Recent progress on two-loop massless pentabox integrals with one off-shell leg

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Abstract

Analytic expressions in terms of polylogarithmic functions for all three families of planar two-loop five-point Master Integrals with one off-shell leg are presented. The Simplified Differential Equations approach is the only known way to fulfil this task due to its unique factorisation property of the symbols of the canonical differential equation. The results are relevant to the study of many $2 \to 3$ scattering processes of interest at the LHC.

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1 Introduction

During the last decade we have learned that in order to discover new phenomena in Nature, from gravitational wave astronomy [1] to high-energy physics [2, 3], we need not only very sophisticated, state-of-the-art instrumentation, but also very precise theoretical predictions.

Next-to-next-to-leading order (NNLO) accuracy is needed for the vast majority of QCD dominated scattering processes at the LHC (see [4] and references therein). Two-loop amplitude computations require the reduction of the scattering matrix element in terms of basis integrals, usually referred to as Master Integrals (MI). Traditional reduction techniques based on integration-by-part identities [5–7] (IBP), *at the integral level*, are now more and more replaced by *integrand-reduction methods* [8–10], following the one-loop paradigm [11].

Results for five-point two-loop amplitudes, relevant for three-jet/photon, $W, Z, H + 2$ jets production have been recently presented [12–16]. Moreover, a complete NNLO calculation for the relatively easy case of three-photon production at the LHC, has been recently published [17, 18]. Despite the progress in understanding amplitude reduction and real radiation corrections at NNLO, a remarkable contradistinction with the NLO case is that the basis of Master Integrals at two loops is still far from complete[1].

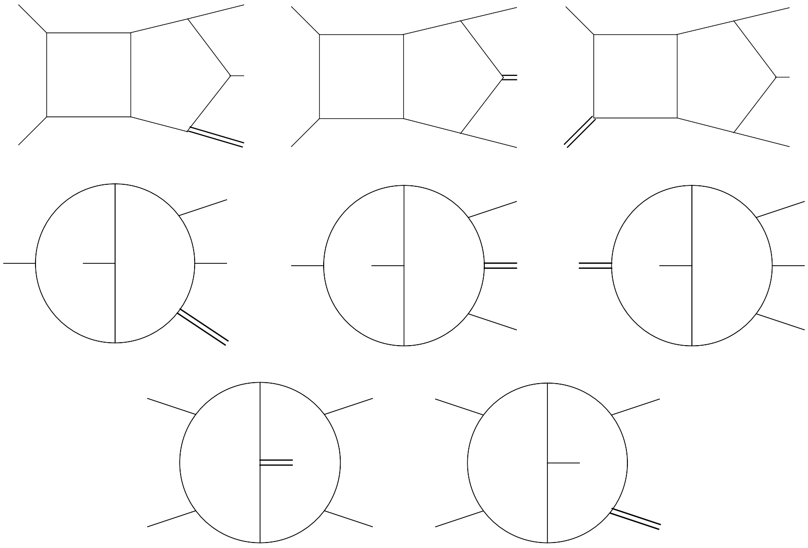

Figure 1: Diagrammatic representation of the planar and non-planar families with one external massive leg (double line). In the first row, P_1 (left), P_2 (middle) and P_3 (right) planar families are shown. In the second and third row, N_1 (top left), N_2 (top middle), N_3 (top right), N_4 (bottom left), N_5 (bottom right) non-planar families are shown. All internal particles are massless.

Five-point two-loop Master Integrals determine the current frontier. The computation of all planar and non-planar five-point two-loop Master Integrals with massless internal propagators and on-shell light-like external momenta, has been completed some time ago [21–25]. The next step on this path of computing the five-point two-loop Master Integrals would be those with one of the external legs being off-shell. The planar and non-planar topologies corresponding to these Master Integrals are shown in Fig. 1. Based on the Simplified Differential Equations (SDE) approach [26], all Master Integrals for the first non-trivial planar family of five-point two-loop Master Integrals with massless internal propagators and one external particle carrying a space- or time-like momentum, P_1 in Fig. 1, as well as the full set of planar five-point two-loop massless Master Integrals with light-like external momenta [21] has been calculated some time ago. Very recently results on all planar families have been reported in reference [27]. In reference [28] we presented fully analytic results in terms of poly-logarithmic functions for all planar families, based on the Simplified Differential Equations approach.

