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Three-loop soft anomalous dimensions in QCD

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Abstract

I present results for soft anomalous dimensions through three loops for many QCD processes. In particular, I give detailed expressions for soft anomalous dimensions in various processes with electroweak and Higgs bosons as well as single top quarks and top-antitop pairs.

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1 Introduction

The calculation of higher-order soft-gluon corrections in perturbative QCD requires calculations of soft anomalous dimensions, Γ_S, for the corresponding processes [1]. The current state-of-the-art for Γ_S for many processes is three loops. In this paper, I present results for Γ_S for various processes at hadron colliders. These include processes with W, Z, γ, and H bosons, as well as single-top and top-pair production, and $2 \to 3$ processes involving top quarks produced in association with electroweak or Higgs bosons.

Soft-gluon corrections are very important because they are typically large and they dominate the perturbative corrections for a multitude of processes, especially those involving top quarks. We consider partonic processes $p_a + p_b \to p_1 + p_2 + \cdots$ and define $s = (p_a + p_b)^2$, $t = (p_a - p_1)^2$, $u = (p_b - p_1)^2$ and $s_4 = s + t + u - \sum m_i^2$. At partonic threshold $s_4 \to 0$, and the soft corrections at order α_s^n involve logarithmic terms of the form $\ln^k(s_4/M^2)/s_4$, with M a hard scale and $k \leq 2n-1$. In order to resum these soft corrections in the (differential) cross section at NLL, NNLL, and N^{3}LL accuracy, we need to calculate soft anomalous dimensions at, correspondingly, one loop, two loops, and three loops.

If we take transforms of the cross section, with transform variable N, then we can write a factorized expression as

$$\sigma^{ab\to 12\cdots}(N) = \mathrm{tr}\left\{H^{ab\to 12\cdots}\, S^{ab\to 12\cdots}\left(\frac{\sqrt{s}}{N\mu_F}\right)\right\}\psi_a(N_a,\mu_F)\,\psi_b(N_b,\mu_F)\prod J_i(N,\mu_F)\,,$$

where the ψ and J functions describe collinear emission from incoming and outgoing partons, $H^{ab\to 12\cdots}$ is a short-distance hard function, and $S^{ab\to 12\cdots}$ is a soft function which describes soft-gluon emission [1] and which satisfies the renormalization group equation

$$\left(\mu_R \frac{\partial}{\partial \mu_R} + \beta(g_s)\frac{\partial}{\partial g_s}\right) S^{ab\to 12\cdots} = -\Gamma_S^{\dagger\, ab\to 12\cdots}\, S^{ab\to 12\cdots} - S^{ab\to 12\cdots}\, \Gamma_S^{ab\to 12\cdots}\,.$$

The soft anomalous dimension $\Gamma_S^{ab\to 12\cdots}$ controls the evolution of the soft function which gives the exponentiation of logarithms of N in the resummed cross section. For a recent review of soft anomalous dimensions for many QCD processes, see Ref. [2].

2 Cusp anomalous dimension

The cusp anomalous dimension [3–9] is the simplest type of Γ_S and a basic ingredient of calculations for QCD processes. For eikonal lines with momenta p_i and p_j we define the cusp angle $\theta = \cosh^{-1}(p_i \cdot p_j/\sqrt{p_i^2 p_j^2})$. The perturbative series is $\Gamma_{\rm cusp} = \sum_{n=1}^{\infty} (\alpha_s/\pi)^n\, \Gamma_{\rm cusp}^{(n)}$ where at one loop $\Gamma_{\rm cusp}^{(1)} = C_F(\theta \coth\theta - 1)$, at two loops

$$\begin{aligned}
\Gamma_{\rm cusp}^{(2)} \;&=\; K_2\, \Gamma_{\rm cusp}^{(1)} + \frac{1}{2} C_F C_A \left\{ 1 + \zeta_2 + \theta^2 - \coth\theta \left[\zeta_2 \theta + \theta^2 + \frac{\theta^3}{3} + {\rm Li}_2\left(1 - e^{-2\theta}\right)\right] \right. \\
&\left. + \coth^2\theta \left[-\zeta_3 + \zeta_2\theta + \frac{\theta^3}{3} + \theta\, {\rm Li}_2\left(e^{-2\theta}\right) + {\rm Li}_3\left(e^{-2\theta}\right)\right] \right\},
\end{aligned}$$

and at three loops $\Gamma_{\rm cusp}^{(3)} = K_3 \Gamma_{\rm cusp}^{(1)} + 2K_2(\Gamma_{\rm cusp}^{(2)} - K_2\, \Gamma_{\rm cusp}^{(1)}) + C^{(3)}$, where K_3 and $C^{(3)}$ have long expressions (see Refs. [2, 9] for explicit expressions) and $K_2 = C_A(67/36 - \zeta_2/2) - (5/18)n_f$.

