PMC$_\infty$: Infinite-Order Scale-Setting method using the Principle of Maximum Conformality and preserving the Intrinsic Conformality

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Abstract

We show results for Thrust and C-parameter in e^+e^- annihilation to 3 jets obtained using the recently developed new method for eliminating the scale ambiguity and the scheme dependence in pQCD namely the Infinite-Order Scale-Setting method using the Principle of Maximum Conformality (PMC$_\infty$). This method preserves an important underlying property of gauge theories: intrinsic Conformality (iCF). It leads to a remarkably efficient method to eliminate the conventional renormalization scale ambiguity at any order in pQCD. A comparison with Conventional Scale Setting method (CSS) is also shown.

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1 Introduction

One of the obstacles in making precision tests of the quantum chromodynamics (QCD) is the uncertainty in setting the renormalization scale μ_R into running coupling $\alpha_s(\mu_R^2)$ for the perturbative expansion of a scale invariant quantity.

The conventional practice (i.e. conventional scale setting - CSS) of simply guessing the scale μ_R of the order of a typical momentum transfer Q in the process, and then varying the scale over a range Q/2 and 2Q, leads to predictions that are affected by large renormalization scale ambiguities.

Additionally, the CSS procedure is not consistent with the Gell-Mann-Low scheme [1] in Quantum Electrodynamics (QED) [2], the pQCD predictions are affected by scheme dependence and the resulting perturbative QCD series is also factorially divergent like $n!\beta_0^n\alpha_s^n$, i.e.

the "renormalon" problem [3]. Given the factorial growth, the hope to suppress scale uncertainties by including higher-order corrections is compromised. We recall that there is no ambiguity in setting the renormalization scale in QED. The standard Gell-Mann-Low scheme determines the correct renormalization scale identifying the scale with the virtuality of the exchanged photon. For example, in electron-muon elastic scattering, the renormalization scale is the virtuality of the exchanged photon, i.e. the spacelike momentum transfer squared $\mu_R^2 = q^2 = t$. Thus

$$\alpha(t) = \frac{\alpha(t_0)}{1 - \Pi(t, t_0)} \tag{1}$$

where

$$\Pi(t, t_0) = \frac{\Pi(t) - \Pi(t_0)}{1 - \Pi(t_0)}.$$

From Eq.1 it follows that the renormalization scale $\mu_R = t$ can be determined by the β_0-term and it sums up all the vacuum polarization contributions into the dressed photon propagator, both proper and improper at all orders. Given that the pQCD and pQED predictions match analytically in the $N_C \to 0$ limit where $C_F \alpha_{QCD} \to \alpha_{QED}$ (see ref. [4]) it would be convenient to extend the same procedure to pQCD. A solution to the scale ambiguity problem is offered by the *Principle of Maximum Conformality* (PMC) [5–10]. This method provides a systematic way to eliminate renormalization scheme-and-scale ambiguities from first principles by absorbing the β terms that govern the behavior of the running coupling via the renormalization group equation. Thus, the divergent renormalon terms cancel, which improves the convergence of the perturbative QCD series. Furthermore, the resulting PMC predictions do not depend on the particular scheme used, thereby preserving the principles of renormalization group invariance [11,12]. The PMC procedure is also consistent with the standard Gell-Mann-Low method in the Abelian limit, $N_c \to 0$ [4]. Besides, in a theory of unification of all forces, electromagnetic, weak and strong interactions, such as the Standard Model, or Grand Unification theories, is highly desirable to use only one method. The PMC offers the possibility to apply the same method in all sectors of a theory, starting from first principles, eliminating the renormalon growth, the scheme dependence, the scale ambiguity, and satisfying the QED Gell-Mann-Low scheme in the zero-color limit $N_c \to 0$.
The recently developed PMC$_\infty$: *Infinite-Order Scale-Setting using the Principle of Maximum Conformality* [13] is a new method based on the PMC and it preserves the property that we define as *Intrinsic Conformality* (*iCF*). This property stems directly from an analysis of the perturbative QCD corrections and leads to scale invariance of an observable calculated at any fixed order independently from the particular process or kinematics. Here we apply this method to the Event Shape Variables : Thrust and C-parameter, showing results and comparison with the CSS.