In section 2, we define the scattering kinematics and derive the form of the canonical differential equation in the SDE approach. The derivation of the boundary terms and the solution for all Master Integrals in terms of Goncharov poly-logarithms (GP), is presented in section 3. In section 4 we show how to obtain numerical results from our analytic expressions in all kinematical regions. Finally in section 5 we summarize our findings and discuss future applications with emphasis on the computation of the remaining non-planar five-point two-loop Master Integrals.

[1] For interesting alternative approaches see references [19, 20].

2 Planar two-loop five-point Master Integrals with one off-shell leg

There are three families of Master Integrals, labelled as P_1, P_2 and P_3, see Fig. 1, associated to planar two-loop five-point amplitudes with one off-shell leg. We adopt the definition of the scattering kinematics following [27], where external momenta q_i, $i = 1\ldots5$ satisfy $\sum_1^5 q_i = 0$, $q_1^2 \equiv p_{1s}$, $q_i^2 = 0$, $i = 2\ldots5$, and the six independent invariants are given by $\{q_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$, with $s_{ij} := \left(q_i + q_j\right)^2$.

In the SDE approach [26] the momenta are parametrized by introducing a dimensionless variable x, as follows

$$q_1 \to p_{123} - x p_{12},\; q_2 \to p_4,\; q_3 \to -p_{1234},\; q_4 \to x p_1\,, \tag{1}$$

where the new momenta p_i, $i = 1\ldots5$ satisfy now $\sum_1^5 p_i = 0$, $p_i^2 = 0$, $i = 1\ldots5$, whereas $p_{i\ldots j} := p_i + \ldots + p_j$. The set of independent invariants is given by $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := \left(p_i + p_j\right)^2$. The explicit mapping between the two sets of invariants is given by

$$\begin{gathered} p_{1s} = (1-x)(S_{45} - S_{12}x),\; s_{12} = \left(S_{34} - S_{12}(1-x)\right)x,\; s_{23} = S_{45},\; s_{34} = S_{51}x, \\ s_{45} = S_{12}x^2,\; s_{15} = S_{45} + (S_{23} - S_{45})x \end{gathered} \tag{2}$$

and as usual the $x = 1$ limit corresponds to the on-shell kinematics.

The P_1 family consists of 74 Master integrals. For P_2 and P_3 the corresponding numbers are 75 and 86. This can easily be verified using standard IBP reduction software, such as `FIRE6` [29] and `Kira` [30,31]. The top-sector integrals are shown in Fig. 2.

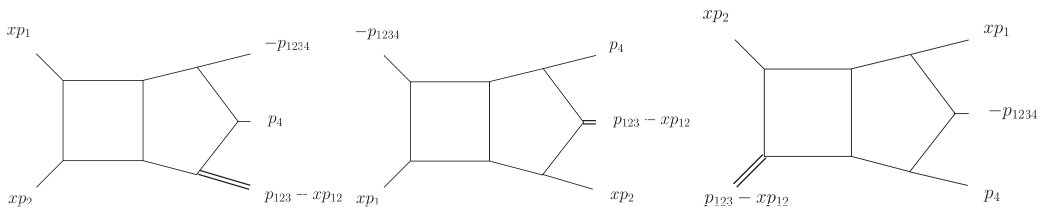

Figure 2: The two-loop diagrams representing the top-sector of the planar pentabox family P_1, P_2 and P_3. All external momenta are incoming.

