unitarity diagrams

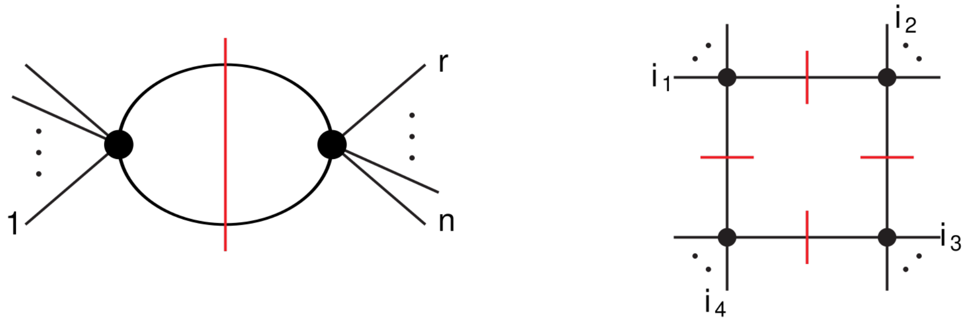

Figure 1: Unitarity cut constructions build loop amplitudes from tree amplitudes, and this is particularly effective with the maximal supersymmetry of $\mathcal{N} = 4$ SYM.

like those on the left-hand side of Figure 1. Together with the simple form of the tree amplitude, this resulted in another amplitude to have been computed up to $\mathcal{O}(\epsilon)$ to arbitrary multiplicity [12, 13]:

$$A_n^{\text{MHV}} = \frac{1}{4}\frac{\delta^{(8)}(|i\rangle\eta_{iA})}{\langle 12\ldots n1\rangle}\sum_{i_1,i_3=1}^{n} \text{tr}(i_1 q_{i_1+1,i_3} i_3 q_{i_3+1,i_1}) I_4^{[i_1,i_1+1,i_3,i_3+1]} + \mathcal{O}(\epsilon)\,, \tag{7}$$

where $q_{ij} = p_i + \cdots + p_{j-1}$ where the counting is defined cyclically in terms of particle labels. The amplitude (7) can also very easily be computed using generalised unitarity cuts [14] which builds the entire amplitude from on-shell products of amplitudes like the one depicted on the right-hand-side of Figure 1.

Astoundingly, multiloop results for $(n < 6)$-points extend to all orders, thanks to the exponentiation of the one-loop result given by the Bern-Dixon-Smirnov (BDS) ansatz [15]. For higher n the problem then turns into fixing the rest of the structure not captured by the BDS ansatz. Two-loop results have recently reached $n = 9$ [16] and $n = 6$ results extending through to seven loops [17].

3 Proving the conjecture

The verification of the relationship (1) was done in [3] to all orders up to $n = 6$ multiplicity. This combined a string-derived formalism for the $\mathcal{N} = 4$ side [18], and D-dimensional unitarity for the QCD side. We use the D-dimensional unitarity on both sides, matching the cuts which capture the full amplitude to all orders in ϵ.

3.1 Statement in terms of D-dimensional cuts

To compute full amplitudes in dimensional regularisation we can use a technique equivalent to taking massive unitarity cuts. D-dimensional unitarity treats a cut in $4 - 2\epsilon$ dimensions by splitting up the loop momentum into a four-dimensional contribution l (which lives in the same space as the external momenta) and the -2ϵ difference

$$\ell = l + \ell^{[-2\epsilon]}\,, \tag{8}$$

and the unitarity-cut (on-shell) condition becomes

$$\ell^2 = l^2 - \mu^2 = 0\,, \tag{9}$$

where we have defined

$$\mu^2 \equiv \left(\ell^{[-2\epsilon]}\right)^2 . \tag{10}$$

Meanwhile, as was originally observed in [3], the vanishing of all-plus gluon amplitudes in theories with supersymmetry implies that

$$A_n^{\mathrm{QCD}}(1^+, 2^+, \ldots, n^+) = 2A_n^{[0]}(1^+, 2^+, \ldots, n^+), \tag{11}$$

where $A_n^{[0]}$ is the amplitude with real scalar gluons circulating in the loop. This implies that the unitarity cuts are given by the product of scalar-gluon tree amplitudes, where the scalars have mass as defined in equation (10).

