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The Soft Anomalous Dimension at four loops in the Regge Limit

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Abstract

The soft anomalous dimension governs the infrared divergences of scattering amplitudes in general kinematics to all orders in perturbation theory. By comparing the recent Regge-limit results for $2 \to 2$ scattering (through Next-to-Next-to-Leading Logarithms) in full colour to a general form for the soft anomalous dimension at four loops we derive powerful constraints on its kinematic dependence, opening the way for a bootstrap-based determination.

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1 Introduction

Scattering amplitudes are crucial to precision physics and our understanding of quantum field theory. Infrared singularities are a salient feature of gauge theory amplitudes, a manifestation of the gluon being massless. Infrared singularities factorise and exponentiate, and are thus governed by a finite and universal quantity – the soft anomalous dimension [1–3]. This quantity is therefore central to understand gauge theory scattering. The soft anomalous dimension for massless parton scattering in general kinematics is known to three-loop order [4]. Furthermore its structure is highly constrained to all orders, owing to its intimate connection to the renormalisation of correlators of Wilson lines [5–7].

The soft anomalous dimension for massless scattering can be defined as a correlator of a product of semi-infinite lightlike Wilson lines. This implies that its colour structure, at any loop order, corresponds to connected graphs and it admits bose symmetry under permutation of any two external lines, which links the kinematic dependence to that of colour. Finally, the kinematic dependence is largely constrained by the rescaling symmetry of individual Wilson line velocities.

The functional form of the three-loop soft anomalous dimension was recovered using a bootstrap approach in [8]. In this approach, one writes down an ansatz for the kinematic functions using a suitable class of iterated integrals, and then fixes the (rational) coefficients in this ansatz using factorisation and symmetry constraints, along with constraints from the collinear and Regge limits. A general form in terms of colour factors multiplying unknown kinematic functions for the soft anomalous dimension for up to four loops was put forward in [9].

The aim of the work reported in this talk is to find new constraints on the unknown functions in the soft anomalous dimension at four loops [9] using information from a highly interesting kinematic limit: the Regge limit. There has been much study of scattering amplitudes in the high-energy (Regge) limit in QCD [10–15] including recent work on $2 \to 2$ scattering [16–21]. The new constraints are found by using results from newly calculated amplitudes in the Regge limit to Next-to-Leading Logarithmic (NLL) accuracy for the even signature amplitude [10,16,18,22] and NNLL accuracy for the odd signature amplitude [21,23,24]. These results are available in full colour for arbitrary representations of the scattered partons. They are compared to the soft anomalous dimension from [9] which is expressed in the Regge limit separated by signature. This work is discussed in detail in an upcoming paper [24] with initial results published in [23].

1.1 Regge limit and signature

Our focus is on $2 \to 2$ scattering shown diagrammatically in Fig. 1 below and expressed in the equation

$$i(p_1, a_1, \lambda_1) + j(p_2, a_2, \lambda_2) \to j(p_3, a_3, \lambda_3) + i(p_4, a_4, \lambda_4), \tag{1}$$

where the partons i, j can each be a quark or a gluon and they respectively represent the target and projectile. Treating the particles with momenta p_1, p_2 as incoming, and p_3, p_4 as outgoing

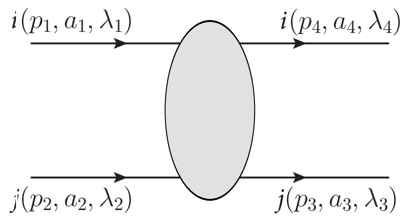

Figure 1: 2 to 2 particle scattering which at tree level correspond to eq. (3). The arrows indicate the direction of the momenta. The colour indices are a_k, while the labels λ_k represent the particle helicities.

in Figure 1, the process is described in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2 > 0 \qquad t = (p_1 - p_4)^2 < 0 \qquad u = (p_1 - p_3)^2 < 0. \tag{2}$$