In the case of the production of heavy-quark pairs, with mass m, we can also write the above expressions in terms of $\beta = \tanh(\theta/2) = \sqrt{1-(4m^2/s)}$, and denote them by $\Gamma_{\rm cusp}^{(n)\beta}$.

If eikonal line i represents a massive quark and eikonal line j a massless quark, then we have simpler expressions. At one loop $\Gamma_{\rm cusp}^{(1)m_i} = C_F[\ln(2p_i \cdot p_j/(m_i\sqrt{s})) - 1/2]$, at two loops $\Gamma_{\rm cusp}^{(2)m_i} = K_2\, \Gamma_{\rm cusp}^{(1)m_i} + (1/4)C_F C_A(1-\zeta_3)$, and at three loops

$$\Gamma_{\rm cusp}^{(3)m_i} = K_3\, \Gamma_{\rm cusp}^{(1)m_i} + \frac{1}{2} K_2 C_F C_A (1-\zeta_3) + C_F C_A^2 \left(-\frac{1}{4} + \frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5 \right).$$

If both eikonal lines are massless, then $\Gamma_{\rm cusp}^{\rm massless} = C_F \ln(2p_i \cdot p_j/s) \sum_{n=1}^{\infty} (\alpha_s/\pi)^n K_n$.

3 Γ_S for some simple processes

For processes with trivial color structure, the soft anomalous dimension is very simple. In fact Γ_S vanishes for the following: Drell-Yan processes $q\bar{q} \to \gamma^*$, $q\bar{q} \to Z$; W-boson production via $q\bar{q}' \to W^{\pm}$; Higgs production via $b\bar{b} \to H$ and $gg \to H$; electroweak-boson pair production $q\bar{q} \to \gamma\gamma$, $q\bar{q} \to ZZ$, $q\bar{q} \to W^+W^-$; production of two different electroweak bosons $q\bar{q} \to \gamma Z$, $q\bar{q}' \to W^{\pm}\gamma$, $q\bar{q}' \to W^{\pm}Z$; charged Higgs production via $b\bar{b} \to H^-W^+$, $b\bar{b} \to H^+H^-$, $gg \to H^+H^-$.

Also, for Deep Inelastic Scattering (DIS), $lq \to lq$ with subprocess $q\gamma^* \to q$, we have at one loop: $\Gamma_S^{(1)\, q\gamma^*\to q} = C_F \ln(-t/s)$; at two loops: $\Gamma_S^{(2)\, q\gamma^*\to q} = K_2\, C_F \ln(-t/s)$; and at three loops: $\Gamma_S^{(3)\, q\gamma^*\to q} = K_3\, C_F \ln(-t/s)$.

More generally, when all external lines in a process are massless, then $\Gamma_S^{(2)}$ is proportional to $\Gamma_S^{(1)}$ [10], but this is not true for processes with massive lines. Furthermore, at three loops for multi-leg scattering there are contributions from four-parton correlations [11].

4 Γ_S for large-p_T W, Z, γ, H production

Let V denote a W or Z boson or a photon or a Higgs boson. The soft anomalous dimension for these processes is a simple function (not a matrix) [12–14] (see also [2]).

For the processes $qg \to W^{\pm}q'$, $qg \to Zq$, $qg \to \gamma q$, and $bg \to Hb$, we have at one loop: $\Gamma_S^{(1)\, qg\to Vq'} = C_F \ln(-u/s) + (C_A/2)\ln(t/u)$; at two loops: $\Gamma_S^{(2)\, qg\to Vq'} = K_2\, \Gamma_S^{(1)\, qg\to Vq'}$; and at three loops: $\Gamma_S^{(3)\, qg\to Vq'} = K_3\, \Gamma_S^{(1)\, qg\to Vq'}$. The same Γ_S also describes the reverse processes such as $\gamma q \to qg$.