2 The Thrust and C-parameter at NNLO and the CSS

The thrust (T) and C-parameter (C) are defined as

$$T = \max_{\vec{n}} \left(\frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \right), \tag{2}$$

$$C = \frac{3}{2} \frac{\sum_{i,j} |\vec{p}_i||\vec{p}_j| \sin^2 \theta_{ij}}{\left(\sum_i |\vec{p}_i|\right)^2}, \tag{3}$$

where $\vec{p}_i$ denotes the three-momentum of particle i. For the thrust, the unit vector $\vec{n}$ is varied to define the thrust axis $\vec{n}_T$ by maximizing the sum on the right-hand side. For the C-parameter, θ_{ij} is the angle between $\vec{p}_i$ and $\vec{p}_j$. It is often used the variable $(1-T)$, which for the LO of the 3 jet production is restricted to the range $(0 < 1-T < 1/3)$ and for the C-parameter is $0 \le C \le 0.75$. (For a review on the Event Shape variables see Refs. [14–26].)

In general a normalized IR safe single variable observable, such as the thrust or the C-parameter distribution for the $e^+e^- \to 3jets$ [27, 28], is given by the sum of pQCD contributions calculated up to NNLO at the initial renormalization scale μ_0:

$$\frac{1}{\sigma_{tot}}\frac{Od\sigma(\mu_0)}{dO} = \left\{ x_0 \cdot \frac{Od\bar{A}_O(\mu_0)}{dO} + x_0^2 \cdot \frac{Od\bar{B}_O(\mu_0)}{dO} + x_0^3 \cdot \frac{Od\bar{C}_O(\mu_0)}{dO} + \mathcal{O}(\alpha_s^4) \right\}, \quad (4)$$

where $x(\mu) \equiv \alpha_s(\mu)/(2\pi)$, O is the selected Event Shape variable, σ the cross section of the process,

$$\sigma_{tot} = \sigma_0 \left(1 + x_0 A_{tot} + x_0^2 B_{tot} + \mathcal{O}\left(\alpha_s^3\right)\right)$$

is the total hadronic cross section and $\bar{A}_O, \bar{B}_O, \bar{C}_O$ are respectively the normalized LO, NLO and NNLO coefficients:

$$\begin{aligned} \bar{A}_O &= A_O \\ \bar{B}_O &= B_O - A_{tot} A_O \\ \bar{C}_O &= C_O - A_{tot} B_O - \left(B_{tot} - A_{tot}^2\right) A_O, \end{aligned} \quad (5)$$

where A_O, B_O, C_O are the coefficients normalized to the tree level cross section σ_0 calculated by MonteCarlo (see e.g. EERAD and Event2 codes [20–24]) and A_{tot}, B_{tot} are:

$$\begin{aligned} A_{tot} &= \frac{3}{2} C_F; \\ B_{tot} &= \frac{C_F}{4} N_c + \frac{3}{4} C_F \frac{\beta_0}{2} (11 - 8\zeta(3)) - \frac{3}{8} C_F^2, \end{aligned} \quad (6)$$

where ζ is the Riemann zeta function.

In general according to CSS the renormalization scale is set to $\mu_0 = \sqrt{s} = M_{Z_0}$ and theoretical uncertainties are evaluated using standard criteria. In this case, we have used the definition given in Ref. [20] of the parameter δ, we define the average error for the event shape variable distributions as:

$$\bar{\delta} = \frac{1}{N} \sum_i^N \frac{\max_\mu(\sigma_i(\mu)) - \min_\mu(\sigma_i(\mu))}{2\sigma_i(\mu = M_{Z_0})}, \quad (7)$$

where i is the index of the bin and N is the total number of bins, the renormalization scale is varied in the range: $\mu \in [M_{Z_0}/2; 2M_{Z_0}]$.

3 The iCF : conformal coefficients and intrinsic scales

We define *Intrinsic Conformality* as the unique property of a renormalizable SU(N)/U(1) gauge theory, like QCD, which yields to a particular structure of the perturbative corrections that can be made explicit representing the perturbative coefficients of Eq. 4 using the following RG invariant parametrization:

$$\begin{aligned} A_O(\mu_0) &= A_{Conf}, \\ B_O(\mu_0) &= B_{Conf} + \frac{1}{2}\beta_0 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right) A_{Conf}, \\ C_O(\mu_0) &= C_{Conf} + \beta_0 \ln\left(\frac{\mu_0^2}{\mu_{II}^2}\right) B_{Conf} + \frac{1}{4}\left[\beta_1 + \beta_0^2 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\right] \ln\left(\frac{\mu_0^2}{\mu_I^2}\right) A_{Conf}, \end{aligned} \tag{8}$$

where the $A_{Conf}, B_{Conf}, C_{Conf}$ are the scale invariant *Conformal Coefficients* (i.e. the coefficients of each perturbative order not depending on the scale μ_0) while we define the μ_N as *Intrinsic Conformal Scales* and β_0, β_1 are the first two coefficients of the β-function [29–33].

