$\mathcal{O}(\alpha_s^3)$ corrections to semileptonic $b \to c$ decays in the heavy daughter approximation

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Abstract

We present our recent calculation of the third order corrections to the semileptonic $b \to c$ and the muon decays. The calculation has been performed in an expansion around the limit $m_c \sim m_b$, but shows decent convergence even for $m_c = 0$ from which the contribution to the muon decay can be extracted. For the semileptonic $b \to c$ decay we find large perturbative corrections in the on-shell scheme which can be significantly reduced by changing to the kinetic scheme for the bottom quark mass. These results are important input for the inclusive determination of $|V_{cb}|$ and the Fermi coupling constant G_F.

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1 Introduction

The Cabbibo-Kobayashi-Maskawa (CKM) matrix elements are fundamental constants in the Standard Model (SM) which describe the flavor mixing in the quark sector and provide the only source of charge-parity (CP) violation. It is therefore important to determine these constants precisely. One way to determine the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$ are inclusive semileptonic B meson decays $B \to X_{c(u)}\ell\overline{\nu}$ using global fits to the experimental values of the semileptonic decay widths and moments of kinematical distributions [1–5]. Here, the presence of the heavy bottom quark allows to describe the decay in the heavy quark effective theory (HQET), where the decay rate can be given in an expansion in the strong coupling constant α_s and in inverse powers of the heavy quark mass $1/m_b$. The leading order in $1/m_b$ is given by the free quark decay $b \to c\ell\overline{\nu}$ which had been known up to $\mathcal{O}(\alpha_s^2)$ [6–8] together with leading terms in the large β_0 approximation to higher orders [9]. Higher terms in the $1/m_b$ expansion are obtained from higher-dimensional operators in the HQE. In these proceedings we review the calculation of the semileptonic decay rate at leading order in $1/m_b$ to order α_s^3 obtained in Ref. [10] and report on recent progress on the extension of the calculation to inclusive moments.

2 Calculation

We compute the process

$$b(p) \to X_c(p_X) l(p_l) \overline{\nu}(p_\nu) , \tag{1}$$

where X_c is a state containing charm, light quarks and gluons. The calculation is based on the optical theorem, this means we have to compute the imaginary part of 5-loop forward scattering diagrams. Some example diagrams are given in Fig. 1. The total leptonic momentum is given by $q = p_\ell + p_\nu$. For the global fits not only the total decay width but also moments of kinematic distributions are used. In the following we will focus on the recently proposed q^2-moments [11] which are defined by

$$Q_i = \int \mathrm{d}q^2 (q^2)^i \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2} . \tag{2}$$

Note that Q_0 corresponds to the total decay rate Γ. Inclusively these moments, including the total rate Γ, can be computed from the imaginary part of the forward scattering diagrams by multiplying the integrands of the individual Feynman diagrams with the appropriate power of q^2 and then integrate. For global fits, moments of the total hadronic invariant mass and the charged lepton energy have been used.

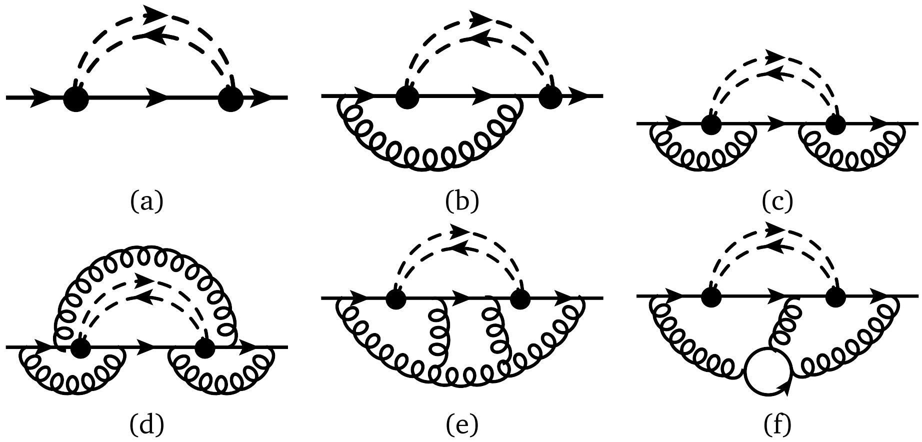

Figure 1: Sample Feynman diagrams which contribute to the forward scattering amplitude of a bottom quark at LO (a), NLO (b), NNLO (c) and N^{3}LO (d-f). Straight, curly and dashed lines represent quarks, gluons and leptons, respectively. The weak interaction mediated by the W boson is shown as a blob.