For the processes $q\bar{q}' \to W^{\pm}g$, $q\bar{q} \to Zg$, $q\bar{q} \to \gamma g$, and $b\bar{b} \to Hg$, we have at one loop: $\Gamma_S^{(1)\, q\bar{q}'\to Vg} = (C_A/2)\ln(tu/s^2)$; at two loops: $\Gamma_S^{(2)\, q\bar{q}'\to Vg} = K_2\, \Gamma_S^{(1)\, q\bar{q}'\to Vg}$; and at three loops: $\Gamma_S^{(3)\, q\bar{q}'\to Vg} = K_3\, \Gamma_S^{(1)\, q\bar{q}'\to Vg}$. The same Γ_S also describes the reverse processes such as $\gamma g \to q\bar{q}$.

5 Γ_S for single-top production

We continue with results for single-top production [15–19] (see also [2, 20].

For single-top t-channel production, $\Gamma_S^{bq\to tq'}$ is a 2×2 matrix [15, 18, 19]. Using a t-channel singlet-octet color basis, the matrix elements are at one loop

$$\Gamma_{S\,11}^{(1)bq\to tq'} = C_F\left[\ln\left(\frac{t(t-m_t^2)}{m_t s^{3/2}}\right) - \frac{1}{2}\right], \qquad \Gamma_{S\,12}^{(1)bq\to tq'} = \frac{C_F}{2N_c}\ln\left(\frac{u(u-m_t^2)}{s(s-m_t^2)}\right),$$

$$\Gamma_{S\,21}^{(1)bq\to tq'} = \ln\left(\frac{u(u-m_t^2)}{s(s-m_t^2)}\right), \qquad \Gamma_{S\,22}^{(1)bq\to tq'} = \frac{C_A}{2}\left[\ln\left(\frac{u(u-m_t^2)}{m_t\, s^{3/2}}\right) - \frac{1}{2}\right]$$
$$+\left(C_F - \frac{C_A}{2}\right)\left[\ln\left(\frac{t(t-m_t^2)}{m_t\, s^{3/2}}\right) - \frac{1}{2} + 2\ln\left(\frac{u(u-m_t^2)}{s(s-m_t^2)}\right)\right],$$

at two loops

$$\Gamma_{S\,11}^{(2)bq\to tq'} = K_2\, \Gamma_{S\,11}^{(1)bq\to tq'} + \frac{1}{4}C_F C_A(1-\zeta_3), \qquad \Gamma_{S\,12}^{(2)bq\to tq'} = K_2\, \Gamma_{S\,12}^{(1)bq\to tq'},$$
$$\Gamma_{S\,21}^{(2)bq\to tq'} = K_2\, \Gamma_{S\,21}^{(1)bq\to tq'}, \qquad \Gamma_{S\,22}^{(2)bq\to tq'} = K_2\, \Gamma_{S\,22}^{(1)bq\to tq'} + \frac{1}{4}C_F C_A(1-\zeta_3),$$

and at three loops

$$\Gamma_{S\,11}^{(3)bq\to tq'} = K_3\, \Gamma_{S\,11}^{(1)bq\to tq'} + \frac{1}{2}K_2 C_F C_A(1-\zeta_3) + C_F C_A^2\left(-\frac{1}{4} + \frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5\right),$$
$$\Gamma_{S\,12}^{(3)bq\to tq'} = K_3\, \Gamma_{S\,12}^{(1)bq\to tq'} + X_{12}^{(3)bq\to tq'}, \qquad \Gamma_{S\,21}^{(3)bq\to tq'} = K_3\, \Gamma_{S\,21}^{(1)bq\to tq'} + X_{21}^{(3)bq\to tq'},$$
$$\Gamma_{S\,22}^{(3)bq\to tq'} = K_3\, \Gamma_{S\,22}^{(1)bq\to tq'} + \frac{1}{2}K_2 C_F C_A(1-\zeta_3) + C_F C_A^2\left(-\frac{1}{4} + \frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5\right)$$
$$+ X_{22}^{(3)bq\to tq'},$$

where the $X_{ij}^{(3)bq\to tq'}$ denote unknown terms from four-parton correlations in the last three matrix elements at three loops. It is important to note that due to the color structure of this process, only the first three-loop matrix element, $\Gamma_{S\,11}^{(3)bq\to tq'}$, contributes to the N^{3}LO soft-gluon corrections; therefore, the unknown terms in the other three-loop matrix elements do not pose a problem in deriving N^{3}LO results.