For the tree-level diagram, the blob is a single-gluon t-channel exchange. The tree-level expression is given by

$$\mathcal{M}^{\text{tree}}_{ij\to ij} = g_s^2 \frac{2s}{t} \mathbf{T}_i \cdot \mathbf{T}_j \, \delta_{\lambda_1\lambda_4} \delta_{\lambda_2\lambda_3}, \tag{3}$$

where the factor $\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}$ represents helicity conservation, and the colour dependence is expressed using the colour-space formalism introduced in [25–28]. Following this notation, a colour operator $\mathbf{T}_k$ corresponds to the colour generator associated with the k-th parton in the scattering amplitude, which acts as an SU(N_c) matrix on the colour indices of that parton. In eq. (3), the colour indices a_k of the incoming and outgoing partons are implicit. The dipole is expressed as $\mathbf{T}_i \cdot \mathbf{T}_j = (T_i^b)_{a_1 a_4}(T_j^b)_{a_2 a_3}$, where $(T_i^b)_{a_1 a_4} = t^b_{a_1 a_4}$ for quarks, $(T_i^b)_{a_1 a_4} = -t^b_{a_4 a_1}$ for antiquarks, and $(T_i^b)_{a_1 a_4} = i f^{a_1 b a_4}$ for gluons.

The high-energy limit is defined by the condition $s \gg -t$, i.e., the centre of mass energy s becomes much larger than the momentum transfer $|t|$. As a result, $u \simeq -s$. Neglecting power-suppressed terms, this introduces an additional *signature* symmetry to the amplitude under the exchange $s \leftrightarrow u$. It is then advantageous to split the amplitude into its even and odd components under $s \leftrightarrow u$:

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big), \tag{4}$$

where $\mathcal{M}^{(+)}$, $\mathcal{M}^{(-)}$ are referred to, respectively, as the *even* and *odd* amplitudes. As demonstrated in ref. [17], using the signature-even combination of logarithms,

$$L \equiv \log\left(\frac{s}{-t}\right) - \frac{i\pi}{2} = \log\left(\frac{s-i0}{-t}\right) + \log\left(\frac{-u-i0}{-t}\right), \tag{5}$$

and expanding the amplitudes $\mathcal{M}_\pm$ according to

$$\mathcal{M}^{(\pm)}_{ij\to ij} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{n} L^m \mathcal{M}^{(\pm,n,m)}_{ij\to ij}, \tag{6}$$

with $\mathcal{M}^{(-,0,0)}_{ij\to ij} \equiv \mathcal{M}^{\text{tree}}_{ij\to ij}$, it can be shown that the odd amplitude coefficients $\mathcal{M}^{(-,n,m)}_{ij\to ij}$ are purely *real*, while the even ones $\mathcal{M}^{(+,n,m)}_{ij\to ij}$ are purely *imaginary*.

1.2 Colour-operator notation

Colour conservation in $2 \to 2$ scattering implies

$$\big(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4\big)\mathcal{M}^{\text{tree}}_{ij\to ij} = 0. \tag{7}$$

In the high-energy limit it is helpful to express the colour generators using the basis of Casimirs corresponding to colour flow through the three channels [15, 29]:

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{T}_3 - \mathbf{T}_4 \qquad \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3 = -\mathbf{T}_2 - \mathbf{T}_4 \qquad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 = -\mathbf{T}_2 - \mathbf{T}_3\,. \tag{8}$$

In order to make the signature symmetry manifest within Bose-symmetric amplitudes $\mathcal{M}^{(\pm)}$, it is useful to introduce a colour operator that is *odd* under $s \leftrightarrow u$ crossing:

$$\mathbf{T}^2_{s-u} \equiv \tfrac{1}{2}\big(\mathbf{T}_s^2 - \mathbf{T}_u^2\big). \tag{9}$$

This will form part of what is called the *Regge-limit basis*, which involves writing the colour operators in terms of $\mathbf{T}^2_{s-u}$ and $\mathbf{T}^2_t$ in nested commutators where possible and can be seen in eqs. (21,24,25). The quartic Casimir which contains a fully symmetrised trace, first appears in the soft anomalous dimension at four loops. It is defined as

$$\frac{d_{RR_i}}{N_{R_i}} = \frac{1}{4}\sum_{\sigma\in\mathcal{S}_4} \mathrm{Tr}_R\left[T^{\sigma(a)}T^{\sigma(b)}T^{\sigma(c)}T^{\sigma(d)}\right]\mathbf{T}_i^a\mathbf{T}_i^b\mathbf{T}_i^c\mathbf{T}_i^d = \boldsymbol{\mathcal{D}}^R_{iiii} \tag{10}$$

where the $\boldsymbol{\mathcal{D}}^R_{iiii}$ notation is adopted from [9].