Since an analytic calculation retaining the full dependence on the charm and bottom mass seems out of reach, we compute the diagrams in an asymptotic expansion around

$$\delta = 1 - \frac{m_c}{m_b} \approx 0.7 . \tag{3}$$

Although this limit seems unnatural for the physical values of the charm and bottom quark masses it has been shown at $\mathcal{O}(\alpha_s^2)$ in Ref. [12] that this expansion converges quite fast at the physical point and can even be extended to $\delta \to 1$. Furthermore, this limit has a couple of technical advantages:

- To calculate the asymptotic expansion we use the method of regions [13]. In the limit $\delta \to 0$ the leptonic momentum has to be ultrasoft $q \sim \delta \cdot m_b$. The number of regions to be considered is therefore reduced.
- When performing the δ-expansion the leptonic system completely factorizes and can be integrated out without any IBP reduction. In the end, we are therefore left with 3-loop integrals, although starting from 5 loops.

The asymptotic expansion has been implemented in dedicated `FORM` [14] routines and we made use of the program `LIMIT` [15] to implement partial fractioning and the minimization of topologies. In the end we are left with integral families where either all loop momenta are hard or ultrasoft. The first class of integrals corresponds to on-shell propagator integrals which are well studied in the literature [16–18]. The second class has recently be considered for the relation between the kinetic and on-shell mass up to $\mathcal{O}(\alpha_s^3)$ [19,20], Here, the relevant master integrals are given to the necessary order in ϵ needed for the present calculation. Due to the large expansion depth of our calculation, we aim at 8 terms of the δ-expansion, huge intermediate expressions of $\mathcal{O}(100\text{GB})$ had to be handled and $\mathcal{O}(10^7)$ scalar integrals with positive or negative indices up to 12 had to be reduced. For this task we used `FIRE` [21] in combination with `LiteRed` [22,23].[1]

3 Results

We parametrize the total decay rate as

$$\Gamma = \Gamma_0 \left(X_0 + C_F \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^i X_i \right), \tag{4}$$

with $\Gamma_0 = A_{\text{ew}} G_F^2 |V_{cb}|^2 m_b^5/(192\pi^3)$, $\alpha_s \equiv \alpha_s^{(5)}(\mu_s)$, $X_0 = 1 - 8\rho^2 - 12\rho^4 \ln(\rho^2) + 8\rho^6 - \rho^8$, $\rho = m_c/m_b$ and $A_{\text{ew}} = 1.014$ is the leading electroweak correction [24]. Our result for the total decay rate at $\mathcal{O}(\alpha_s^3)$ reads

$$\begin{aligned} X_3 = {} & \delta^5 \Bigg(\frac{266929}{810} - \frac{5248 a_4}{27} + \frac{2186\pi^2\zeta_3}{45} - \frac{4094\zeta_3}{45} - \frac{1544\zeta_5}{9} - \frac{656 l_2^4}{81} + \frac{1336}{405}\pi^2 l_2^2 \\ & + \frac{44888\pi^2 l_2}{135} - \frac{9944\pi^4}{2025} - \frac{608201\pi^2}{2430} \Bigg) + \delta^6 \Bigg(-\frac{284701}{540} + \frac{2624 a_4}{9} - \frac{1093\pi^2\zeta_3}{15} \\ & + \frac{391\zeta_3}{3} + \frac{772\zeta_5}{3} + \frac{328 l_2^4}{27} - \frac{668}{135}\pi^2 l_2^2 - \frac{1484\pi^2 l_2}{3} + \frac{4972\pi^4}{675} \\ & + \frac{591641\pi^2}{1620} \Bigg) + \mathcal{O}(\delta^7 \ln^2(\delta)), \end{aligned} \tag{5}$$

where we specified the color factors to QCD, set $\mu_s = m_b$ and only show the first two expansion terms. Furthermore we use the notations $l_2 = \ln(2)$, $a_4 = \text{Li}_4(1/2)$ and ζ_i is Riemanns zeta function. The full result expressed in terms of $SU(N)$ color factors and up to $\mathcal{O}(\delta^{12})$ can be found in the ancillary file to Ref. [10]. Recently the results of three color factors up to $\mathcal{O}(\delta^9)$ have been confirmed in Ref. [25].

