A question which the authors may want to address, to help me out of my confusion. Figures 2c and 2d show a topological change in the Fermi surface before and after the gap closing transition. It is argued that an equivalent topological transition appears in the density of states as a function of frequency, where figure 2c corresponds to figure 2a and figure 2d to 2b. I am unsure how this comparison works.

Figure 2c is at a fixed value of the parameter ζ<1, say ζ=1/2, while figure 2d is for a fixed ζ>1, say ζ=3/2 and these two figures are indeed topologically distinct. Now to compare with figures 2a and 2b I would look at the curves N(ω) versus ω at ζ=1/2 and ζ=3/2. These two curves are topologically identical, both show a cusp at ω=0. So in what sense does the topological distinction of figures 2c,2d carry over to figures 2a,2b?

I ask because this topological correspondence is the central point of the paper. Basically I don't see how the 2D surfaces of figures 2c,2d correspond to 1D cuts in figures 2a,2b.

I notice that in an earlier version of this manuscript on arXiv:2105.01934v1 there was indeed a comparison of surfaces at fixed ζ. In this version figures 2a,b contain ζ on one of the axes, so the comparison is to a surface cut and then I don't see a topological distinction.

Thank you so much for the valuable comment. You have raised a crucial question.

First, we would like to note that we have been excited and surprised to reveal that known for a long-time (more than 60 years) transition separating gap and gapless phases (Abrikosov-Gor'kov transition) of superconductivity is of the Lifshitz type, i.e, it can possess the topological nature. To the best of our knowledge there is no evidence of this fact in the literature.

Our conclusions were based on the two facts:

i) the presence of the characteristic discontinuity of the third derivative of the free energy in respect to ζ at the point $\zeta = 1$ with the square root singularity from the gapless side;

ii) the density of states has a typical cusp $\sqrt{\zeta - 1}$ at ω=0.

These two features are completely analogous to those ones at the Lifshitz 2½ order phase transition that allows to assume the topological nature of Abrikosov-Gork'kov transition.

To prove our assumption and to find the corresponding topological invariant for such kind of transition, we compared it to the standard Lifshitz transition of the neck disruption type, where the number of the components of topological connectivity changes. In the case of the transition between the gap and gapless phases we found that some nontrivial topological changes occur which can be attributed to it.

Since the Fermi surface is irrelevant for the transition under consideration, we addressed to the characteristics evidently related to such a transition: the DOS as the function of energy (ω), gap in absence of impurities (Δ_0), and the parameter which governs the transition (ζ). Namely, we used the expressions

$$N_s(\omega) = N(0)\zeta^{-1}\,\mathrm{Im}\,u,$$

$$\frac{\omega}{\Delta} = u\left(1 - \frac{\zeta}{\sqrt{1-u^2}}\right)$$

and

$$\ln\left(\frac{\Delta}{\Delta_0}\right)=\begin{cases}-\frac{\pi}{4}\zeta,\ \zeta\le 1\\ -\operatorname{arccosh}\zeta-\frac{1}{2}\left(\zeta\arcsin\zeta^{-1}-\sqrt{1-\zeta^{-2}}\right),\ \zeta>1,\end{cases}$$

The analysis of the corresponding surfaces in the $\{N,\omega,\Delta_0\}$ space for different values of $\zeta \gtrless 1$ evidently demonstrated that the transition is accompanied by the alteration the Euler characteristic[1] for the circumscribed polyhedron $N(\omega,\Delta_0)$ (see our ArXiv publication, version 2 and figure 1 here).