1.3 Soft anomalous dimension for $2 \to 2$ scattering

The long-distance singularities of $2 \to 2$ massless amplitudes factorise following the infrared factorisation theorem for fixed-angle scattering [1,3,4,9,28,30–34]

$$\mathcal{M}_4(s,t;\epsilon) = \mathbf{Z}_4(s,t;\epsilon)\,\mathcal{H}_4(s,t;\epsilon)\,, \tag{11}$$

where ϵ is the dimensional regulator and the subscript is for $n=4$ partons. The infrared divergences are captured in $\mathbf{Z}_4$ which acts on the finite, so-called *hard* function $\mathcal{H}_4$. The operator $\mathbf{Z}_4$ exponentiates

$$\mathbf{Z}_4(s,t;\epsilon) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_4(s,t,\lambda,\alpha_s(\lambda^2))\right\}, \tag{12}$$

where $\mathbf{\Gamma}_4$ is the soft anomalous dimension. The latter is a finite quantity depending on the d-dimensional running coupling $\alpha_s(\lambda^2)$. Its expansion is

$$\mathbf{\Gamma}_4(\{s_{ij}\},\lambda,\alpha_s) = \sum_{\ell=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^{\ell}\mathbf{\Gamma}_4^{(\ell)}(\{s_{ij}\},\lambda), \tag{13}$$

where ℓ is the loop order. All kinematic functions within the soft anomalous dimension are expanded in a similar way. The soft anomalous dimension depends on the colour generators $\mathbf{T}_i$ associated to the external particles, which have been defined after eq. (3). The soft anomalous dimension also depends on the kinematic invariants

$$(-s_{ij}) \equiv 2|p_i \cdot p_j|e^{-i\pi\sigma_{ij}}, \tag{14}$$

where p_i represents the momentum of the particle i, and $\sigma_{ij}=1$ if both partons are in the initial or final state, otherwise $\sigma_{ij}=0$. Specifically $\mathbf{\Gamma}_4$ depends on the invariants s_{ij} either through the logarithms

$$l_{ij} \equiv \log\frac{-s_{ij}}{\lambda^2}, \tag{15}$$

with λ being the renormalisation scale, or via conformally invariant cross-ratios (CICRs)

$$\beta_{ijkl} = \log\rho_{ijkl} = \log\frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})} = l_{ij}+l_{kl}-l_{ik}-l_{jl}, \tag{16}$$

whose symmetries are discussed in [9,35]. It was shown in [1,3,33] that owing to soft-collinear factorisation and rescaling invariance of the Wilson-line velocities, the dependence on scale as in eq. (15) is directly linked with collinear singularities and these are linear in l_{ij} and are generated exclusively by the cusp anomalous dimension [1, 3, 9, 33]. The cusp anomalous dimension has the form [1,3,7,9,33]

$$\Gamma_i^{\text{cusp}}(\alpha_s(\lambda^2)) = \frac{1}{2}\gamma_K(\alpha_s(\lambda^2))C_i + \sum_R g_R(\alpha_s(\lambda^2))\frac{d_{RR_i}}{N_i} + O(\alpha_s^5), \tag{17}$$

where γ_K multiplies the quadratic Casimir C_i in the representation of parton i, while the component g_R, starting only at four loops, multiplies the quartic Casimir (defined in eq. (10)). Γ_i^{cusp} is known to four loops in QCD [36–40]. In contrast, dependence on the kinematics through conformal cross ratios in eq. (16) is not constrained by factorisation and can be complicated. It has been computed at three loops in refs. [4,8] and is yet unknown at four loops.