In particular, the dimension-shifting relationship can be stated at the integrand level in terms of the parameter μ^2. In considering the full integral in dimensional regularisation we see that the "dimension shift" comes from μ^2 terms present in the numerator

$$\begin{aligned} I^{4-2\epsilon}\left[\mu^{2r}\right] &= -\int \frac{d^4 l d\mu^2}{(-\mu^2)^{1+\epsilon}} \frac{\left(\mu^2\right)^r}{(l^2-\mu^2)\cdots((l-q)^2-\mu^2)} \\ &= -\epsilon(1-\epsilon)\cdots(r-1-\epsilon) I^{4+2r-2\epsilon}\left[1\right] \end{aligned} \tag{12}$$

which means we can restate (1) in terms of a relationship between unitarity cuts. Thus, as suggested in [3] we prove the relationship

$$A_n^{\mathrm{QCD}}\Bigg|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0} = A^{\mathcal{N}=4}\left[\frac{2\mu^4}{\langle ij\rangle^4}\right]\Bigg|_{q_{rs}\ \mathrm{cut}}^{\mu^2\neq 0} \tag{13}$$

for cuts in all momentum channels q_{rs}^2.

3.2 Necessary tree amplitudes

Proving equation (13) requires understanding the D-dimensional cuts on the $\mathcal{N}=4$ side. The amplitudes needed are known [19–22], and through Coulomb-branch supersymmetry they can be bundled together into the "MHV-band" amplitudes [22–24], which admit a delta-function representation analogous to the one given in equation (5),

$$A_{\mathrm{tree}}^{\mathrm{MHV-band}} = \frac{[\lambda_n\lambda_1]^2\delta_{12}^{\chi}\delta_{34}^{\chi}}{m^2 q_{n2}^4} A(\mathbf{1}^0, 2^+, 3^+, ..., (n-1)^+, \mathbf{n}^0)\,, \tag{14}$$

where $A(\mathbf{1}^0, 2^+, 3^+, ..., (n-1)^+, \mathbf{n}^0)$ are precisely the massive-scalar-gluon amplitudes needed for the QCD side of equation (13). Here $q_{n2} = p_n + p_1$, and $[\lambda_i|$ are the spinors obtained from the projection of p_i against a null reference spinor (or q-frame) p_χ in the Dittmaier massive-spinor-helicity formalism [25]. The delta functions δ_{ij}^{χ} are defined in [23]. They depend on the reference momentum p_χ and encode the remaining supersymmetry after it is partially broken in the Coulomb branch by the introduction of the mass m [23].

3.3 Proof

The tree amplitudes which can be extracted from equation (14) are most compactly expressed in the gauge constrained such that $p_\chi \cdot q_{n2} = 0$. We consider the sum of the helicity states in this gauge which correspond to the configurations of the general types depicted in Figure 2.

Cuts of type (a) vanish as a consequence of the vanishing of the all-plus-helicity tree amplitudes on the left-hand side. Cuts of type (b) give a non-vanishing contribution for the