[1] We thank A. Smirnov for providing a private version of `Fire` which was essential for the reduction.

Analogously, we can give the result for Q_1:

$$Q_1 \;=\; \Gamma_0 m_b^2 \left(Y_0 + C_F \sum_{i=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^i Y_i \right), \tag{6}$$

with $Y_0 = 3/10(1-\rho^{10}) - 9/2(1-\rho^6)\rho^2 - 24(1-\rho^2)\rho^4 - 18(1+\rho^2)\rho^4 \ln(\rho^2)$. Our result for the $\mathcal{O}(\alpha_s^3)$ correction reads:

$$\begin{aligned} Y_3 \;=\; & \delta^7 \Big(-\frac{52480 a_4}{567} + \frac{4372\pi^2\zeta_3}{189} - \frac{8188\zeta_3}{189} - \frac{15440\zeta_5}{189} - \frac{6560 l_2^4}{1701} + \frac{2672\pi^2 l_2^2}{1701} \\ & + \frac{89776\pi^2 l_2}{567} - \frac{19888\pi^4}{8505} - \frac{608201\pi^2}{5103} + \frac{266929}{1701} \Big) + \delta^8 \Big(\frac{26240 a_4}{189} - \frac{2186\pi^2\zeta_3}{63} \\ & + \frac{3910\zeta_3}{63} + \frac{7720\zeta_5}{63} + \frac{3280 l_2^4}{567} - \frac{1336}{567}\pi^2 l_2^2 - \frac{2120\pi^2 l_2}{9} + \frac{9944\pi^4}{2835} + \frac{591641\pi^2}{3402} \\ & - \frac{284701}{1134} \Big) + \mathcal{O}(\delta^9 \ln^2(\delta)) . \end{aligned} \tag{7}$$

Since the leptonic momentum q has to be ultrasoft, i.e. $q \sim \delta \cdot m_b$, the n-th q^2 moment is suppressed by $2n$ additional powers of δ as compared to the leading δ^5 term in Eq. 5.

The convergence of the δ-expansion is studied in Fig. 2. For both, the total rate Γ and the first q^2 moment Q_1, one observes that the convergence at the physical point $\rho \sim 0.28$ is fast and does not vary much starting from order δ^9 for the total decay rate and δ^{11} for Q_1. Also at $\rho \to 0$ one sees a convergence for the high expansion terms, although much slower than at the physical point. We get:

$$X_3(\rho = 0.28) = -68.4 \pm 0.3 , \qquad Y_3(\rho = 0.28) = -14.41 \pm 0.03 . \tag{8}$$

The uncertainty due to the truncation of the series has been determined from the difference of the last two expansion orders including a safety factor of 5. At 2-loop order this approach leads to a conservative error approximation.

Using the on-shell masses $m_c = 1.3\,\text{GeV}$ and $m_b = 4.7\,\text{GeV}$ and setting the renormalization scale $\mu_s = m_b$, we find

$$\Gamma(m_b, m_c) \;=\; \Gamma_0 X_0 \left[1 - 1.72 \frac{\alpha_s}{\pi} - 13.09 \left(\frac{\alpha_s}{\pi}\right)^2 - 162.82 \left(\frac{\alpha_s}{\pi}\right)^3 \right], \tag{9}$$

$$Q_1(m_b, m_c) \;=\; \Gamma_0 m_b^2 Y_0 \left[1 - 1.61 \frac{\alpha_s}{\pi} - 12.83 \left(\frac{\alpha_s}{\pi}\right)^2 - 168.34 \left(\frac{\alpha_s}{\pi}\right)^3 \right]. \tag{10}$$