The soft anomalous dimension for $2 \to 2$ scattering through four loops may be expressed as [9]

$$\begin{aligned}\mathbf{\Gamma}_4(\{s_{ij}\},\lambda,\alpha_s(\lambda^2)) = &-\frac{1}{4}\gamma_K(\alpha_s)\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\log\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_i^4\gamma_i(\alpha_s)\\ &+f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{ijki}+\sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,\mathcal{F}(\beta_{ijlk},\beta_{iklj};\alpha_s)\\ &-\sum_R\frac{g_R(\alpha_s)}{2}\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\log\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\right]\\ &+\sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,\mathcal{G}_R(\beta_{ijlk},\beta_{iklj};\alpha_s)+\sum_{(i,j,k,l)}\mathcal{T}_{ijkli}\,\mathcal{H}_1(\beta_{ijlk},\beta_{iklj};\alpha_s),\end{aligned} \tag{18}$$

the summation is over permutations of lines $(1,2,3,4)$. The various terms in eq. (18) correspond to colour connected structures as dictated by the non-Abelian exponentiaion theorem [41–43]. The colour structures have been defined in [9] as

$$\begin{aligned}\mathcal{T}_{ijkl} &= \frac{1}{4!}f^{ade}f^{bce}\sum_{\sigma\in S_4}\mathbf{T}_i^{\sigma(a)}\mathbf{T}_j^{\sigma(b)},\mathbf{T}_k^{\sigma(c)}\mathbf{T}_l^{\sigma(d)},\\ \mathcal{D}^R_{ijkl} &= \frac{1}{4!}\sum_{\sigma\in S_4}\mathrm{Tr}_R\left(T^{\sigma(a)}T^{\sigma(b)}T^{\sigma(c)}T^{\sigma(d)}\right)\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d,\\ \mathcal{T}_{ijklm} &= \frac{1}{5!}if^{adf}f^{bcg}f^{efg}\sum_{\sigma\in S_5}\mathbf{T}_i^{\sigma(a)}\mathbf{T}_j^{\sigma(b)}\mathbf{T}_k^{\sigma(c)}\mathbf{T}_l^{\sigma(d)}\mathbf{T}_m^{\sigma(e)}.\end{aligned} \tag{19}$$

There is an explicit sum over representations R of the particle content of the specific theory considered e.g QCD, in the functions g_R and $\mathcal{G}_R$ in the third and fourth lines. The coupling depends on the scale $\alpha_s(\lambda^2)$ throughout but we will just write α_s from here onwards.

At one and two loop order, only the first line of eq. (18), the so-called dipole formula contributes. The collinear anomalous dimension for parton i is γ_i, studied in refs. [44–47]. It has recently been computed at four loops in QCD [48]. The terms on the second line first appear at three loops with their known expressions from [4]. At four-loop order there is an implicit sum over the representations contributing to the f and $\mathcal{F}$ colour structures. The terms from the third line onwards in eq. (18) appear for the first time at four loops. The fully-symmetric function $\mathcal{G}_R(\beta_{ijlk},\beta_{iklj};\alpha_s)$ depends on CICRs and the representation. The function $\mathcal{H}_1$ depends on CICRs and does not depend on the representation at four-loop order with an expected sum appearing at five loops and beyond. Our focus will be on understanding the high-energy limit of the unknown functions that depend on CICRs at four loops.

2 Soft Anomalous Dimension in the high-energy limit

In the high-energy limit of $2 \to 2$ scattering, the soft anomalous dimension $\mathbf{\Gamma}_4$ takes the form [15, 17, 49]

$$\begin{aligned}\mathbf{\Gamma}_4\left(\alpha_s(\lambda^2),L,\frac{-t}{\lambda^2}\right) = &\frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T}_t^2+i\pi\mathbf{T}_{s-u}^2\right]+\Gamma_i\left(\alpha_s,\frac{-t}{\lambda^2}\right)+\Gamma_j\left(\alpha_s,\frac{-t}{\lambda^2}\right)\\ &+\sum_{\ell=3}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^{\ell}\sum_{m=0}^{\ell-1}L^m\mathbf{\Delta}^{(\ell,m)},\end{aligned} \tag{20}$$

where L is the signature-even log of eq. (5). Γ_i captures the collinear singularities for parton i and contains the collinear anomalous dimension γ_i and the cusp anomalous dimension eq. (17). The dipole formula in the first line of eq. (20) is exact up to two loops [1,33]. All terms that are not part of the dipole formula are collected in the second line. The known coefficients $\boldsymbol{\Delta}^{(\ell,m)}$ are given in Appendix B.