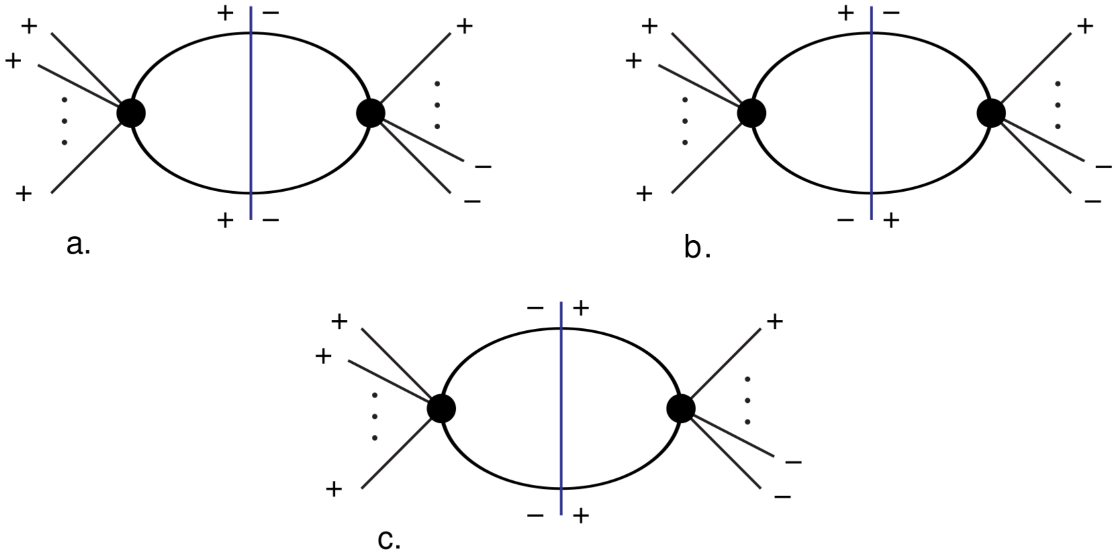

Figure 2: The three types of cut which reproduce the $\mathcal{N}=4$ amplitude.

fermionic, gluonic, and scalar loop content in the supermultiplet, but the total cancels so that these cuts do not contribute to the full amplitude.

So the only type of cuts contributing are (c), which are MHV on both sides and capture the epsilon-truncated one-loop amplitude in equation (7). In this case they are actually capturing the structure to all orders in ϵ through the Coulomb-branch amplitudes.

The proof then proceeds as follows. The cut integral is

$$\begin{aligned}&\int d^4\eta_{l_r} d^4\eta_{l_s} A_L^{\text{MHV; tree}}(\text{-}\boldsymbol{l}_r^1, r, \ldots, (s-1), \boldsymbol{l}_s^1) \times A_R^{\text{MHV; tree}}(\text{-}\boldsymbol{l}_r^1, r, \ldots, s-1, \boldsymbol{l}_s^1) = \\ &\quad \int d^4\eta_{l_r} d^4\eta_{l_s} \frac{\delta^{(8)}(L)}{\mu^2\langle\lambda_{l_s}\lambda_{l_r}\rangle^2} A_L(\text{-}\boldsymbol{l}_r^0, r, \ldots, (s-1), \boldsymbol{l}_s^0) \times \frac{\delta^{(8)}(R)}{\mu^2\langle\lambda_{l_s}\lambda_{l_r}\rangle^2} A_R(\text{-}\boldsymbol{l}_r^0, s, \ldots, r-1, \boldsymbol{l}_r^0)\,, \\ &\qquad L \equiv |i\rangle\eta_{iA},\ i \in \{\lambda_{\text{-}l_r}, r, \ldots, s-1, \lambda_{l_s}\}\ ;\ R \equiv |i\rangle\eta_{jA},\ i \in \{\lambda_{l_s}, r, \ldots, s-1, \lambda_{\text{-}l_r}\}\,,\end{aligned} \tag{15}$$

and applying the Grassmann integration and delta functions gives

$$A^{\text{MHV}}\Big|_{q_{rs}\text{ cut}} = \frac{\delta^{(8)}(\langle\lambda_i|\eta_{iA})}{\mu^4} A_L(\text{-}\boldsymbol{l}_r^0, r^+, \ldots, (s-1)^+, \boldsymbol{l}_s^0) A_R(\text{-}\boldsymbol{l}_s^1, s^+, \ldots, (r-1)^+, \boldsymbol{l}_r^1)\,. \tag{16}$$

Applying the functional derivatives then gives $\frac{\delta^4}{\delta\eta_i^4}\frac{\delta^4}{\delta\eta_j^4}$,

$$\begin{aligned}A_n^{\mathcal{N}=4}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)\Big|_{q_{rs}\text{ cut}} &= \frac{\delta^4}{\delta\eta_i^4}\frac{\delta^4}{\delta\eta_j^4}\left[A_n^{\text{MHV}}\Big|_{q_{rs}\text{ cut}}\right] \\ &= \frac{\langle ij\rangle^4}{2\mu^4} A_n^{\text{QCD}}\Big|_{q_{rs}\text{ cut}}\end{aligned} \tag{17}$$

which proves the conjecture (1).