2.1 Canonical basis and Differential Equations

In order to express all planar five-point integrals, the easiest way is to define a basis that satisfies a canonical differential equation. By basis we mean a combination of Feynman Integrals with coefficients depending on the set of invariants and the dimensionality of space-time $d = 4 - 2\epsilon$. Let us assume that such a basis is known, then the DE is written in general as

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}\,, \tag{3}$$

where $\vec{g}$ represents a vector containing all elements of the canonical basis, W_a are functions of the kinematics and $\tilde{M}_a$ are matrices independent of the kinematical invariants, whose matrix elements are pure rational numbers. Notice that Eq. (3) is a multi-variable equation and in the case under consideration the differentiation is understood with respect to the six-dimensional array of independent kinematical invariants, $\{q_1^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$. Since W_a are in general

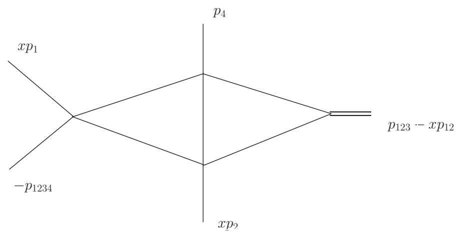

Figure 3: The two-loop diagram representing the decoupling basis element.

algebraic functions of the kinematical invariants a straightforward integration of Eq. (3) in terms of generalized poly-logarithms is not an easy task.

In the SDE approach though, Eq. (3) takes the much simpler form

$$\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - l_b} M_b \vec{g}\,, \tag{4}$$

where M_b are again rational matrices independent of the kinematics, and the so-called letters, l_b, are independent of x, depending only on the five invariants, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. Notice that the number of letters in x is generally smaller than the number of letters in Eq. (3). Since the Eq. (4) is a Fuchsian system of ordinary differential equations, it is straightforwardly integrated in terms of Goncharov poly-logarithms, $\mathcal{G}(l_1, l_2, \ldots; x)$.

As is seen from Eq. (4) the main achievement of the SDE approach is the factorisation of the letters W_a in Eq. (3) in terms of the x−variable, in the form $x - l_b$, where l_b depend on the underlying kinematical variables. The knowledge of the canonical basis is enough within the SDE approach to derive the form of the corresponding canonical differential equation, Eq. (4), by explicitly differentiating with respect to x and using IBP identities to express the resulting combinations of Feynman integrals in terms of basis elements.

In the present calculation this factorisation is achieved for all basis elements of the P_1 family. For P_2 and P_3 families Eq. (4) is applicable after eliminating a special basis element whose leading singularity is proportional to a non-rationalizable square root in terms of x. The corresponding integral is shown in Fig. 3 and it is the same for the two families. Its expression in terms of poly-logarithmic functions is already known from the double-box families with two off-shell legs[2]. For a detailed discussion see reference [28].

[2] The basis element is numbered as 46 in the P_2 family and 53 in the P_3 family and is given in terms of the double-box P_{23} family variables [32].

3 Boundary Conditions and Analytic Expressions

The solution of Eq. (4) up to order $\mathcal{O}\left(\epsilon^4\right)$ can be written as follows:

$$\begin{aligned}
\mathbf{g} = {}& \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \epsilon^2\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \\
&+ \epsilon^3\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)}\right) \\
&+ \epsilon^4\Bigg(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)} \\
&+ \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)}\Bigg) \qquad (5) \\
\mathcal{G}_{ab\ldots} :={}& \mathcal{G}(l_a, l_b, \ldots; x),
\end{aligned}$$

where $\mathbf{g}$ and $\mathbf{M}$ represent $\vec{g}$ and M appearing in Eq. (4) and $\mathbf{b}_0^{(i)}$ are the boundary values of the basis elements in the limit $x \to 0$ (see Eq.(3.6) of reference [21]) at order ϵ^i, $i = 0 \ldots 4$. In the above equation $\mathcal{G}(l_a, l_b, \ldots; x)$ stands for Goncharov polylogarithms. Since all the data of the above equation, namely the letters $l_a, l_b, \ldots$ and the matrices $\mathbf{M}_a, \mathbf{M}_b, \ldots$ are already given, the only remaining task is the computation of the boundary values, $\mathbf{b}_0^{(i)}$, in terms of poly-logarithmic functions.