In the high-energy limit, all the terms in the anomalous dimension of eq. (20) are expanded in powers of the large logarithm L. Furthermore, we separate terms by their signature under crossing $s \leftrightarrow u$. The contributions proportional to $\mathbf{T}_t^2, \Gamma_i, \Gamma_j$ in the first line are manifestly even under $s \leftrightarrow u$, while $\mathbf{T}_{s-u}^2$ is odd by definition. The coefficients $\boldsymbol{\Delta}^{(\ell,m)}$ decompose according to their signature in $\boldsymbol{\Delta}^{(\pm,\ell,m)}$, similarly to the complete amplitude in eq. (4). All the coefficients in eq. (20) at three loops are known by expanding the result in general kinematics [17]. It is worth recalling that, upon expansion, the soft anomalous dimension will be multiplied by the odd tree-level amplitude in eq. (3): for this reason, odd signature in the amplitude corresponds to even signature in the soft anomalous dimension. At four loops, the explicit calculations of the NLL in the signature-even amplitude [16, 18–20] and of the NNLL in the signature-odd one [23, 24] yield

$$\begin{aligned}\boldsymbol{\Gamma}_4^{(4)}(L) = &-L^3 i\pi \frac{\zeta_3}{24}\Big[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\Big]\mathbf{T}_t^2 + L^2\boldsymbol{\Delta}^{(-,4,2)} \\ &+ L^2\zeta_2\zeta_3\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24} - \frac{1}{4}\mathbf{T}_t^2[(\mathbf{T}_{s-u}^2)^2,\mathbf{T}_t^2] + \frac{3}{4}[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2\right) \\ &+ \mathcal{O}(L),\end{aligned} \tag{21}$$

where $\boldsymbol{\Delta}^{(-,4,2)}$ and corrections at $O(L)$ and $O(L^0)$ are still to be determined. Despite this result being valid for QCD, there are no n_f terms present in eq. (21). This is so because the relevant contributions (at this logarithmic accuracy) are generated solely by gluon diagrams. Below we use the result in eq. (21) to constrain the unknown functions in eq. (18).

2.1 Separating the soft anomalous dimension by signature and colour operators

In the previous section, at four-loop order, we have presented a newly calculated expression for the soft anomalous dimension in high-energy limit to NNLL which contains only colour adjoint contributions. In comparing eq. (21) with eq. (18) it becomes clear that any sums over the representations R collapse to the adjoint representation and the expression becomes

$$\begin{aligned}\boldsymbol{\Gamma}_4^{(4)}(\{\beta_{ijkl}\}) = &\sum_{(i,j,k,l)\in S_4} \mathcal{T}_{ijkl}\,\mathcal{F}_A^{(4)}(\beta_{ijlk},\beta_{iklj}) + \sum_{(i,j,k,l)\in S_4} \mathcal{D}_{ijkl}^A\,\mathcal{G}_A^{(4)}(\beta_{ijlk},\beta_{iklj}) \\ &+ \sum_{(i,j,k,l)\in S_4} \mathcal{T}_{ijkli}\,\mathcal{H}_1^{(4)}(\beta_{ijlk},\beta_{iklj}) + \mathcal{O}(L).\end{aligned} \tag{22}$$

The superscript is for the loop order $\ell = 4$. The four-loop order coefficients $(\gamma_{K,R}^{(4)}, g_R^{(4)}, f_R^{(4)})$ contributing at $O(L)$ and $O(L^0)$ are suppressed and are discussed in ref. [24].

Taking the Regge limit of the kinematic functions involves an analytical continuation to the physical region, and an expansion of the functions in powers of the high-energy signature-even logarithm L. This procedure has been discussed in detail for the three-loop case in [8]. Here we consider the four-loop case, where an explicit calculation in general kinematics is still missing. In the Regge limit, an expansion in the signature-even logarithm L can be performed on each of the kinematic functions. For example

$$\mathcal{F}_A^{(+,4)}(L) = \mathcal{F}_A^{(+,4,3)}L^3 + \mathcal{F}_A^{(+,4,2)}L^2 + \mathcal{F}_A^{(+,4,1)}L + \mathcal{F}_A^{(+,4,0)}, \tag{23}$$