For single-top s-channel production, $\Gamma_S^{q\bar{q}'\to t\bar{b}}$ is also a 2×2 matrix [15, 16, 19]. Using an s-channel singlet-octet color basis, we have at one loop

$$\Gamma_{S\,11}^{(1)q\bar{q}'\to t\bar{b}}=C_F\left[\ln\left(\frac{s-m_t^2}{m_t\sqrt{s}}\right)-\frac{1}{2}\right], \qquad \Gamma_{S\,12}^{(1)q\bar{q}'\to t\bar{b}}=\frac{C_F}{2N_c}\ln\left(\frac{t(t-m_t^2)}{u(u-m_t^2)}\right),$$
$$\Gamma_{S\,21}^{(1)q\bar{q}'\to t\bar{b}}=\ln\left(\frac{t(t-m_t^2)}{u(u-m_t^2)}\right), \qquad \Gamma_{S\,22}^{(1)q\bar{q}'\to t\bar{b}}=\frac{C_A}{2}\left[\ln\left(\frac{t(t-m_t^2)}{m_t\,s^{3/2}}\right)-\frac{1}{2}\right]$$
$$+\left(C_F-\frac{C_A}{2}\right)\left[\ln\left(\frac{s-m_t^2}{m_t\sqrt{s}}\right)-\frac{1}{2}+2\ln\left(\frac{t(t-m_t^2)}{u(u-m_t^2)}\right)\right],$$

at two loops

$$\Gamma_{S\,11}^{(2)q\bar{q}'\to t\bar{b}}=K_2\,\Gamma_{S\,11}^{(1)q\bar{q}'\to t\bar{b}}+\frac{1}{4}C_F C_A(1-\zeta_3), \qquad \Gamma_{S\,12}^{(2)q\bar{q}'\to t\bar{b}}=K_2\,\Gamma_{S\,12}^{(1)q\bar{q}'\to t\bar{b}},$$
$$\Gamma_{S\,21}^{(2)q\bar{q}'\to t\bar{b}}=K_2\,\Gamma_{S\,21}^{(1)q\bar{q}'\to t\bar{b}}, \qquad \Gamma_{S\,22}^{(2)q\bar{q}'\to t\bar{b}}=K_2\,\Gamma_{S\,22}^{(1)q\bar{q}'\to t\bar{b}}+\frac{1}{4}C_F C_A(1-\zeta_3),$$

and at three loops

$$\Gamma_{S\,11}^{(3)q\bar{q}'\to t\bar{b}}=K_3\,\Gamma_{S\,11}^{(1)q\bar{q}'\to t\bar{b}}+\frac{1}{2}K_2 C_F C_A(1-\zeta_3)+C_F C_A^2\left(-\frac{1}{4}+\frac{3}{8}\zeta_2-\frac{\zeta_3}{8}-\frac{3}{8}\zeta_2\zeta_3+\frac{9}{16}\zeta_5\right),$$
$$\Gamma_{S\,12}^{(3)q\bar{q}'\to t\bar{b}}=K_3\,\Gamma_{S\,12}^{(1)q\bar{q}'\to t\bar{b}}+X_{12}^{(3)q\bar{q}'\to t\bar{b}}, \qquad \Gamma_{S\,21}^{(3)q\bar{q}'\to t\bar{b}}=K_3\,\Gamma_{S\,21}^{(1)q\bar{q}'\to t\bar{b}}+X_{21}^{(3)q\bar{q}'\to t\bar{b}},$$
$$\Gamma_{S\,22}^{(3)q\bar{q}'\to t\bar{b}}=K_3\,\Gamma_{S\,22}^{(1)q\bar{q}'\to t\bar{b}}+\frac{1}{2}K_2 C_F C_A(1-\zeta_3)+C_F C_A^2\left(-\frac{1}{4}+\frac{3}{8}\zeta_2-\frac{\zeta_3}{8}-\frac{3}{8}\zeta_2\zeta_3+\frac{9}{16}\zeta_5\right)$$
$$+X_{22}^{(3)q\bar{q}'\to t\bar{b}},$$

where the $X_{ij}^{(3)q\bar{q}'\to t\bar{b}}$ denote unknown terms in the last three matrix elements. Again, we note that only the first three-loop matrix element, $\Gamma_{S\,11}^{(3)q\bar{q}'\to t\bar{b}}$, contributes to the N^{3}LO soft-gluon corrections.