4 Closed forms to all multiplicities

Through the conjecture, we need only compute one side to get the general all-orders-in-ϵ, all-n form of the amplitude in both theories. This amounts to extracting coefficients of box and pentagon integrals.

4.1 Box coefficients

These coefficients are remarkably simple to compute on the $\mathcal{N} = 4$ side, as the boxes are given from four-dimensional cuts and we can read the coefficients directly from equation (7). Namely a coefficient b_4 is given by

$$b_4^{\text{QCD};\,[i_1,i_3-1,i_3,i_1-1]} = \frac{1}{2}\frac{\text{tr}(i_1 q_{i_1 i_3} i_3 q_{i_3+1,i_1-1})}{\langle 12...n1\rangle}\,, \tag{18}$$

but we can also see these emerge on the QCD side from imposing ultraviolet constraints on the box-integral basis and demonstrating that only "two-mass-easy" ($i_2 = i_3 - 1$, $i_4 = i_1 - 1$) box coefficients contribute. The coefficients are then given by the formula

$$b_4^{\text{QCD};\,[i_1,i_3-1,i_3,i_1-1]} = \frac{1}{\mu^4}\left[2\times\frac{1}{2}\sum_{\alpha_\pm} A^{\text{tree}}\times A^{\text{tree}}\times A^{\text{tree}}\times A^{\text{tree}}\Big|_{\mathcal{O}(\mu^6)}\right]. \tag{19}$$

4.2 Pentagon coefficients

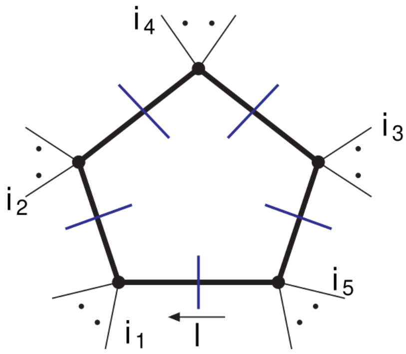

Figure 3: The five-particle cut reproduces pentagon coefficients.

The pentagon coefficients can be easily solved for thanks to the generalised D-dimensional unitarity penta-cut solution presented in [26], which simply gives an explicit solution for the loop momentum given five massive (or D-dimensional) cuts depicted in Figure 3

$$\begin{aligned} l_{i_1}^{\mu} &= -\frac{\text{tr}_5\left(q_{i_1i_2}q_{i_2i_3}q_{i_3i_4}q_{i_4i_5}q_{i_5i_1}\gamma_\mu\right)}{2\,\text{tr}_5(q_{i_1i_2}q_{i_2i_3}q_{i_3i_4}q_{i_4i_5})} \\ \mu^2 &= \frac{\text{tr}(q_{i_1i_2}q_{i_2i_3}q_{i_3i_4}q_{i_4i_5}q_{i_5i_1}q_{i_1i_2}q_{i_2i_3}q_{i_3i_4}q_{i_4i_5}q_{i_5i_1}) - 2\prod_{k=1}^{5} q^2_{i_ki_{k+1}}}{\text{tr}_5^2(q_{i_1i_2}q_{i_2i_3}q_{i_3i_4}q_{i_4i_5})} \end{aligned} \tag{20}$$

and this need only be plugged into the product of the five on-shell amplitudes to give the coefficients:

$$\begin{aligned} b_5^{\text{QCD};\,[i_1,i_2,i_3,i_4,i_5]} = c_0^3 A^{\text{tree}}(\text{-}l^0_{i_1}, i_1^+, ..., (i_2-1)^+, l^0_{i_2}) \times A^{\text{tree}}(\text{-}l^0_{i_2}, i_2^+, ..., (i_3-1)^+, l^0_{i_3})\times \\ A^{\text{tree}}(\text{-}l^0_{i_3}, i_3^+, ..., (i_4-1)^+, l^0_{i_3}) \times A^{\text{tree}}(\text{-}l^0_{i_4}, i_4^+, ..., (i_5-1)^+, l^0_{i_3})\times \\ A^{\text{tree}}(\text{-}l^0_{i_5}, i_5^+, ..., (i_1-1)^+, l^0_{i_3}) \quad . \end{aligned} \tag{21}$$

A non-trivial check is that this reproduces the parity-odd contributions to the finite QCD amplitude, namely the tr_5 piece of the QCD amplitude in equation (4) where

$$\mathrm{tr}_-(i_1 i_2 i_3 i_4) = \frac{1}{2}\left(\mathrm{tr}(i_1 i_2 i_3 i_4) - \mathrm{tr}_5(i_1 i_2 i_3 i_4)\right) \tag{22}$$

matches the b_5 contributions upon substituting the pentagon integrals for their values in the $\epsilon \to 0$ limit

$$I_5[\mu^4] \underset{\epsilon=0}{\rightarrow} -\frac{1}{24}\,. \tag{23}$$

We confirm this numerically up to to the $n = 17$ case.

5 Conclusion

The dimension shift relationship between QCD all-plus-helicity amplitudes and $\mathcal{N} = 4$ SYM MHV amplitudes has been proven at one-loop to all multiplicities. We have also given all-n all-orders-in-ϵ expressions in terms of pentagon and box integrals, through fixing their coefficients through generalised D-dimensional unitarity cuts. The origin of the box coefficients is particularly distinct, with the $\mathcal{N} = 4$ computation falling out automatically from four-dimensional cuts while the QCD requires UV truncation of the integral basis in D dimensions to fully constrain.

Further work towards studying the analytic structure of these amplitudes to all orders in ϵ could yield deeper insight at higher-loop order. In particular, we observe that the "purity" (polylogarithmic simplicity) of the amplitude is broken at high orders in ϵ in $\mathcal{N} = 4$ super Yang-Mills. Future work will involve getting a stronger analytic control of these all-n all-orders-in-ϵ expressions.

Acknowledgements

Diagrams were drawn using Axodraw. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 647356 "CutLoops.")

References

[1] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, *Two-loop six-gluon maximally helicity violating amplitude in maximally supersymmetric Yang-Mills theory*, Phys. Rev. D **78**, 045007 (2008), doi:10.1103/PhysRevD.78.045007.

[2] Z. Bern, J. Joseph Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, arXiv:1909.01358.

[3] Z. Bern, L. Dixon, D. C. Dunbar and D. A. Kosower, *One-loop self-dual and N = 4 super Yang-Mills*, Phys. Lett. B **394**, 105 (1997), doi:10.1016/S0370-2693(96)01676-0.

[4] R. Britto, G. R. Jehu and A. Orta, *The dimension-shift conjecture for one-loop amplitudes*, J. High Energy Phys. **04**, 276 (2021), doi:10.1007/JHEP04(2021)276.

[5] M. T. Grisaru and H. N. Pendleton, *Some Properties of Scattering Amplitudes in Supersymmetric Theories*, Nucl. Phys. B **124**, 81 (1977), doi:10.1016/0550-3213(77)90277-2.

[6] G. Mahlon, *Multigluon helicity amplitudes involving a quark loop*, Phys. Rev. D **49**, 4438 (1994), doi:10.1103/PhysRevD.49.4438.

[7] Z. Bern, G. Chalmers, L. J. Dixon and D. A. Kosower, *One loop N gluon amplitudes with maximal helicity violation via collinear limits*, Phys. Rev. Lett. **72**, 2134 (1994), doi:10.1103/PhysRevLett.72.2134.