To derive the $x \to 0$ limit of basis elements we first exploit the canonical differential equation in x, Eq. (3), which in the limit takes the form

$$\frac{d\vec{g}}{dx} = \epsilon \frac{1}{x} M_0 \vec{g} + \mathcal{O}(x^0), \qquad (6)$$

with the solution ($\mathbf{b} := \sum_{i=0}^{4} \epsilon^i \mathbf{b}_0^{(i)}$)

$$\mathbf{g}_0 = \mathbf{S} e^{\epsilon \log(x) \mathbf{D}} \mathbf{S}^{-1} \mathbf{b} \qquad (7)$$

and the matrices $\mathbf{S}$ and $\mathbf{D}$ are obtained through Jordan decomposition of the $\mathbf{M}_0$ matrix, $\mathbf{M}_0 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. We call the matrix $\mathbf{R}_0 = \mathbf{S} e^{\epsilon \log(x) \mathbf{D}} \mathbf{S}^{-1}$, the resummed matrix at $x = 0$. Since the biggest Jordan block of it has dimension two, it can be written in the form

$$\mathbf{R}_0 = \sum_i x^{n_i \varepsilon} \left(\mathbf{R}_{0i} + \varepsilon \log(x) \mathbf{R}_{0i0}\right), \qquad (8)$$

with $\mathbf{R}_{0i}$ and $\mathbf{R}_{0i0}$ matrices of rational numbers and the exponents n_i are the eigenvalues of the matrix $\mathbf{D}$ (equivalently $\mathbf{M}_0$).

On the other hand through IBP reduction the elements of the canonical basis can be related to a set of Master Integrals,

$$\mathbf{g} = \mathbf{T}\mathbf{G}. \qquad (9)$$

The list of Feynman Integrals $\mathbf{G}$ chosen as Master Integrals in the IBP reduction as well as the expression of the basis elements in terms of Feynman Integrals for all families is given in the ancillary files of reference [28].

We have used the expansion by regions techniques [33] in order to write each Master Integral in the form of a sum over region-integrals,

$$G_i \underset{x \to 0}{=} \sum_j x^{b_j + a_j \varepsilon} G_i^{(j)}, \qquad (10)$$

with a_j and b_j being integers, by making use of the `FIESTA4` [34] public code. Combining Eqs. (7) and (9) we get

$$\mathbf{g}_0 := \mathbf{R}_0\mathbf{b} = \lim_{x\to 0} \mathbf{TG}|_{\mathcal{O}(x^{0+a_j\epsilon})} , \tag{11}$$

where, since the dependence of the left-hand side on x is only through Eq. (8), in the right-hand side, except for the terms of the form $x^{a_j\epsilon}$ arising from Eq. (10), we expand around $x = 0$, keeping only terms of order x^0. Notice also, that the left-hand side of the equation contains the boundary values of the basis elements that are pure functions of the underlying kinematics $\vec{S} := \{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$ whereas in the right hand side the matrix $\mathbf{T}$ is an algebraic function of $\vec{S}$. The consistency of Eq. (11) implies that the right-hand side should also be a pure function of $\vec{S}$. Therefore, in order to determine the matrix $\mathbf{T}$ entering in Eq. (11), we can employ solutions of IBP identities using numerical, actually integer values for $\vec{S}$, keeping x and d in a symbolic form. This results to a significant reduction in complexity and CPU time, taking into account that there are several basis elements in $\mathbf{g}$, that are given in terms of Baikov polynomials [27], $\mu_{11}, \mu_{12}, \mu_{22}$, which when expressed in terms of inverse propagators, contain Feynman Integrals with up to fourth powers of irreducible inverse propagators. For details see reference [28].

4 Numerical Results and Validation

In order to numerically evaluate the solution given in Eq. (5), Goncharov poly-logarithms up to weight 4 need to be computed. To understand the complexity of the expressions at hand, we present in Table 1, the number of poly-logarithmic functions entering in the solution. In parenthesis we give the corresponding number for the non-zero top-sector basis elements. The weight W=1...4 is identified as the number of letters l_a in GP $\mathcal{G}(l_a, \ldots; x)$.

Table 1: Number of GP entering in the solution, as explained in the text.