For associated tW production the soft anomalous dimension is a simple function [15, 17, 19]. At one loop

$$\Gamma_S^{(1)bg\to tW}=C_F\left[\ln\left(\frac{m_t^2-t}{m_t\sqrt{s}}\right)-\frac{1}{2}\right]+\frac{C_A}{2}\ln\left(\frac{u-m_t^2}{t-m_t^2}\right),$$

at two loops

$$\Gamma_S^{(2)bg\to tW}=K_2\,\Gamma_S^{(1)bg\to tW}+\frac{1}{4}C_F C_A(1-\zeta_3),$$

and at three loops

$$\Gamma_S^{(3)bg\to tW}=K_3\,\Gamma_S^{(1)bg\to tW}+\frac{1}{2}K_2 C_F C_A(1-\zeta_3)+C_F C_A^2\left(-\frac{1}{4}+\frac{3}{8}\zeta_2-\frac{\zeta_3}{8}-\frac{3}{8}\zeta_2\zeta_3+\frac{9}{16}\zeta_5\right).$$

The same soft anomalous dimension applies for the process $bg\to tH^-$, and for the FCNC processes, via anomalous top-quark couplings, $qg\to tZ$, $qg\to tZ'$, and $qg\to t\gamma$.

6 Γ_S for top-antitop pair production

We continue with soft anomalous dimension matrices for $t\bar{t}$ production [1, 7, 21, 22] (see also [2, 20]).

For top-antitop pair production via the $q\bar{q} \to t\bar{t}$ channel, $\Gamma_S^{q\bar{q}\to t\bar{t}}$ is a 2×2 matrix and we use an s-channel singlet-octet color basis. At one loop for $q\bar{q} \to t\bar{t}$

$$\begin{aligned}
\Gamma_{S\,11}^{(1)q\bar{q}\to t\bar{t}} &= \Gamma_{\rm cusp}^{(1)\beta}\,, \quad \Gamma_{12}^{(1)q\bar{q}\to t\bar{t}} = \frac{C_F}{C_A}\ln\left(\frac{t-m_t^2}{u-m_t^2}\right), \quad \Gamma_{21}^{(1)q\bar{q}\to t\bar{t}} = 2\ln\left(\frac{t-m_t^2}{u-m_t^2}\right), \\
\Gamma_{22}^{(1)q\bar{q}\to t\bar{t}} &= \left(1-\frac{C_A}{2C_F}\right)\Gamma_{\rm cusp}^{(1)} + 4C_F\ln\left(\frac{t-m_t^2}{u-m_t^2}\right) - \frac{C_A}{2}\left[1+\ln\left(\frac{s m_t^2(t-m_t^2)^2}{(u-m_t^2)^4}\right)\right],
\end{aligned}$$

and at two loops

$$\begin{aligned}
&\Gamma_{S\,11}^{(2)q\bar{q}\to t\bar{t}} = \Gamma_{\rm cusp}^{(2)\beta}\,, \quad \Gamma_{12}^{(2)q\bar{q}\to t\bar{t}} = \left(K_2 - C_A N_2^\beta\right)\Gamma_{12}^{(1)q\bar{q}\to t\bar{t}}\,, \quad \Gamma_{21}^{(2)q\bar{q}\to t\bar{t}} = \left(K_2 + C_A N_2^\beta\right)\Gamma_{21}^{(1)q\bar{q}\to t\bar{t}}\,, \\
&\Gamma_{22}^{(2)q\bar{q}\to t\bar{t}} = K_2\Gamma_{22}^{(1)q\bar{q}\to t\bar{t}} + \left(1-\frac{C_A}{2C_F}\right)\left(\Gamma_{\rm cusp}^{(2)\beta} - K_2\Gamma_{\rm cusp}^{(1)\beta}\right) + \frac{1}{4}C_A^2(1-\zeta_3)\,,
\end{aligned}$$

where

$$N_2^\beta = \frac{1}{4}\ln^2\left(\frac{1-\beta}{1+\beta}\right) + \frac{(1+\beta^2)}{8\beta}\left[\zeta_2 - \ln^2\left(\frac{1-\beta}{1+\beta}\right) - \mathrm{Li}_2\left(\frac{4\beta}{(1+\beta)^2}\right)\right].$$

At three loops for $q\bar{q} \to t\bar{t}$ we can write the last matrix element as

$$\begin{aligned}
\Gamma_{S\,22}^{(3)q\bar{q}\to t\bar{t}} &= K_3\,\Gamma_{S\,22}^{(1)q\bar{q}\to t\bar{t}} + \left(1-\frac{C_A}{2C_F}\right)\left(\Gamma_{\rm cusp}^{(3)\beta} - K_3\Gamma_{\rm cusp}^{(1)\beta}\right) + \frac{1}{2}K_2C_A^2(1-\zeta_3) \\
&\quad + C_A^3\left(-\frac{1}{4}+\frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5\right) + X_{22}^{(3)q\bar{q}\to t\bar{t}}\,,
\end{aligned}$$

where $X_{22}^{(3)q\bar{q}\to t\bar{t}}$ denotes unknown three-loop contributions from four-parton correlations. The other matrix elements are also not fully known at three loops, but they have an analogous structure to that at two loops (essentially, replace (2)'s by (3)'s in the superscripts as well as replace K_2's by K_3's, and add X terms for unknown contributions).