As expected we find a bad convergence of the perturbative series using the on-shell scheme for the quark masses. To mitigate this problem various so-called threshold masses have been proposed. We want to focus on the scheme used for the latest extraction of $|V_{cb}|$. Here, the bottom mass is expressed in the kinetic scheme, while the charm quark is expressed in the $\overline{\text{MS}}$ scheme at the scale $\mu_c = 3\,\text{GeV}$. With the input values $m_b^{\text{kin}} = 4.526\,\text{GeV}$ and $\overline{m}_c(3\,\text{GeV}) = 0.993\,\text{GeV}$, we find

$$\Gamma(m_b^{\text{kin}}, \overline{m}_c(3\,\text{GeV})) \;=\; \Gamma_0 X_0 \left[1 - 1.67 \frac{\alpha_s^{(4)}}{\pi} - 7.25 \left(\frac{\alpha_s^{(4)}}{\pi}\right)^2 - 28.6 \left(\frac{\alpha_s^{(4)}}{\pi}\right)^3 \right], \tag{11}$$

$$Q_1(m_b^{\text{kin}}, \overline{m}_c(3\,\text{GeV})) \;=\; \Gamma_0 Y_0 \left(m_b^{\text{kin}}\right)^2 \left[1 - 1.83 \frac{\alpha_s^{(4)}}{\pi} - 8.45 \left(\frac{\alpha_s^{(4)}}{\pi}\right)^2 - 34.7 \left(\frac{\alpha_s^{(4)}}{\pi}\right)^3 \right], \tag{12}$$

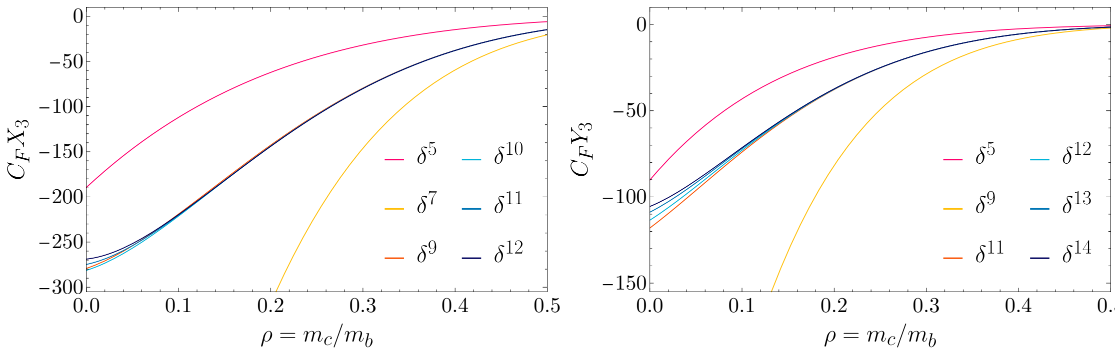

Figure 2: The $\mathcal{O}(\alpha_s^3)$ contribution to the total decay rate $b \to s\ell\overline{\nu}$ (left) and its first q^2-moment (right) for different expansions depths in δ.

where $\mu_s = m_b^{\text{kin}}$ is used. Note that the conversion to the kinetic scheme also contains the renormalization of the HQET parameters μ_π and ρ_D, which formally only enter at order $1/m_b^2$ and $1/m_b^3$ respectively. The scale dependence of the two quantities can be studied in Fig. 3. One observes a much better behavior of the perturbative series and a reduced dependence on the renormalization scale.

The results for the total cross section together with the improvement of the relation between the on-shell and kinetic mass to $\mathcal{O}(\alpha_s^3)$ have recently been used to update the inclusive determination of $|V_{cb}|$ [5]

$$|V_{cb}| = 42.16(30)_{th}(32)_{exp}(25)_\Gamma \times 10^{-3}\,. \tag{13}$$

The inclusion of the higher order calculations resulted in a small shift of the central value and a reduced theory uncertainty. Especially the uncertainty due to the width Γ was halved.

The heavy daughter limit allows us also to estimate the $O(\alpha_s^3)$ correction to the $b \to u\ell\overline{\nu}_\ell$ decay by setting $\delta \to 1$:

$$X_3^u = -202 \pm 20\,, \tag{14}$$

where the relative 10% uncertainty is estimated for unknown higher δ terms in the expansion. We can study the convergence of the perturbative series also in this case. We use the exact one and two loop results in the massless limit from Ref. [26] and the three loop correction estimated above to derive the total rate for $b \to u\ell\overline{\nu}_\ell$

$$\Gamma_{b\to u}(m_b^{\text{kin}}, \overline{m}_c(3\text{ GeV})) = \Gamma_0 X_0\left[1 - 0.27\frac{\alpha_s^{(4)}}{\pi} + 4.0\left(\frac{\alpha_s^{(4)}}{\pi}\right)^2 + 95.4\left(\frac{\alpha_s^{(4)}}{\pi}\right)^3\right]. \tag{15}$$

We observe an apparent worse behavior of the α_s expansion compared to $b \to c$. Note that the result depends in this case also on the Weak-Annihilation scale entering in the Wilson coefficient of ρ_D at order $1/m_b^3$. We set $\mu_{\text{WA}} = m_b^{\text{kin}}/2$.

