1. In the abstract, the authors should try to highlight the system they investigate. I understand it is a spin- triplet superconductor, but what type of system it is? Otherwise, it remains unclear.

=> We thank the referee for the valuable comment. The systems of interest in our work are thin films of equal-spin pairing spin-triplet superconductors with spin degeneracy. We have revised the pertinent part of the abstract as follows:

> By considering equal-spin-pairing spin-triplet superconductors with bulk spin degeneracy, we show how all the transitions can be characterized by the relation between the voltage drop and the spin-polarized current bias.

2. In the first part of the introduction, the authors highlight spin-triplet superconductors. Here, I would recommend to include into the discussion other spin-triplet superconductors such as those due to spin-orbit coupling. See for instance:
Phys. Rev. Lett. 87, 037004 (2001)
Phys. Rev. B 92, 134512 (2015)
Phys. Rev. B 98, 075425 (2018)

=> We thank the referee for the valuable comment. We have revised the introduction as follows:

> Recent years have seen newer candidates for spin-triplet superconductors in uranium-based materials [5–9], doped topological insulator [10,11], spin-orbit-coupled materials [12–14], and magic angle graphene [15–20], notwithstanding controversies concerning older candidate Sr2RuO4 [3,21,22].

We have also cited the mentioned papers as Refs. [12-14].

3. In the second paragraph of the introduction, it is not clear to me what is the motivation of the authors for carrying out this study. In particular, I recommend the authors to expand the discussion on how a two component condensate in 2D, with multiple BKT transitions involving charge and spin degrees of freedom, “naturally” raises the question of robustness of spin transport.

=> We thank the referee for the valuable comment. We have expanded the pertinent part of the introduction as follows:

> The multifaceted aspect of spin-triplet superconductivity also gives rise to additional complexity in fluctuations. An especially telling case would be the spin-triplet superconductor in two dimensions (2D) with easy-plane anisotropy, which, as we shall show, has the like-spin

pairing when the spin quantization axis is perpendicular. Such two component condensates in 2D would allow multiple types of Berezinskii-Kosterlitz-Thouless (BKT) phase transitions [28,29] involving both the charge and the spin degrees of freedom. While the effects of vortex fluctuations on charge transport in superconductors have now been understood well [53], the effects of the critical fluctuations on charge and spin transport in spin-triplet superconductors have not been studied yet. In particular, since spin-triplet superconductors have been proposed as efficient spin-transport medium in superconducting spintronics, it would be crucial to investigate the robustness of superfluid spin transport in 2D spin-triplet superconductors for realizing superconductor-based spintronics with minimal dissipation [30]. Previously, we have shown that quasi-long range ordering is sufficient for spin superfluid transport in the 2D XY magnets [31]. Whether the same robustness exists for spin transport in spin-triplet superconductors has remained an open question.

In this Letter, we study the effects of critical fluctuations associated with three distinct types of vortices—conventional vortex, **d**-vector meron, and fractional vortex—on charge and spin transport in equal-spin-pairing spin-triplet superconductors. Specifically, we show that superfluid spin transport of spin-triplet superconductors at finite temperatures fundamentally differs from that of XY magnets due to the existence of fractional vortices, which are topological defects intertwining charge and spin currents. More specifically, we show that the fractional vortex sets an upper bound to spin current, and this upper bound decreases algebraically with distance. Our results indicate the possibility of transport detection of fractional vortices in spin-triplet superconductors. Also, the identified vulnerability of superfluid spin transport to topological defects calls for further investigations of fluctuation effects on promised superconductor-based spintronic devices.

4. In the introduction, I recommend to put in a single paragraph the text starting with "In this Letter ....". This will help the readers to spot what is done in the manuscript. I also recommend here to briefly say what is the system the authors study.

=> We thank the referee for the comment. We have revised the introduction such that the last paragraph starts with "In this Letter"; Please see our response above for the details of the revision.

5. In several instances of the work, I found that the authors just directly write down mathematical expressions. While at some point it is fine, I think it would help the reader if the authors include a sort of derivation or guide on how to obtain those expressions. This includes equations before (2), (2), (4), (5), (6). The authors should guide the reader.

=> We appreciate the critical comment. We have revised the sentences of all the above equations to provide guidance on how they were derived. To indicate the physical meaning of the first term of Eq. (2), a phrase has been added just after Eq. (2):

> where the first term arises from the transverse Magnus force that external current applies on vortices

To show how Eq. (3) arises from the combination of Eq. (2) and the assumption of the purely longitudinal vortex mobility, a phrase has been added just before Eq. (3):

> *i.e.* the vortex current density is obtained by multiplying the vortex mobility μ to the transverse Magnus force that can be derived from the first term of Eq. (2)

To indicate how Eq. (4) is derived from Eq. (2), the following has been added just before Eq. (4):

> for the vortex pair energy, rather than increasing monotonically with distance, reaches its maximum when the inverse distance pair attraction is at equilibrium with the Magnus force applied by the external current, as can be derived from Eq. (2). The energy barrier per vortex would be half of this maximum, which, for weak external current, is

To provide a better clarity for Eq. (6), a step has been added to Eq. (6):

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{J}_{\mathrm{sp}}\right)=\frac{1}{\tau}\boldsymbol{\nabla}s_z=2\pi\frac{K_{\mathrm{sp}}}{\tau v_{\mathrm{sp}}{}^{2}}\mathbf{j}_{\mathrm{sp}}\times\hat{\boldsymbol{z}}\,,$$

The guidance to Eq. (5) is provided by the revision made in response to the next query.