For top-antitop pair production via the $gg \to t\bar{t}$ channel, $\Gamma_S^{gg\to t\bar{t}}$ is a 3×3 matrix, and we use a color basis $c_1 = \delta^{ab}\delta_{12}$, $c_2 = d^{abc}T_{12}^c$, $c_3 = if^{abc}T_{12}^c$. We have

$$\Gamma_S^{gg\to t\bar{t}} = \left[\begin{array}{ccc}
\Gamma_{S\,11}^{gg\to t\bar{t}} & 0 & \Gamma_{S\,13}^{gg\to t\bar{t}} \\
0 & \Gamma_{S\,22}^{gg\to t\bar{t}} & \Gamma_{S\,23}^{gg\to t\bar{t}} \\
\Gamma_{S\,31}^{gg\to t\bar{t}} & \Gamma_{S\,32}^{gg\to t\bar{t}} & \Gamma_{S\,22}^{gg\to t\bar{t}}
\end{array}\right].$$

At one loop for $gg \to t\bar{t}$

$$\begin{aligned}
\Gamma_{S\,11}^{(1)gg\to t\bar{t}} &= \Gamma_{\rm cusp}^{(1)\beta}\,, \quad \Gamma_{S\,13}^{(1)gg\to t\bar{t}} = \ln\left(\frac{t-m_t^2}{u-m_t^2}\right), \quad \Gamma_{S\,31}^{(1)gg\to t\bar{t}} = 2\ln\left(\frac{t-m_t^2}{u-m_t^2}\right), \\
\Gamma_{S\,22}^{(1)gg\to t\bar{t}} &= \left(1-\frac{C_A}{2C_F}\right)\Gamma_{\rm cusp}^{(1)\beta} + \frac{C_A}{2}\left[\ln\left(\frac{(t-m_t^2)(u-m_t^2)}{s\,m_t^2}\right) - 1\right], \\
\Gamma_{S\,23}^{(1)gg\to t\bar{t}} &= \frac{C_A}{2}\ln\left(\frac{t-m_t^2}{u-m_t^2}\right), \quad \Gamma_{S\,32}^{(1)gg\to t\bar{t}} = \frac{(N_c^2-4)}{2N_c}\ln\left(\frac{t-m_t^2}{u-m_t^2}\right),
\end{aligned}$$

and at two loops

$$\Gamma_{S\,11}^{(2)gg\to t\bar{t}} = \Gamma_{\rm cusp}^{(2)\beta}, \; \Gamma_{S\,13}^{(2)gg\to t\bar{t}} = \left(K_2 - C_A N_2^\beta\right)\Gamma_{S\,13}^{(1)gg\to t\bar{t}}, \; \Gamma_{S\,31}^{(2)gg\to t\bar{t}} = \left(K_2 + C_A N_2^\beta\right)\Gamma_{S\,31}^{(1)gg\to t\bar{t}},$$
$$\Gamma_{S\,22}^{(2)gg\to t\bar{t}} = K_2\,\Gamma_{S\,22}^{(1)gg\to t\bar{t}} + \left(1 - \frac{C_A}{2C_F}\right)\left(\Gamma_{\rm cusp}^{(2)\beta} - K_2\Gamma_{\rm cusp}^{(1)\beta}\right) + \frac{1}{4}C_A^2(1-\zeta_3),$$
$$\Gamma_{S\,23}^{(2)gg\to t\bar{t}} = K_2\,\Gamma_{S\,23}^{(1)gg\to t\bar{t}}, \quad \Gamma_{S\,32}^{(2)gg\to t\bar{t}} = K_2\,\Gamma_{S\,32}^{(1)gg\to t\bar{t}}.$$

At three loops for $gg \to t\bar{t}$, we can write the 22 matrix element as

$$\begin{aligned}\Gamma_{S\,22}^{(3)gg\to t\bar{t}} = {} & K_3\,\Gamma_{S\,22}^{(1)gg\to t\bar{t}} + \left(1 - \frac{C_A}{2C_F}\right)\left(\Gamma_{\rm cusp}^{(3)\beta} - K_3\Gamma_{\rm cusp}^{(1)\beta}\right) + \frac{1}{2}K_2 C_A^2(1-\zeta_3) \\ & + C_A^3\left(-\frac{1}{4} + \frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5\right) + X_{22}^{(3)gg\to t\bar{t}},\end{aligned}$$

where $X_{22}^{(3)gg\to t\bar{t}}$ denotes unknown three-loop contributions from four-parton correlations. The other matrix elements are, again, also not fully known at three loops, but they have an analogous structure to that at two loops.

