This short note is a response to the following referee comment and shows that the capacity of entanglement *does* vanish even if the groundstate is degenerate:

After (3.49) the authors say that if "$\beta_2 \to \infty, \rho_2$ reduces to the ground state, and the relative entropy variance vanishes (along with $C(\beta_2) \to 0$)" This is only true if the ground state of the Hamiltonian is non-degenerate.

1 Thermal state capacity for degenerate ground states

Consider a quantum system with Hilbert space dimension D and with energy eigenvalues E_n whose Hamiltonian takes the form

$$H = \sum_{n=0}^{N} E_n P_n \tag{1.1}$$

where P_n is the orthogonal projector onto the eigensubspace of eigenvalue E_n with $E_0 < E_1 < \ldots < E_N$. They satisfy

$$P_n P_m = \delta_{nm}, \quad P_n^\dagger = P_n. \tag{1.2}$$

Notice that the groundstate E_0 is degenerate. We denote the orthogonal set of eigenvectors corresponding to the eigenvalue E_n by $\vec{v}_{n,i}$ with $i = 1, \ldots, d_n$. (Note that $\sum_{i=1}^N d_i = D$). Let A_n be the $D \times d_n$ matrix with each column being an eigenvector:

$$A_n = \begin{pmatrix} \vec{v}_{n,1} & \cdots & \vec{v}_{n,d_n} \end{pmatrix}. \tag{1.3}$$

Then the projectors can be written as

$$P_n = A_n A_n^\dagger. \tag{1.4}$$

Consider now a thermal state

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}} \tag{1.5}$$

and the corresponding capacity of entanglement

$$C(\rho) = \langle (\log \rho)^2 \rangle_\rho - \langle \log \rho \rangle_\rho^2 \tag{1.6}$$

which can be written as thermal heat capacity

$$C(\rho) = \beta^2 \left[\mathrm{Tr}\,(\rho H^2) - (\mathrm{Tr}\,(\rho H))^2 \right]. \tag{1.7}$$

We can write

$$\rho = \frac{1}{Z} \sum_{n=0}^{N} P_n\, e^{-\beta E_n}, \quad Z = \sum_{n=0}^{N} d_n\, e^{-\beta E_n} \tag{1.8}$$

where we used $\mathrm{Tr}\, P_n = \mathrm{rank}\, A_n = d_n$. Substituting gives

$$C(\rho) = \beta^2 \left[\frac{\sum_{n=0}^{\infty} E_n^2\, d_n\, e^{-\beta E_n}}{\sum_{n=0}^{N} d_n\, e^{-\beta E_n}} - \left(\frac{\sum_{n=0}^{\infty} E_n\, d_n\, e^{-\beta E_n}}{\sum_{n=0}^{N} d_n\, e^{-\beta E_n}} \right)^2 \right] \tag{1.9}$$

which can be written as

$$C(\rho) = \beta^2 \left[\frac{E_0^2 + \sum_{n=1}^{N} E_n^2\, (d_n/d_0)\, e^{-\beta(E_n - E_0)}}{1 + \sum_{n=1}^{N} (d_n/d_0)\, e^{-\beta(E_n - E_0)}} - \left(\frac{E_0 + \sum_{n=1}^{N} E_n\, (d_n/d_0)\, e^{-\beta(E_n - E_0)}}{1 + \sum_{n=1}^{N} (d_n/d_0)\, e^{-\beta(E_n - E_0)}} \right)^2 \right]. \tag{1.10}$$

In the zero temperature limit $\beta \to \infty$, we can expand it as

$$C(\rho) = \beta^2 \left[E_0^2 + \mathcal{O}(e^{-\beta(E_1 - E_0)}) - \left(E_0 + \mathcal{O}(e^{-\beta(E_1 - E_0)}) \right)^2 \right] \tag{1.11}$$

which becomes

$$C(\rho) = \beta^2(E_0^2 - E_0^2) + \mathcal{O}(\beta^2 e^{-\beta(E_1 - E_0)}). \tag{1.12}$$

Since $\lim_{\beta\to\infty} \beta^2 e^{-\beta x} = 0$ for $x > 0$, we get

$$\lim_{\beta\to\infty} C(\rho) = 0. \tag{1.13}$$

Intuitively capacity measures the variation of energy due to thermal fluctuations, but at zero temperature, all such fluctuations vanish (even if the ground state is degenerate).