with all other functions contributing at NNLL accuracy in the Regge limit having a similar expansion. The colour structures are expressed in a *Regge-limit basis* with the steps elaborated in [24]. The subscript Regge denotes functions after the Regge limit has been taken. The signature-even part is

$$\begin{aligned}\mathbf{\Gamma}^{(+,4)}_{4,\mathrm{Regge}} =& 2\mathcal{F}^{(+,4)}_A(L)\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big] \\ &+\mathcal{G}^{(+,4)}_A(L)\bigg(2\left(\frac{d_{AA}}{N_A}-\frac{C^4_A}{24}\right)-\frac{1}{2}\mathbf{T}^2_t[(\mathbf{T}^2_{s-u})^2,\mathbf{T}^2_t]+\frac{3}{2}[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\mathbf{T}^2_t\mathbf{T}^2_{s-u} \\ &+\frac{C_A}{2}\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big]\bigg)+\mathcal{H}^{(+,4)}_1(L)\bigg(-\frac{1}{2}C_A\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big] \\ &+\frac{3}{4}C_A\mathbf{T}^2_{s-u}[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]-\frac{1}{4}\mathbf{T}^2_t[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\mathbf{T}^2_{s-u}\bigg)+\mathcal{O}(L),\end{aligned} \tag{24}$$

and the signature-odd part

$$\begin{aligned}\mathbf{\Gamma}^{(-,4)}_{4,\mathrm{Regge}} =& -\mathcal{F}^{(-,4)}_A(L)\Big[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\Big] \\ &+\mathcal{H}^{(-,4)}_1(L)\bigg(-\frac{1}{2}\Big[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},[\mathbf{T}^2_{s-u},\mathbf{T}^2_t]\Big]+\frac{1}{8}\Big[\mathbf{T}^2_t,[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]\Big]\bigg) \\ &+\tilde{\mathcal{H}}^{(-,4)}_1(L)\left(\frac{1}{4}\Big[\mathbf{T}^2_t,[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]\Big]\right)+\mathcal{O}(L).\end{aligned} \tag{25}$$

The definitions of the even and odd functions of $\mathcal{F}^{(\pm)}$ and $\mathcal{H}^{(\pm)}_1$ are given in Appendix A. $\mathcal{G}_A$ is a completely symmetric function so $\mathcal{G}^{(-)}_A = 0$ and it only contributes to the signature-even part of the soft anomalous dimension. The quartic Casimir of $\frac{d_{AA}}{N_A}$ is defined in eq. (10). Both eqs. (24,25) are fully non-planar. The planar contributions appear at $\mathcal{O}(L^1)$ and are associated with the cusp anomalous dimension with more discussion in [24].

3 Constraint Results

We now compare eqs. (24,25), which correspond to the most general form for the soft anomalous dimension at four-loop order, expanded in the Regge limit through quadratic terms in L, with the results of explicit calculations of the soft anomalous dimension in eq. (21). We obtain the set of constraints for the unknown kinematic functions summarised in Table. 1. In the left

Table 1: Regge constraints separated by signature for the kinematic functions at four-loop order for $2 \to 2$ scattering in soft anomalous dimension ansatz of eq. (18).

$+$	L^3	L^2	$-$	L^3	L^2
$\mathcal{F}^{(+,4)}_A$	0	$-C_A\frac{\zeta_2\zeta_3}{8}$	$\mathcal{F}^{(-,4)}_A$	$i\pi C_A\frac{\zeta_3}{24}$	?
$\mathcal{F}^{(+,4)}_F$	0	0	$\mathcal{F}^{(-,4)}_F$	0	?
$\mathcal{G}^{(+,4)}_A$	0	$\frac{\zeta_2\zeta_3}{2}$	$\mathcal{G}^{(-,4)}_R$	0	0
$\mathcal{G}^{(+,4)}_F$	0	0	$\mathcal{H}^{(-,4)}_1$	0	?
$\mathcal{H}^{(+,4)}_1$	0	0	$\tilde{\mathcal{H}}^{(-,4)}_1$	0	?

columns of the table, we show the signature-even functions to NNLL accuracy. The signature-odd functions are given in the right columns of the table with constraints to NLL accuracy. The question marks represent quantities which have not been calculated. Because the even part of the soft anomalous dimension is two-loop exact at NLL and since the functions in eq. (24) all multiply independent colour structures, they are each individually zero at $O(L^3)$. Since $\mathcal{G}_R$ is a purely symmetric function, any antisymmetric component would be zero at all orders of L. We notice that only two functions contribute to $\Gamma^{(+,4)}$ through the terms proportional to L^2.