By collecting together the terms identified by the same conformal coefficient, we obtain the observable written in *conformal subset* (σ_n) :

$$\begin{aligned} \sigma_I &= \left\{\left(\frac{\alpha_s(\mu_0)}{2\pi}\right) + \frac{1}{2}\beta_0 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 \right. \\ &+ \left. \frac{1}{4}\left[\beta_1 + \beta_0^2 \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\right] \ln\left(\frac{\mu_0^2}{\mu_I^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 + \ldots \right\} A_{Conf} \\ \sigma_{II} &= \left\{\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^2 + \beta_0 \ln\left(\frac{\mu_0^2}{\mu_{II}^2}\right)\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 + \ldots \right\} B_{Conf} \\ \sigma_{III} &= \left\{\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^3 + \ldots \right\} C_{Conf}, \\ &\vdots \qquad \qquad \ddots \\ \sigma_n &= \left\{\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n \right\} \mathcal{L}_{nConf}. \end{aligned} \tag{9}$$

Any combination of the conformal subsets, $\sigma_I, \sigma_{II}, \sigma_{III}, \ldots$ such as $\sigma_N = \sum_i \sigma_i$ is still conformal :

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right) \sigma_N = 0. \tag{10}$$

We define here this property of Eq. 9 of separating an observable in the union of ordered scale invariant disjoint subsets $\sigma_I, \sigma_{II}, \sigma_{III}, \ldots$ as *ordered scale invariance*.

The coefficients of Eq. 8 can be identified from a numerical either theoretical perturbative calculation. For the purpose we use the NNLO results calculated in Refs. [23, 24]. Since the leading order is already (A_{Conf}) void of β-terms we start with NLO coefficients. A general numerical/theoretical calculation keeps tracks of all the color factors and the respective coefficients:

$$B_O(N_f) = C_F\left[C_A B_O^{N_c} + C_F B_O^{C_F} + T_F N_f B_O^{N_f}\right], \tag{11}$$

where $C_F = \frac{(N_c^2-1)}{2N_c}$, $C_A = N_c$ and $T_F = 1/2$. We can determine the conformal coefficient B_{Conf} of the NLO order straightforwardly, by fixing the number of flavors N_f in order to kill the β_0 term:

$$\begin{aligned} B_{Conf} &= B_O\left(N_f \equiv \frac{33}{2}\right), \\ B_{\beta_0} \equiv \log\frac{\mu_0^2}{\mu_I^2} &= 2\frac{B_O - B_{Conf}}{\beta_0 A_{Conf}} \end{aligned} \tag{12}$$

we would achieve the same results in the usual PMC way, i.e. identifying the N_f coefficient with the β_0 term and then determining the conformal coefficient. At the NNLO a general coefficient is made of the contribution of six different color factors:

$$
\begin{aligned}
C_O(N_f) &= \frac{C_F}{4}\left\{N_c^2 C_O^{N_c^2} + C_O^{N_c^0} + \frac{1}{N_c^2} C_O^{\frac{1}{N_c^2}}\right. \\
&\left. + N_f N_c \cdot C_O^{N_f N_c} + \frac{N_f}{N_c} C_O^{N_f/N_c} + N_f^2 C_O^{N_f^2}\right\}. \qquad (13)
\end{aligned}
$$

In order to identify all the terms of Eq.8 we notice first that the coefficients of the terms β_0^2 and β_1 are already given by the NLO coefficient B_{β_0}, then we need to determine only the β_0- and the conformal C_{Conf}-terms. In order to determine the latter coefficients we use the same procedure we used for the NLO , i.e. we set the number of flavors $N_f \equiv 33/2$ in order to drop off all the β_0 terms. We have then:

$$
\begin{aligned}
C_{Conf} &= C_O\left(N_f \equiv \frac{33}{2}\right) - \frac{1}{4}\overline{\beta}_1 B_{\beta_0} A_{Conf}, \\
C_{\beta_0} \equiv \log\left(\frac{\mu_0^2}{\mu_{II}^2}\right) &= \frac{1}{\beta_0 B_{Conf}}\left(C_O - C_{Conf}\right. \\
&\left. - \frac{1}{4}\beta_0^2 B_{\beta_0}^2 A_{Conf} - \frac{1}{4}\beta_1 B_{\beta_0} A_{Conf}\right), \qquad (14)
\end{aligned}
$$