6. The authors refer to Eq.(5) a the Josephson relation. However, most of the community is familiar with the 1t and 2nd Josephson relations involving the superconducting phase. I think the authors refer here to the Josephson energy. Please clarify.

=> We appreciate the critical comment. Equation (5) is the second Josephson relation. The same equation has been invoked in other publications, e.g., Eq.

(S3) of [Breznay, Steiner, Kivelson, and Kapitulnik, Proc. Natl. Acad. Sci 113, 280 (2016)], which we have newly cited in the revised manuscript. To avoid a confusion, we have revised the pertinent part as follows:

> The electric field E induced by the charge vortex current density j_C can be derived from the second Josephson relation,
>
> $$\boldsymbol{E} = 2\pi \frac{\hbar}{2e} \hat{\mathbf{z}} \times \mathbf{j}_\mathrm{c} = \frac{\pi\hbar}{e} \hat{\mathbf{z}} \times \sum_{m^c, m^{sp}} m^c \mathbf{j}(m^c, m^{sp}) \,,$$
>
> indicating that the qualitative change of current-voltage relation across the BKT transition represents the corresponding change of the charge vorticity current.

Note that we have changed the symbol for the electric field from $\boldsymbol{E}_J$ to $\boldsymbol{E}$ to avoid its misinterpretation as the Josephson energy that is commonly denoted by E_J in literature. We have also added the derivation of Eq. (5) as Appendix A, which can be found in the revised manuscript.

7. In section 3, the authors use a junction setup to get signatures of each BKT transition. As mentioned before, the authors write down expressions Eqs.(7-9) but I believe they need to make it simpler for the reader. My concern is that the authors should try to make the paper self-contained and should guide the reader in the derivation of these expressions without implying to write lengthy and complicated equations.

=> We appreciate the critical comment. For the sentence of Eq. (7), we have revised its beginning to

> The logical starting point for deriving the bulk current-voltage relation which will be measured in the setup of Fig. 3 is

to state the motivation for Eq. (7) and its ending to

> where the second Josephson relation is modified by the voltage drop at the contact.

to indicate how Eq. (7) was derived. For the sentence of Eq. (8), we have revised its beginning to

> Adding and subtracting two spin components of Eq. (7) gives us the spin current boundary condition,

added a step to Eq. (8):

$$I_{l,r}^{\mathrm{sp}} = \frac{\hbar}{2e} p_{l,r} I \pm \left(1 - {p_{l,r}}^2\right) \frac{\hbar^2}{4e^2} \partial_t \alpha_{l,r} = \frac{\hbar}{2e} p_{l,r} I \pm \left(1 - {p_{l,r}}^2\right) \tilde{g}_{l,r} \frac{d}{dx} I_{l,r}^{\mathrm{sp}} \,,$$

to better clarify Eq. (8), and revised its ending by inserting in phrases that were originally in the sentence of Eq. (9)

> and we have used …; note that, as shown in Fig. 3, … for the left / right lead.

to further clarify Eq. (8). For the sentence of Eq. (9), we have revised its beginning to

> Again by adding and subtracting Eq. (7), we obtain

to indicate how Eq. (9) was derived.

8. In relation to the results, the authors only provide expressions. While it is nice to have expressions, I believe the authors need to make some representative plots showing their main results in subsections 3.2, 3.3, and 3.4.

=> We appreciate the referee's valuable comment. By following the suggestion of the referee, we have added Fig. 4 which schematically illustrates our main results of the subsections 3.2, 3.3, and 3.4, i.e., the change of the length dependence of the magnetoresistance across three different types of BKT transitions. See the revised manuscript for the added figure.

9. It would be very helpful for the reader if the authors include some additional physical discussion of their results. For instance, why the authors look at the magnetoresistance voltages, etc.

=> We thank the referee for the critical comment. By following the referee's suggestion, we have added the following sentences in the revised manuscript:

> The proposed setup is referred to as a spin-valve structure, which allows for electrical measurement of spin transport and thus has been widely used in spintronics to study spin-dependent transport properties of magnetic metals. In this work, we are interested in the changes of charge and spin transport across three types of BKT transitions, and these can be detected through spin-dependent current-voltage relation in the proposed setup as will be detailed below.