

II. REPLY TO REFEREE 2

- *“The authors provide an alternative method to derive the density of states in 1d lattices when the hopping amplitudes and the on-site potentials are position dependent. The method employ a combination of Bloch and Wannier functions by decimating the lattice into pieces (with equal number of sites in each) and then using a quasi-momentum within the small chain (Bloch-type) and localized orbitals for each of the smaller chains (Wannier). Then the Hamiltonian is rewritten using the new decimation and labelling and subsequently the authors carefully do perturbation theory to extract the energy spectrum and the DOS in terms of the hopping parameters that are position dependent, the local chemical potential and their derivatives (as well as the quasi-momentum) . I find the work very interesting, timely and the paper well-organised and clearly written. It should be published in SciPost after some minor considerations.”*

We thank the referee for their report and comments which have helped us to improve the clarity and accessibility of our paper. In the following we address all the points raised by the referee.

- 1. *“It is straightforward to add disorder into a perfectly translational invariant system. This approach can provide some light into such problems. The authors could comment on that as it is an extension within the limits of the theory. Can the same integral as their result of the DOS obtained by real space Green’s functions methods?”*

We thank the referee for pointing out the possible connection of our work to the study of disorder. The calculation of the matrix elements in our partial Wannier decomposition relies on a Taylor expansion of the hopping as a function of real space, and therefore on the fact that it changes continuously. It therefore seems difficult at first sight to generalize our results to disordered systems with abrupt variations in the potential and/or hopping profile. Having said that, we do appreciate that a smooth yet random profile for $\mu(x)$ or $t(x)$ can be treated within our approach. We agree with the referee that the question of how such smooth disorder influences the spectral and/or transport features of the system is interesting. It is beyond the scope of the current manuscript however, and we will reserve it for future study.

The question about Green’s functions method was also raised by the Referee 1, and we repeat our reply: It is indeed possible to re-formulate our method in terms of Green’s functions. We can begin by writing the Green’s function in the partial Wannier basis and then apply perturbation theory to that, which yields

$$G_{m_c, \vartheta; m'_c, \vartheta'} = G_{m_c, \vartheta}^0 \delta_{m_c m'_c} \delta_{\vartheta \vartheta'} + G_{m_c, \vartheta}^0 \langle m_c, \vartheta | \mathcal{H}_1 | m'_c, \vartheta' \rangle G_{m'_c, \vartheta'}^0 + \dots \quad (3)$$

The lowest order Green’s function is diagonal in the PWF basis where $G_{m_c, \vartheta}^0 = (\omega + i0^+ - E_{m_c, \vartheta}^{(0)})^{-1}$. Therefore, using the basic relation of DOS in terms of Green’s function i.e.

$$D(\omega) = \frac{1}{\pi} \Im \text{Tr} \hat{G} = \frac{1}{\pi} \sum_{m_c, \vartheta} \Im G_{m_c, \vartheta; m_c, \vartheta} \quad (4)$$

in the lowest order, the Eqs. (2) or equivalently (18) of the main text can be obtained again.

- 2. *“In 2D there are more recent studies and classifications of Van-Hove singularities and real systems that exhibit this physics, probably the authors should comment on those and connect their work (e.g. PRRResearch 2, 013355 (2020), PRB 101, 125120 (2020), PRL 123, 207202 (2019)).”*

We thank the referee for bringing these relevant papers to our attention. We have included the references and an additional sentence in the introduction.

- 3. *“What are the physical requirements for a smooth $t(x)$ and $\mu(x)$? Can the authors envisage a possible experiment that can test the results? I believe this would strengthen the work considerably.”*

We assume the smoothness is defined by small variations on the atomic scale, namely, $|t(x_n) - t(x_{n-1})| \ll t(x_n)$ for the hoppings. In solids, the smoothness is then guaranteed by the absence of stacking faults, vacancies, or strong disorder. We should emphasize that even in the presence of few isolated points with abrupt changes or even discontinuous profiles, we can still rely on our perturbative approach. The only possible modification would be to add

the contribution of bound states trapped at the points of abrupt change, which may result in the addition of a few sharp localized peaks in the DOS.

We thank the referee for mentioning possible experimental tests of our result, and include a new paragraph in the conclusions section addressing this point. There are various ways to realize position-dependent hoppings in experimental settings, including local engineering of the chemical structure, position-dependent doping, and of course inhomogeneous strain. In addition, the effective hopping parameters can be easily implemented in various metamaterial settings (photonic, acoustic, topoelectric, etc.). In all these systems, the total DOS can be probed by transport measurements, while the local crystal structure and the hopping parameters can be extracted using STM or other local probes.