with $\overline{\beta}_1 \equiv \beta_1(N_f = 33/2) = -107$. This procedure can be extended to all orders and one may also decide whether to cancel the β_0, β_1 or β_2 by fixing the appropriate number of flavors. We point out that extending the Intrinsic Conformality to all orders we can predict at this stage the coefficients of all the color factors of the higher orders related to the β-terms except those related to the higher order conformal coefficients and β_0-terms (e.g. at NNNLO the D_{Conf} and D_{β_0}). The β-terms are coefficients that stem from UV-divergent diagrams connected with the running of the coupling constant and not from UV-finite diagrams. UV-finite N_F terms may arise but would not contribute to the β-terms. These terms should be considered as conformal terms.

4 The PMC$_\infty$ renormalization scales

According to the PMC$_\infty$, renormalization scales are set to the intrinsic scales, and Eq.4 becomes:

$$
\frac{1}{\sigma_{tot}} \frac{O d\sigma(\mu_I, \tilde{\mu}_{II}, \mu_0)}{dO} = \left\{\overline{\sigma}_I + \overline{\sigma}_{II} + \overline{\sigma}_{III} + \mathcal{O}(\alpha_s^4)\right\}, \qquad (15)
$$

where the $\overline{\sigma}_N$ are normalized conformal subsets that are given by:

$$
\begin{aligned}
\overline{\sigma}_I &= A_{Conf} \cdot x_I \\
\overline{\sigma}_{II} &= \left(B_{Conf} + \eta A_{\rm tot} A_{Conf}\right) \cdot x_{II}^2 - \eta A_{\rm tot} A_{Conf} \cdot x_0^2 \\
&\quad - A_{\rm tot} A_{Conf} \cdot x_0 x_I \\
\overline{\sigma}_{III} &= \left(C_{Conf} - A_{\rm tot} B_{Conf} - (B_{\rm tot} - A_{\rm tot}^2) A_{Conf}\right) \cdot x_0^3, \qquad (16)
\end{aligned}
$$

where x_I, x_{II}, x_0 are the couplings determined at the $\mu_I, \tilde{\mu}_{II}, \mu_0$ scales respectively.

Normalized conformal subsets for the region $(1-T) > 0.33$ and $C > 0.75$ can be achieved simply by setting $A_{Conf} \equiv 0$ in the Eq. 16. The PMC$_\infty$ scales, μ_N , are given by:

$$\begin{aligned} \mu_I &= \sqrt{s} \cdot e^{f_{sc}-\frac{1}{2}B_{\beta_0}}, \qquad\qquad {\scriptstyle (1-T)<0.33\,,\, C<0.75} \\ \tilde{\mu}_{II} &= \begin{cases} \sqrt{s} \cdot e^{f_{sc}-\frac{1}{2}C_{\beta_0}\cdot \frac{B_{Conf}}{B_{Conf}+\eta\cdot A_{tot}A_{Conf}}}, & \\ & {\scriptstyle (1-T)<0.33\,,\, C<0.75,} \\ \sqrt{s} \cdot e^{f_{sc}-\frac{1}{2}C_{\beta_0}}, & \\ & {\scriptstyle (1-T)>0.33\,,\, C>0.75} \end{cases} \end{aligned} \tag{17}$$

and $\mu_0 = M_{Z_0}$. The renormalization scheme factor for the QCD results is set to $f_{sc} \equiv 0$.

The η parameter is a regularization term in order to cancel the singularities of the NLO scale, μ_{II}, in the range $(1-T) < 0.33$ and $C < 0.75$, depending on non-matching zeroes between numerator and denominator in the C_{β_0}. In general this term is not mandatory for applying the PMC$_\infty$, it is necessary only in case one is interested to apply the method all over the entire range covered by the thrust, or any other observable. Its value has been determined to $\eta = 3.51$ for both thrust and C-parameter distribution and it introduces no bias effects up to the accuracy of the calculations and the related errors are totally negligible up to this stage. The LO and NLO PMC$_\infty$ scales for thrust and C-parameter are shown in Fig.1. The