7 Γ_S for tqH, tqZ, $tq\gamma$, tqW production

We consider processes $bq \to tq'H$ as well as $bq \to tq'Z$, $bq \to tq'\gamma$, $bq \to tqW^-$, $qq \to tq'W^+$. We use a t-channel singlet-octet color basis, and we further define $s' = (p_1+p_2)^2$, $t' = (p_b-p_2)^2$, $u' = (p_a-p_2)^2$. All these processes have the same soft anomalous dimension matrix [23]. We have at one loop

$$\Gamma_{S\,11}^{(1)\,bq\to tq'H} = C_F\left[\ln\left(\frac{t'(t-m_t^2)}{m_t s^{3/2}}\right) - \frac{1}{2}\right],$$
$$\Gamma_{S\,12}^{(1)\,bq\to tq'H} = \frac{C_F}{2N_c}\ln\left(\frac{u'(u-m_t^2)}{s(s'-m_t^2)}\right), \quad \Gamma_{S\,21}^{(1)\,bq\to tq'H} = \ln\left(\frac{u'(u-m_t^2)}{s(s'-m_t^2)}\right),$$
$$\Gamma_{S\,22}^{(1)\,bq\to tq'H} = C_F\left[\ln\left(\frac{t'(t-m_t^2)}{m_t s^{3/2}}\right) - \frac{1}{2}\right] - \frac{1}{N_c}\ln\left(\frac{u'(u-m_t^2)}{s(s'-m_t^2)}\right) + \frac{N_c}{2}\ln\left(\frac{u'(u-m_t^2)}{t'(t-m_t^2)}\right),$$

at two loops

$$\Gamma_{S\,11}^{(2)\,bq\to tq'H} = K_2\,\Gamma_{S\,11}^{(1)\,bq\to tq'H} + \frac{1}{4}C_F C_A(1-\zeta_3), \quad \Gamma_{S\,12}^{(2)\,bq\to tq'H} = K_2\,\Gamma_{S\,12}^{(1)\,bq\to tq'H},$$
$$\Gamma_{S\,21}^{(2)\,bq\to tq'H} = K_2\,\Gamma_{S\,21}^{(1)\,bq\to tq'H}, \quad \Gamma_{S\,22}^{(2)\,bq\to tq'H} = K_2\,\Gamma_{S\,22}^{(1)\,bq\to tq'H} + \frac{1}{4}C_F C_A(1-\zeta_3),$$

and at three loops

$$\Gamma_{S\,11}^{(3)bq\to tq'H} = K_3\,\Gamma_{S\,11}^{(1)bq\to tq'H} + \frac{1}{2}K_2 C_F C_A(1-\zeta_3) + C_F C_A^2\left(-\frac{1}{4} + \frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5\right),$$
$$\Gamma_{S\,12}^{(3)bq\to tq'H} = K_3\,\Gamma_{S\,12}^{(1)bq\to tq'H} + X_{12}^{(3)bq\to tq'H}, \qquad \Gamma_{S\,21}^{(3)bq\to tq'H} = K_3\,\Gamma_{S\,21}^{(1)bq\to tq'H} + X_{21}^{(3)bq\to tq'H},$$
$$\begin{aligned}\Gamma_{S\,22}^{(3)bq\to tq'H} = {} & K_3\,\Gamma_{S\,22}^{(1)bq\to tq'H} + \frac{1}{2}K_2 C_F C_A(1-\zeta_3) + C_F C_A^2\left(-\frac{1}{4} + \frac{3}{8}\zeta_2 - \frac{\zeta_3}{8} - \frac{3}{8}\zeta_2\zeta_3 + \frac{9}{16}\zeta_5\right) \\ & + X_{22}^{(3)bq\to tq'H},\end{aligned}$$

where the $X_{ij}^{(3)bq\to tq'H}$ denote unknown terms in the last three matrix elements which, however, do not contribute to the soft-gluon corrections at N^{3}LO.