If we specify the color factors to QED and set $\delta = 1 - m_e/m_\mu \approx 0.005$ we obtain a prediction for the muon lifetime τ_μ via

$$\frac{1}{\tau_\mu} \equiv \Gamma(\mu^- \to e^- \nu_\mu \overline{\nu}_e) = \frac{G_F^2 m_\mu^5}{192\pi^3}(1 + \Delta q)\,. \tag{16}$$

Precise measurements of the muon lifetime together with accurate QED predictions therefore allow the extraction of the Fermi constant G_F. The various correction terms, see for example Ref. [27] for a review, are usually parametrized via

$$\Delta q = \sum_{i\geq 0} \Delta q^{(i)}\,. \tag{17}$$

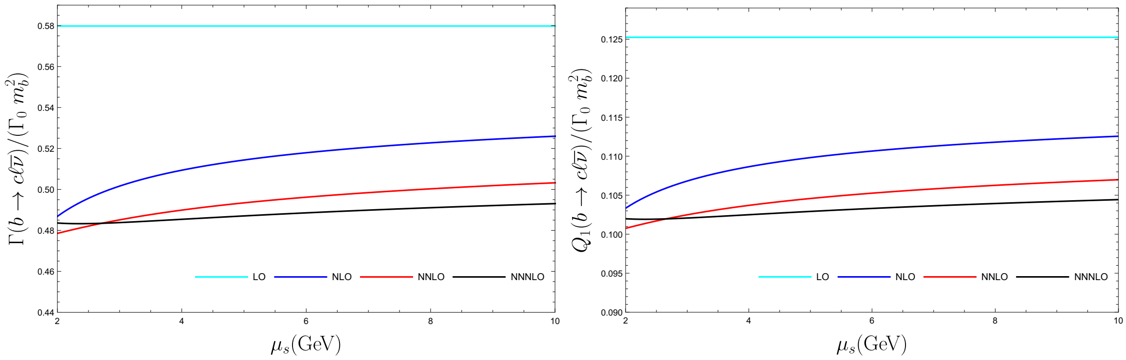

Figure 3: The dependence on μ_s of the total rate Γ (left) and the first q^2-moment Q_1 (right) for different orders in the strong coupling constants with the bottom quark expressed in the kinetic scheme and the charm mass in the $\overline{\mathrm{MS}}$ scheme. The scale of the charm quark is set to $\mu_c = 3\,\mathrm{GeV}$.

We find for the QED corrections

$$\Delta q^{(3)} \approx \left(\frac{\alpha(m_\mu)}{\pi}\right)^3 (-15.3 \pm 2.3)\,, \tag{18}$$

where the error is estimated from the convergence properties at 1- and 2-loop order for which exact calculations are available [28–30]. This translates to a shift in the muon lifetime of $\Delta\tau_\mu \approx (-9 \pm 1) \times 10^{-8}\,\mu\mathrm{s}$. Comparing this with the current experimental value given by $\tau_\mu = (2.1969811 \pm 0.0000022)\,\mu\mathrm{s}$ [31], we see that the new corrections are two orders of magnitude smaller than the current experimental uncertainty. A new extraction of the Fermi constants therefore needs an improvement of the experimental data.

4 Conclusions

In these proceedings we reviewed the calculation of the $\mathcal{O}(\alpha_s^3)$ corrections to the process $b \to c\ell\overline{\nu}$ retaining finite charm quark effects through an expansion around the equal mass case obtained in Ref. [10]. Furthermore, we showed an extension of our method to inclusive q^2 moments, which can be used to further constrain the global fits from which also $|V_{cb}|$ is extracted. We showed that the expansions converge fast at the physical point and can even be extended down to $\delta \to 1$. Although we find a badly converging prediction using the on-shell masses for charm and bottom, the predictions are improved by changing into the kinetic scheme for the bottom quark. Since the knowledge of other moments, like moments of the lepton energy or the hadronic mass, is desireable for the global fits we plan to extend our calculation. After specifying our results to QED we also obtain $\mathcal{O}(\alpha^3)$ predictions for the muon decay.

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