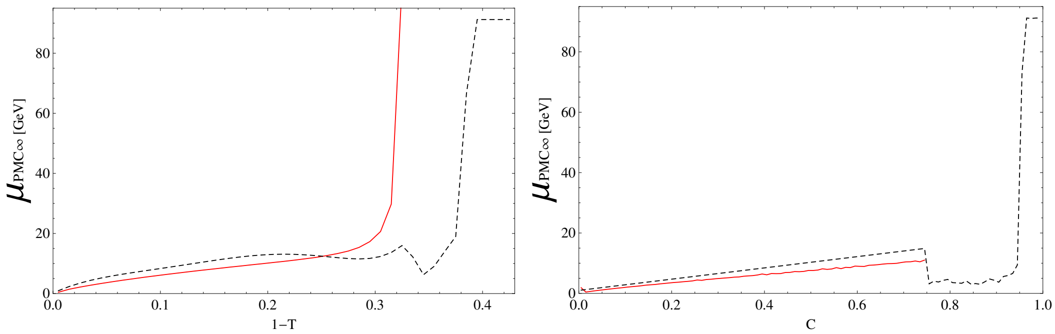

Figure 1: The LO-PMC$_\infty$ (Solid Red) and the NLO-PMC$_\infty$ (Dashed Black) scales for thrust (Left) and C-parameter (Right) (Ref. [13].)

PMC$_\infty$ scales are functions of the center-of mass-energy $\sqrt{s}$ and of the event shape variable. We notice that LO and NLO PMC$_\infty$ scales have similar behaviors in the range $(1-T) < 0.33$ and $C < 0.75$ going to zero at the lower boundary.

5 Comparison of the CSS and PMC$_\infty$ Results

We show in Fig.2 and Fig. 3 results for the thrust and C-parameter with a direct comparison of the PMC$_\infty$ with the the CSS method. In addition we have shown also the results of the first PMC approach used in [39,40] that we indicate as PMC($\mu_{\rm LO}$) extended to the NNLO accuracy. In this approach the last unknown PMC scale $\mu_{\rm NLO}$ of the NLO has been set to the last known PMC scale $\mu_{\rm LO}$ of the LO, while the NNLO scale $\mu_{\rm NNLO}\equiv \mu_0$ has been set to the kinematic scale $\mu_0 \equiv \sqrt{s}$. This analysis has been performed in order to show that the procedure of setting the last unknown scale to the last known one leads to stable and precise results and is consistent with proper PMC method in a wide range of values of the $(1-T)$ and C variable. Using the PMC$_\infty$, average errors in the range $0. < (1-T) < 0.42$ of the thrust improve from $\bar{\delta} \simeq 7.36\%$ to 1.95% and in the range $0 < (C) < 1$ of the C-parameter from $\bar{\delta} \simeq 7.26\%$ to 2.43% from NLO to NNLO respectively. Average errors calculated in different regions of the

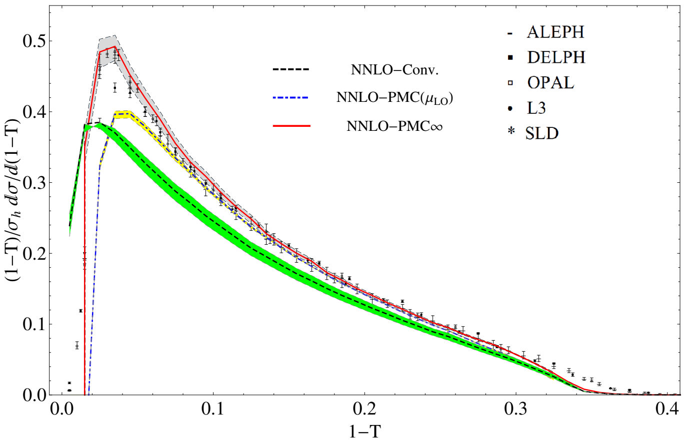

Figure 2: The thrust distribution at NNLO under the Conventional (Dashed Black), the PMC($\mu_{\rm LO}$) (DotDashed Blue) and the PMC$_\infty$ (Solid Red). The experimental data points are taken from the ALEPH, DELPHI,OPAL, L3, SLD experiments [34–38]. The shaded areas show theoretical errors predictions at NNLO (Ref. [13].)