4 Conclusion

Using the state-of-the art knowledge of $2 \to 2$ scattering amplitudes in the high-energy limit we determined NLL and NNLL contributions to the soft anomalous dimension at four loops. These provide constraints on the unknown functions in the soft anomalous dimension. We derive new inhomogeneous constraints to the general form of the soft anomalous dimension in general kinematics in [9], reported in Table 1.

Our results are consistent with the work of [50], which argues that the function $\mathcal{H}_1$ must vanish exactly on the basis of symmetry considerations which exclude terms with an odd number of generators in the soft anomalous dimension. These constraints from the Regge limit along with those from the collinear limit in [9] are important input for a bootstrap approach to determine the unknown functions in the soft anomalous dimension in general kinematics at four loops with a similar method to [8].

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A Even and odd functions in the soft anomalous dimension

The even and odd functions under $s \leftrightarrow u$ $(2 \leftrightarrow 3)$ are

$$\mathcal{F}^{(+)}(\{\beta_{ijkl}\}, \alpha_s) \equiv \frac{1}{2}\Big\{\mathcal{F}(\beta_{1324}, \beta_{1423}; \alpha_s) + \mathcal{F}(\beta_{1234}, \beta_{1432}; \alpha_s)\Big\}, \tag{26}$$

$$\mathcal{F}^{(-)}(\{\beta_{ijkl}\}, \alpha_s) \equiv \frac{1}{2}\Big\{\mathcal{F}(\beta_{1234}, \beta_{1432}; \alpha_s) - \mathcal{F}(\beta_{1324}, \beta_{1423}; \alpha_s)\Big\} + \mathcal{F}(\beta_{1243}, \beta_{1342}; \alpha_s). \tag{27}$$

$$\mathcal{H}_1^{(+)}(\{\beta_{ijkl}\}, \alpha_s) \equiv \frac{1}{2}\Big\{\mathcal{H}_1(\beta_{1324}, \beta_{1423}; \alpha_s) + \mathcal{H}_1(\beta_{1234}, \beta_{1432}; \alpha_s)\Big\}, \tag{28}$$

$$\mathcal{H}_1^{(-)}(\{\beta_{ijkl}\}, \alpha_s) \equiv \frac{1}{2}\Big\{\mathcal{H}_1(\beta_{1324}, \beta_{1423}; \alpha_s) - \mathcal{H}_1(\beta_{1234}, \beta_{1432}; \alpha_s)\Big\}, \tag{29}$$

$$\tilde{\mathcal{H}}_1^{(-)}(\{\beta_{ijkl}\}, \alpha_s) \equiv \mathcal{H}_1(\beta_{1243}, \beta_{1342}; \alpha_s). \tag{30}$$

These are required in order to separate the soft anomalous dimension by signature in the high-energy limit and appears in eqs. (24,25).

B Soft Anomalous dimension in the high-energy limit from three loops

The soft anomalous dimension at one and two loops is captured by the dipole formula [1,3]

$$\mathbf{\Gamma}_n^{\text{dip.}}(\{s_{ij}\},\lambda,\alpha_s)=-\frac{\gamma_K(\alpha_s)}{4}\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\, l_{ij}+\sum_i^n\gamma_i(\alpha_s), \tag{31}$$

with the $l_{ij}=\log\frac{-s_{ij}}{\lambda^2}$. The anomalous dimension of parton i is $\gamma_i(\alpha_s)$, one for each external particle i. The soft anomalous dimension in the Regge limit for $2\to 2$ scattering is given in eq. (20) and at three-loop order, it reads

$$\mathbf{\Gamma}^{(3)}_{ij\to ij}\left(L,\frac{-t}{\lambda^2}\right)=\sum_R\gamma_{K,R}^{(3)}\left[L\mathbf{T}_t^2+i\pi\mathbf{T}_{s-u}^2\right]+\Gamma_i^{(3)}\left(\frac{-t}{\lambda^2}\right)+\Gamma_j^{(3)}\left(\frac{-t}{\lambda^2}\right)+\sum_{m=0}^{2}\mathbf{\Delta}^{(3,m)}L^m \tag{32}$$