[8] S. Badger, H. Frellesvig and Y. Zhang, *A two-loop five-gluon helicity amplitude in QCD*, J. High Energy Phys. **12**, 045 (2013), doi:10.1007/JHEP12(2013)045.

[9] D. C. Dunbar, W. B. Perkins and J. M. W. Strong, *n-point QCD two-loop amplitude*, Phys. Rev. D **101**, 076001 (2020), doi:10.1103/PhysRevD.101.076001.

[10] S. J. Parke and T. R. Taylor, *Amplitude forn-Gluon Scattering*, Phys. Rev. Lett. **56**, 2459 (1986), doi:10.1103/PhysRevLett.56.2459.

[11] M. Bianchi, H. Elvang and D. Z. Freedman, *Generating tree amplitudes in N=4 SYM and $N=8$ SG*, J. High Energy Phys. **09**, 063 (2008), doi:10.1088/1126-6708/2008/09/063.

[12] Z. Bern, L. Dixon, D. C. Dunbar and D. A. Kosower, *One-loop n-point gauge theory amplitudes, unitarity and collinear limits*, Nucl. Phys. B **425**, 217 (1994), doi:10.1016/0550-3213(94)90179-1.

[13] Z. Bern, L. Dixon, D. C. Dunbar and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl. Phys. B **435**, 59 (1995), doi:10.1016/0550-3213(94)00488-Z.

[14] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in super-Yang–Mills*, Nucl. Phys. B **725**, 275 (2005), doi:10.1016/j.nuclphysb.2005.07.014.

[15] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, Phys. Rev. D **72**, 085001 (2005), doi:10.1103/PhysRevD.72.085001.

[16] J. Golden and A. J. McLeod, *The two-loop remainder function for eight and nine particles*, J. High Energy Phys. **06**, 142 (2021), doi:10.1007/JHEP06(2021)142.

[17] S. Caron-Huot, L. J. Dixon, F. Dulat, M. von Hippel, A. J. McLeod and G. Papathanasiou, *The cosmic Galois group and extended Steinmann relations for planar $\mathcal{N}=4$ SYM amplitudes*, J. High Energy Phys. **09**, 061 (2019), doi:10.1007/JHEP09(2019)061.

[18] Z. Bern and D. A. Kosower, *The computation of loop amplitudes in gauge theories*, Nucl. Phys. B **379**, 451 (1992), doi:10.1016/0550-3213(92)90134-W.

[19] D. Forde and D. A. Kosower, *All-multiplicity amplitudes with massive scalars*, Phys. Rev. D **73**, 065007 (2006), doi:10.1103/PhysRevD.73.065007.

[20] G. Rodrigo, *Multigluonic scattering amplitudes of heavy quarks*, J. High Energy Phys. **09**, 079 (2005), doi:10.1088/1126-6708/2005/09/079.

[21] P. Ferrario, G. Rodrigo and P. Talavera, *Compact Multigluonic Scattering Amplitudes with Heavy Scalars and Fermions*, Phys. Rev. Lett. **96**, 182001 (2006), doi:10.1103/PhysRevLett.96.182001.

[22] M. Kiermaier, *The Coulomb-branch S-matrix from massless amplitudes*, arXiv:1105.5385.

[23] N. Craig, H. Elvang, M. Kiermaier and T. R. Slatyer, *Massive amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM*, J. High Energy Phys. **12**, 097 (2011), doi:10.1007/JHEP12(2011)097.

[24] H. Elvang, D. Z. Freedman and M. Kiermaier, *Integrands for QCD rational terms and N=4 SYM from massive CSW rules*, J. High Energy Phys. **06**, 015 (2012), doi:10.1007/JHEP06(2012)015.

[25] S. Dittmaier, *Weyl–van der Waerden formalism for helicity amplitudes of massive particles*, Phys. Rev. D **59**, 016007 (1998), doi:10.1103/PhysRevD.59.016007.

[26] G. R. Jehu, *Symmetric reduction of high-multiplicity one-loop integrals and maximal cuts*, J. High Energy Phys. **02**, 013 (2021), doi:10.1007/JHEP02(2021)013.