Family	W=1	W=2	W=3	W=4
P_1 (g_{72})	17 (14)	116 (95)	690 (551)	2740 (2066)
P_2 (g_{73})	25 (14)	170 (140)	1330 (1061)	4950 (3734)
P_3 (g_{84})	22 (12)	132 (90)	1196 (692)	4566 (2488)

The computation of GPs is performed using their implementation in `GiNaC` [35]. This implementation is capable to evaluate the GPs at an arbitrary precision. The computational cost to numerically evaluate a GP function, depends of course on the number of significant digits required as well as on their weight and finally on their structure, namely how many of its letters, Eq. (5), satisfy $l_a \in [0, x]$. We refer to reference [36] for more details. In Table 2 results for a Euclidean point where all GP functions with real letters are real, namely no letter is in $[0, x]$, are presented. The CPU time running the `GiNaC` Interactive Shell `ginsh`, is given by 1.9, 3.3, and 2 seconds for P_1, P_2 and P_3 respectively and for a precision of 32 significant digits.

In order to obtain numerical results for scattering kinematics, we need to properly analytically continue the GPs and logarithms involved in our solution, Eq. (5). The easiest way is to determine for each physical point under consideration, the real parameters δ_{ij} and δ_x so that the substitution, $S_{ij} \to S_{ij} + i\delta_{ij}\eta$, $x \to x + i\delta_x\eta$, $\eta \to 0$, of the variables used in our solution, properly accounts for the analytic continuation. As detailed in references [21, 32], δ_{ij} and δ_x should satisfy analyticity constraints stemming (a) from the second graph polynomial $\mathcal{F}$ of the top-sector Feynman integral and (b) from the representation of the one-scale integrals in terms of the variables x and S_{ij}. For details and results for all physical regions see reference [28].

Table 2: Numerical results for the non-zero top sector element of each family with 32 significant digits at $S_{12} \to -2, S_{23} \to -3, S_{34} \to -5, S_{45} \to -7, S_{51} \to -11, x \to \frac{1}{4}$.

P_1	g_{72}	ϵ^0: 3/2 ϵ^1: -2.2514604753379400332169314784961 ϵ^2: -17.910593443812320786572184851867 ϵ^3: -26.429770706459534336624681550003 ϵ^4: 21.437938934510558345847354772412
P_2	g_{73}	ϵ^1: 2.8124788185742741402751457351382 ϵ^2: 5.4813042746593704203645729908938 ϵ^3: 11.590234540689191439870956817546 ϵ^4: -5.9962816226829136730734255754596
P_3	g_{84}	ϵ^0: 1/2 ϵ^1: 3.2780415861887284967738281876762 ϵ^2: 0.11455863130537720411162743574627 ϵ^3: -16.979642659429606120982671925458 ϵ^4: -48.101985355625914648042310964575

We have also compared our results for all families, all basis elements and all physical points with those of reference [27] and found perfect agreement to the precision used, ($N_{digits} = 16, 32$). We also checked our results, not only at the level of basis elements but also at the level of Master Integrals, against `FIESTA4` [34] and found agreement within the numerical integration errors provided by it.

5 Conclusions and Outlook

We have presented analytic expressions in terms of poly-logarithmic functions, Goncharov Polylogarithms, of all planar two-loop five-point integrals with a massive external leg [28]. This has been achieved by using the Simplified Differential Equations approach and the data for the canonical basis provided in reference [27]. Moreover, the necessary boundary values of all basis elements have been computed, based mainly on the form of the canonical differential equation, Eq. (4) and, in few cases, on the expansion by regions approach. The ability to straightforwardly compute the boundary values at $x = 0$ and to even more straightforwardly express the solution in terms of Goncharov Polylogarithms, is based on the unique property of the SDE approach that the scattering kinematics is effectively factorised with respect to x, in noticeable contradistinction with the standard differential equation approach, where such an analytic realisation of the solution is prohibitively difficult.

Obviously, the next step, is to extend the current work in the case of the remaining five non-planar families, shown in Fig. 1. The hexa-box families N_1, N_2, N_3 are currently under investigation and we have succeeded in obtaining a full analytic representation for the N_1 family in terms of Goncharov polylogarithms[3]. Results will be published in the near future. When the computation of all five non-planar families in Fig. 1 is completed, a library of all two-loop Master Integrals with internal massless particles and up to five (four) external legs, among which one (two) massive legs will be provided: this will constitute a significant milestone towards the knowledge of the full basis of two-loop Feynman Integrals.

[3]We are grateful to the authors of reference [37] for providing the canonical basis of the hexa-box families, as well as their numerical results with which we found perfect agreement for N_1.

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