LIST OF MODIFICATIONS

1. The introduction has been rewritten and reorganised. In particular the end of the introduction answer to the referee suggestion

   I suggest a reorganisation of the paper, with a clear introduction, where the problem addressed in the paper is stated, the theoretical approach described, and the main result discussed. In the same way, the paper should contain a proper concluding section with a summary of the results and the perspectives opened by the paper.

2. The presentation of the results of numerical simulations has been revised to answer the question:

   Moreover, the simulations the way they are presented in this paper do not meet the minimal standards to be published in a physics journal. In the present form it almost lacks any quantitative analysis on the presented data, and it is impossible to imagine how the figures are representative of generic/general situations. A much more systematic study is needed to be convincing of anything.

   In particular:

   i) We have detailed better the way simulations have been implemented (see, in particular, the discussion below Eq. (42)). ii) We have performed new simulations, corresponding to different parameters, to investigate the generality of our results (which we confirm). Some of these new data have been included in the resubmitted paper, see for instance the new Fig. 1 which now has been enriched with two insets displaying the effect of changing ζ and τ_1. Other choices of parameters considered, which are not represented in figures, have been mentioned in the text. iii) We discussed better the results at a qualitative level and, when possible, we have given quantitative informations on the data displayed in the figures. For example, we provide now best fits to the slopes of the fluctuation-dissipation plots (see, for instance, at the end of Secs. IIC and IID).

3. Concerning the question *Dynamically, multithermalization can only occur in a limiting sense if the time scales of the baths have an extreme separation and heat flows tend to zero. The authors of course understand this, and mention time separation several times in the paper. Unfortunately however in several places, they give the impression that multithermalization can occur in finite times.*

   we have, at pag. 15, end of section IV.A, added the clarifying statement: It may furthermore multi-thermalize in times that are large but still do not diverge with N.

4. Concering the question:

   One needs to start with finite time scales to achieve stationarization, and in that case heat would flow. The limit of in finite time scale should be *taken next to reduce heat flows to zero. This is very much like in (mean-field) ageing except that there is no multi-bath and the effective temperatures, as well as the in finite time scales emerge spontaneously. Similarities between slow stationarization and ageing were previously investigated by one of the authors. It is not clear what is the novelty here -except maybe to pass from 1RSB to full-RSB-.*

at pag.16, beginning of section V we added the sentence: The appearance of effective temperatures in aging glassy systems, even in the absence of a multibath, has long been known [11, 16]. The real power of the multi-bath appears when we use the fact of multi-thermalization with a slowly evolving bath. This allows us to infer the underlying probability distribution of a system, even in a numerical simulation of a realistic system, as we shall see in Section VIII.

5. Concering the question:

   A related point is the characterisation of the system that can multithermalize. Is it true that the systems that can multithermalize are the ones like the SK model, that develop their own eVective temperatures? What is the role of time-scales and emerging time-reparametrization invariance? And the strength of the coupling between fast and slow variables? In sec. II C it would seem that the multibath could 'impose' its temperature to the system. How general is this? What is the interest in 'imposing' an external temperature?

   at end of pag 13 we added the sentence: Let us anticipate when we expect multithermalization to happen. i) Any system with short timescales in contact with a multibath (with suitably separated timescales) develops the scales that are thermalized with those of the multibath. ii) Systems that do not become stationary (they age) in contact with an ordinary bath, may become stationary in contact with a multibath. iii) However, only if the multibath's temperatures coincide with the natural aging temperatures of the system multithermalization may be achieved with minimal energy transport, as we shall see in the dynamical version of Guerra's scheme in Sec. VII. The possibility that the system synchronizes its timescales with those of the bath so as to make temperatures match, exists if the system has reparametrization invariances [7].

6. Concering the question: *The questions then are (1) multithermalization being only possible in a limiting sense the hypothesis is highly non-trivial, not just an innocent hypotheis as suggested in the text. (2) why to care about the interpolating model in dynamics? (3) why should we be happy if the remainder is positive? Is it only positive for large times? etc.*

   - we added at pag. 20 beginning of section VII

   The multibath measure generated by (13) is the core of the Guerra's interpolation scheme for the SK model [14]. In section II A we showed that a multibath measure can be viewed as a stationary measure for a dynamical system in contact with different thermal baths and widely separated timescales, hence it is natural to look for a dynamical analogous of Guerra interpolation. In this section we investigate this analogy by addressing in particular the following question: is there a dynamical counterpart of the positivity property in Guerra's scheme? We will show that if the system multithermalizes (see section III B for the precise meaning) along the interpolating path then the answer to the previous question is positive thanks to the property of a multi-reversible transformation (described in section V B).

- we changed at pag 23 end of section VII the sentence: and since $\frac{dx}{\tau}$ has by construction a negative sign, the negativity of the Guerra's remainder for dynamical average is obtained in dynamical setting with the assumption of multithermalization. We may now perform thermodynamic integration of the l.h.s., and because of multithermalization we obtain a dynamical version of the Guerra's bound (86).

- we added at pag 23 end of section VII the sentence: The relevance of such positivity in a dynamical setting is still to be understood. On one hand, in the equilibrium picture the positivity of the remainder has provided an excellent guide to search for the rigorous proof of the Parisi solution in the mean field case. On the other hand the dynamical setting described here provides a bridge with experimentally accessible computations and thus makes possible to test the robustness of the positivity property also beyond the assumption of multi-reversible thermalization.

References added:

[7] J. Kurchan, Preprint arXiv:2101.12702

[28] A. C. C. Coolen, R. W. Penney, and D. Sherrington,Phys. Rev. B 48, 16116, (1993)