with $\mathbf{\Delta}^{(3,m)}$ being corrections to the dipole formula starting at three loops. The corrections below were calculated explicitly in [4], after which colour expressions with signature in the Regge limit were found in [18]

$$\mathbf{\Delta}^{(-,3,2)}=\text{Im}\left[\mathbf{\Delta}^{(3,2)}\right]=0 \tag{33}$$

$$\mathbf{\Delta}^{(+,3,2)}=\text{Re}\left[\mathbf{\Delta}^{(3,2)}\right]=0 \tag{34}$$

$$\mathbf{\Delta}^{(-,3,1)}=i\pi\left[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\right]\frac{11}{4}\zeta_3 \tag{35}$$

$$\mathbf{\Delta}^{(+,3,1)}=\text{Re}\left[\mathbf{\Delta}^{(3,1)}\right]=0 \tag{36}$$

$$\mathbf{\Delta}^{(-,3,0)}=i\pi\left[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\right]\frac{11}{4}\zeta_4 \tag{37}$$

$$\begin{aligned}\mathbf{\Delta}^{(+,3,0)}=&\frac{1}{4}\left[\mathbf{T}_{s-u}^2,[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\right]\left(4\zeta_2\zeta_3-\zeta_5\right)-\frac{\zeta_5+2\zeta_2\zeta_3}{8}\Bigg\{-\frac{5}{8}C_A^2\mathbf{T}_t^2+f^{abe}f^{cde}\\&\times\left[\{\mathbf{T}_t^a,\mathbf{T}_t^d\}\left(\{\mathbf{T}_{s-u}^b,\mathbf{T}_{s-u}^c\}+\{\mathbf{T}_{s+u}^b,\mathbf{T}_{s+u}^c\}\right)+\{\mathbf{T}_{s-u}^a,\mathbf{T}_{s-u}^d\}\{\mathbf{T}_{s+u}^b,\mathbf{T}_{s+u}^c\}\right]\Bigg\}.\end{aligned} \tag{38}$$

The generators in the line above are defined as

$$\mathbf{T}_{s-u}^a\equiv\frac{1}{\sqrt{2}}\left(\mathbf{T}_s^a-\mathbf{T}_u^a\right)\qquad\mathbf{T}_{s+u}^a\equiv\frac{1}{\sqrt{2}}\left(\mathbf{T}_s^a+\mathbf{T}_u^a\right). \tag{39}$$

At four-loop order, the soft anomalous dimension in the high-energy limit is

$$\mathbf{\Gamma}^{(4)}_{ij\to ij}\left(L,\frac{-t}{\lambda^2}\right)=\sum_R\gamma_{K,R}^{(4)}\left[L\mathbf{T}_t^2+i\pi\mathbf{T}_{s-u}^2\right]+\Gamma_i^{(4)}\left(\frac{-t}{\lambda^2}\right)+\Gamma_j^{(4)}\left(\frac{-t}{\lambda^2}\right)+\sum_{m=0}^{3}\mathbf{\Delta}^{(4,m)}L^m. \tag{40}$$

At four loops, the specific corrections are

$$\mathbf{\Delta}^{(+,4,3)}=\text{Re}\left[\mathbf{\Delta}^{(3,2)}\right]=0, \tag{41}$$

$$\mathbf{\Delta}^{(-,4,3)}=-i\pi\frac{\zeta_3}{24}\left[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\right]\mathbf{T}_t^2, \tag{42}$$

$$\mathbf{\Delta}^{(+,4,2)}=\zeta_2\zeta_3\left\{-\frac{C_A^4}{24}+\frac{d_{AA}}{N_A}+\frac{1}{4}(-\mathbf{T}_t^2[(\mathbf{T}_{s-u}^2)^2,\mathbf{T}_t^2]+3[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2)\right\}, \tag{43}$$

where $\mathbf{\Delta}^{(-,4,2)}$ is still to be found and corrections at $O(L)$ and $O(L^0)$ still to be determined.

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