We next consider the processes $q\bar{q}' \to t\bar{b}H$ as well as $q\bar{q}' \to t\bar{b}Z$, $q\bar{q}' \to t\bar{b}\gamma$, $q\bar{q} \to t\bar{b}W^-$, $q\bar{q}' \to t\bar{q}''W^+$, which all have the same soft anomalous dimension matrix [23], and we use an s-channel singlet-octet color basis. We have at one loop

$$\Gamma_{S\,11}^{(1)\,q\bar{q}'\to t\bar{b}H} = C_F\left[\ln\left(\frac{s'-m_t^2}{m_t\sqrt{s}}\right)-\frac{1}{2}\right],$$
$$\Gamma_{S\,12}^{(1)\,q\bar{q}'\to t\bar{b}H} = \frac{C_F}{2N_c}\ln\left(\frac{t'(t-m_t^2)}{u'(u-m_t^2)}\right), \qquad \Gamma_{S\,21}^{(1)\,q\bar{q}'\to t\bar{b}H} = \ln\left(\frac{t'(t-m_t^2)}{u'(u-m_t^2)}\right),$$
$$\Gamma_{S\,22}^{(1)\,q\bar{q}'\to t\bar{b}H} = C_F\left[\ln\left(\frac{s'-m_t^2}{m_t\sqrt{s}}\right)-\frac{1}{2}\right]-\frac{1}{N_c}\ln\left(\frac{t'(t-m_t^2)}{u'(u-m_t^2)}\right)+\frac{N_c}{2}\ln\left(\frac{t'(t-m_t^2)}{s(s'-m_t^2)}\right),$$

at two loops

$$\Gamma_{S\,11}^{(2)\,q\bar{q}'\to t\bar{b}H} = K_2\,\Gamma_{S\,11}^{(1)\,q\bar{q}'\to t\bar{b}H}+\frac{1}{4}C_F C_A(1-\zeta_3), \qquad \Gamma_{S\,12}^{(2)\,q\bar{q}'\to t\bar{b}H} = K_2\,\Gamma_{S\,12}^{(1)\,q\bar{q}'\to t\bar{b}H},$$
$$\Gamma_{S\,21}^{(2)\,q\bar{q}'\to t\bar{b}H} = K_2\,\Gamma_{S\,21}^{(1)\,q\bar{q}'\to t\bar{b}H}, \qquad \Gamma_{S\,22}^{(2)\,q\bar{q}'\to t\bar{b}H} = K_2\,\Gamma_{S\,22}^{(1)\,q\bar{q}'\to t\bar{b}H}+\frac{1}{4}C_F C_A(1-\zeta_3),$$

and at three loops

$$\Gamma_{S\,11}^{(3)q\bar{q}'\to t\bar{b}H} = K_3\,\Gamma_{S\,11}^{(1)q\bar{q}'\to t\bar{b}H}+\frac{1}{2}K_2 C_F C_A(1-\zeta_3)+C_F C_A^2\left(-\frac{1}{4}+\frac{3}{8}\zeta_2-\frac{\zeta_3}{8}-\frac{3}{8}\zeta_2\zeta_3+\frac{9}{16}\zeta_5\right),$$
$$\Gamma_{S\,12}^{(3)q\bar{q}'\to t\bar{b}H} = K_3\,\Gamma_{S\,12}^{(1)q\bar{q}'\to t\bar{b}H}+X_{12}^{(3)q\bar{q}'\to t\bar{b}H}, \qquad \Gamma_{S\,21}^{(3)q\bar{q}'\to t\bar{b}H} = K_3\,\Gamma_{S\,21}^{(1)q\bar{q}'\to t\bar{b}H}+X_{21}^{(3)q\bar{q}'\to t\bar{b}H},$$
$$\Gamma_{S\,22}^{(3)q\bar{q}'\to t\bar{b}H} = K_3\,\Gamma_{S\,22}^{(1)q\bar{q}'\to t\bar{b}H}+\frac{1}{2}K_2 C_F C_A(1-\zeta_3)+C_F C_A^2\left(-\frac{1}{4}+\frac{3}{8}\zeta_2-\frac{\zeta_3}{8}-\frac{3}{8}\zeta_2\zeta_3+\frac{9}{16}\zeta_5\right)$$
$$+X_{22}^{(3)q\bar{q}'\to t\bar{b}H},$$

where the $X_{ij}^{(3)q\bar{q}'\to t\bar{b}H}$ denote unknown terms in the last three matrix elements which, however, do not contribute to the soft-gluon corrections at N^{3}LO.

8 Conclusion

Soft anomalous dimensions are fundamental in describing soft-gluon emission in QCD processes. In this contribution, I presented results for soft anomalous dimensions for many processes through three loops. These results are needed in calculations of high-order corrections.

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