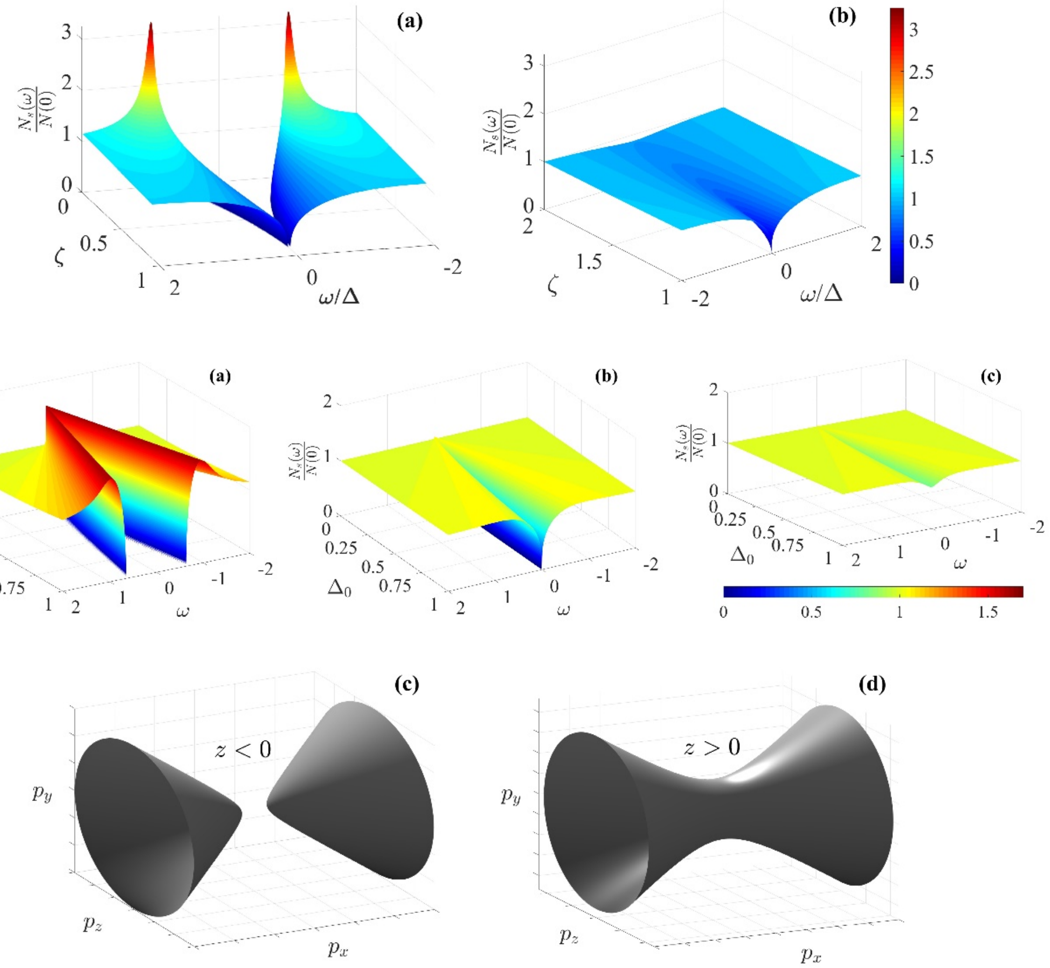

Fig. 1. Top row: Topological evolution of the DOS in the ω-ζ space from the gap state with ζ < 1 (a) to the gapless state with ζ ≥ 1 (b). Middle row: Topological evolution of the DOS in the ω -Δ_0 space from the gap state with ζ = 0.1 (a) through the state where ζ = 1 and the collapse of the energy gap occurs (b) to the gapless state with ζ = 1.75 (c). Bottom row: topological behaviour of the Fermi surface in the momentum space (c,d) for the Lifshitz transition.

[1] In the modern interpretation, the Lifshitz transition can be considered as the alteration of the topological charge in the momentum space, calculated by means of the line, 2D or 3D integration of the expression with a Green function (see details in the review of Volovik https://iopscience.iop.org/article/10.3367/UFNe.2017.01.038218 or Ref. 13 in our manuscript).

We presented the corresponding results at the conference “Superconducting Hybrids at Extreme” and in result of discussions with the colleagues we turned to the idea to proceed with our analysis in the space N, ω, ζ at fixed Δ_0. We want to emphasize that the obtained surfaces in the domain $0 < \zeta < 1$ and the half-space $\zeta > 1$ also have different Euler characteristics, yet your criticism means that the chosen form of presentation is not the optimal. Probably it worth to present in the present version of the article all three sets of figures: $N(\omega, \Delta_0)$, $N(\omega, \zeta)$ with the Lifshitz “neck disruption” transition (see figure 1) to ensure the topological nature of the phase transition.

Regarding the comment that in our current representation that these two curves are topologically identical let’s look at the cross-section of DOS surfaces along the zeta-axis in Fig. 2 (a) and (b) in the manuscript (see Fig. 2 here).

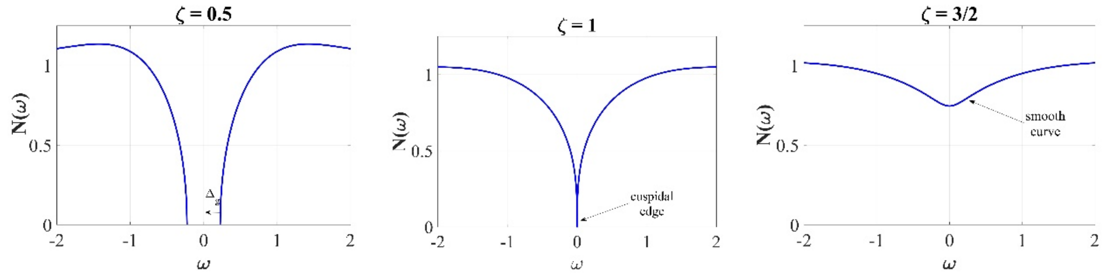

Fig. 2. The evolution of cross-sections along the ζ-axis of Fig. 2 (a) and (b) in the present version of the manuscript.

As one can see from the left-side figure until $\zeta < 1$ the DOS curve clearly shows the gap (like a gap between two sheets of the Fermi surface) without the emergence of the cuspidal point. The cuspidal point occurs only for $\zeta = 1$ (middle figure) that formally corresponds to the merge of the Fermi surface (the Dirac line when z=0), while for $\zeta > 1$ we observe the smooth curve only (right-side figure).

The last point of the comment: in the initial version of this manuscript on arXiv:2105.01934v1 our attempt to provide a topological background was incorrect. According to Eq. (10) in the present version of the manuscript (see also the same equation above for the order parameter), the order parameter Δ is the function of ζ, and thus at fixed ζ the density of states depends only on single variable omega. There is no surface, and therefore there is no topology related to the surface.