spectrum are reported in Table 1 for thrust and C-parameter. From the comparison with the CSS we notice that the PMC$_\infty$ prescription significantly improves the theoretical predictions. Besides, results are in remarkable agreement with the experimental data in a wider range of values for both the $1-T$ and C variables and they show an improvement of the PMC(μ_{LO}) results when the two-jets and the multi-jets regions are approached, i.e. the region near the lower and the upper boundary respectively. The use of the PMC$_\infty$ approach on perturbative thrust QCD-calculations restores the correct behavior of the thrust distribution in the region $(1-T) > 0.33$ and $C > 0.75$ and this is a clear effect of the iCF property. Comparison with the experimental data has been improved all over the spectrum and the introduction of the N^3LO order correction would improve this comparison especially in the multi-jet region. In the PMC$_\infty$ method theoretical errors are given by the unknown intrinsic conformal scale of the last order of accuracy. We expect this scale not to be significantly different from that of the previous orders. In this particular case, as shown in Eq.16, we have also a dependence on the initial scale $\alpha_s(\mu_0)$ left due to the normalization and to the regularization terms. These errors represent the 12.5% and 1.5% respectively of the whole theoretical errors in the range $0 < (1-T) < 0.42$ and they could be improved by means of a correct normalization.

Table 1: Average error, $\bar{\delta}$, for NNLO Thrust and C-parameter distributions under CSS, PMC($\mu_{\rm LO}$) and PMC$_\infty$ scale settings calculated in different intervals.

$\bar{\delta}$[%]	CSS	PMC($\mu_{\rm LO}$)	PMC$_\infty$
$0.00 < (1-T) < 0.33$	5.34	1.33	1.77
$0.00 < (1-T) < 0.42$	6.00	-	1.95
$0.00 < (C) < 1.00$	6.47	1.55	2.43

6 Conclusion

In this article we have shown results for thrust and C-parameter for $e^+e^- \to 3jets$, comparing the two methods for setting the renormalization scale in pQCD: the Conventional Scale Setting (CSS) and the infinite-order scale setting based on the Principle of Maximum Conformality

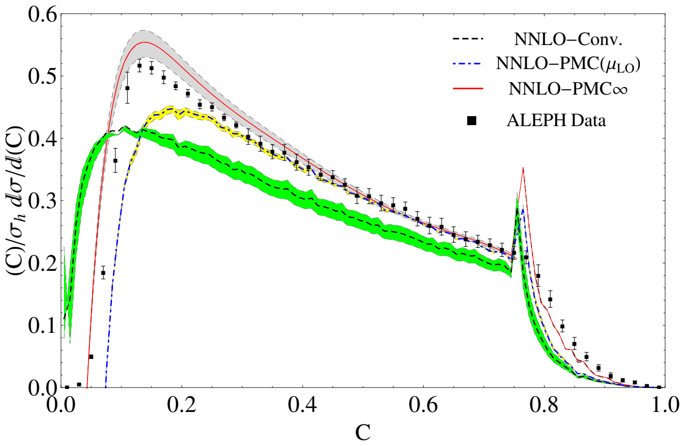

Figure 3: The NNLO C-parameter distribution under the Conventional Scale Setting (Dashed Black), the PMC(μ_{LO}) (DotDashed Blue) and the PMC$_\infty$ (Solid Red). The experimental data points (Black) are taken from the ALEPH experiment [34]. The shaded area shows theoretical errors predictions at NNLO (Ref. [13].)

(PMC$_\infty$). The PMC$_\infty$ method preserves the unique property of the Intrinsic Conformality (iCF). This property leads to a RG invariant parametrization which underlies the *ordered scale invariance*. The PMC$_\infty$ method solves the renormalization scale ambiguity, eliminates the scheme dependence and is consistent with the Gell-Mann and Low scheme in QED. We point out that in fixed order calculations the PMC$_\infty$ last scale is set to the kinematic scale of the process: in this case $\mu_{III} = \sqrt{s} = M_{Z_0}$. As shown in Eq. 9, the scale dependence on the initial scale is totally confined in the last subset σ_n. Thus the the last term in the iCF determines the level of *conformality* reached by the expansion and is entangled with theoretical uncertainties given by higher order uncalculated terms. Any variation of the last scale has to be intended to evaluate theoretical uncertainties given by higher order contributions and not an ambiguity of the PMC$_\infty$ method [41]. Evaluation of the theoretical errors using standard criteria shows that the PMC$_\infty$ significantly improves the precision of the pQCD calculations for thrust and C-parameter. We remark that an improved analysis of theoretical errors might be obtained by giving a prediction on the contributions of higher order terms using a statistical approach as shown in Ref. [42,43]. This would lead to a more rigorous method to evaluate errors and thus to restrict the range of the last PMC$_\infty$ scale that, as we have shown here, can also be fixed to the last known PMC$_\infty$ one leading to precise and stable predictions.

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