Dear Prof. Jiang,

Thank you for your interest in our work. We are happy to see you find our work to be "an elegant theory that connects two realms of topological physics". Bellow we provide answers to your remarks.

Sincerely,

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Remark: *I am wondering how such a connection depends microscopic details: for instance, the size of the HOTI, the couplings between the waveguides and the HOTI, the band gap of the HOTI.*

Our response: We start with the question on how the size of the HOTI impacts the observation of the Floquet topological phases simulated by r. We consider a 2D scattering region of length L and width W to which leads are attached as in Fig. 1 of our manuscript. Just like the topological end-modes of first-order topological phases [Phys. Rev. B **100**, 075415], the spatial profile of topological corner states has an exponential decay with the distance from the respective corners [Nature Physics **14**, p. 925–929(2018), Nature Materials **18**, p. 113–120(2019)]. This implies that the splitting in their energies due to the overlap of their wavefunctions decreases exponentially with the system size.

Whether the reflected phase is quantized to π or not depends on this splitting in energies. Under assumption that L is large enough (later we discuss what this means), reducing W will continuously change the phase difference between the incoming and the reflected modes of the lead away from π, to values of ϕ related to reflections from the bulk of the y edge. Finally, the estimate of the minimal value of W required for the detection of quantized π modes of r depends also on how much are the hoppings of the 2D HOTI dimerized. The smaller the ratio of γ_y/λ_y, the smaller W is needed. On the other side, the length L of the system has a different impact on the topology of the reflection matrix. The probability for the particle to be transmitted through a gapped region of finite length decays exponentially with this length. For this reason, longer 2D HOTIs have reflection matrices that are closer to the unitary limit $rr^\dagger - 1 = 0$. For $\gamma_x = \gamma_y = 0.1$ and $\lambda_x = \lambda_y = 1$, our numerical simulations reveal that size $L = 10$ and $W = 10$ (5×5 unit cells) is sufficient to get a unitary r with π-modes. Here, the deviation from unitarity of r is of the order 10^{-10}, while boundary modes differ from $\phi = \pi$ by less than 10^{-3}.

Next, we answer the question of the band gap of HOTIs. For our discussion, it is important to distinguish between bulk, x edge and y edge band gaps in these systems. First, the vanishing bulk band gap of a 2D system will lead to a subunitary reflection matrix that cannot then be interpreted as a Floquet operator. This is because some incoming modes of one lead will get transmitted to the other lead. If the bulk is gapped but the x edge is gapless, the reflection matrix will again be subunitary ($\det r = 0$) due to the nonzero edge conductance between the leads. Finally, closing the band gap along the y edge keeps the unitarity of the reflection matrix, see our Appendix E.

In fact, the question of the band gap is very much related to the system size for the following reason. The two-terminal electrical conductance can be written as

$G = G_0 \text{tr}[\text{tt}^\dagger]$, where $G_0 = 2e^2/\hbar$. In an insulator, $\text{tr}[\text{tt}^\dagger] \propto \exp{[-\text{L}/\xi]}$, where ξ is the localization length. This length scales with the bandgap Δ as $\xi \propto \hbar v_F/\Delta$ [B. A. Bernevig, *Topological insulators and topological superconductors*, Princeton University Press, 2013]. At the phase transition point where $\Delta \to 0$, the localization length diverges i.e., $\text{tr}[\text{tt}^\dagger] \to 1$. This implies a subunitary r according to the relation $rr^\dagger = 1 - tt^\dagger$, obtained from the unitary constraint on the scattering matrix. Therefore, both the bulk band gap and the length L of the HOTI contribute to producing a unitary reflection matrix.

Finally, we comment on the importance of the system-lead coupling strength t_{ls}. In our numerical simulations, we find that the size of the topological gaps decrease with reducing t_{ls}. This however does not influence the stability of topological π-modes and 0-modes (provided that $t_{ls} \neq 0$, because otherwise the lead is not attached to the system). The reason is that their localization length depends only on the parameters of the scattering region. That this is case can be seen from Fig. 2 of our manuscript, where π-modes and 0-modes have the same spatial profile no matter what is the size of the topological gap that protects them.

Remark: *In addition, there are various 2D HOTIs, how do they map to 1D Floquet topological insulators in various phases?*

Our response: In Appendix C of our manuscript we explain the dimensional reduction map between HOTIs and Floquet topological phases in more detail. We show that the connection between these two types of topology is universal, since it applies to HOTIs in any dimension and symmetry class, provided that their corner states are robust against lattice symmetry breaking.