Label	Character	h	Label	Character	h	Label	Character	h
φ_0	$\frac{1}{2}\chi_{0,2}+\frac{1}{2}\theta_{0,\frac{1}{2}}$	0	φ_3	$\frac{1}{2}\chi_{2,2}$	$\frac{1}{2}$	φ_6	$\frac{1}{2}\theta_{\frac{1}{4},0}+\frac{1}{2}\theta_{\frac{1}{4},\frac{1}{2}}$	$\frac{1}{16}$
φ_1	$\frac{1}{2}\chi_{0,2}-\frac{1}{2}\theta_{0,\frac{1}{2}}$	1	φ_4	$\chi_{1,2}$	$\frac{1}{8}$	φ_7	$\frac{1}{2}\theta_{\frac{1}{4},0}-\frac{1}{2}\theta_{\frac{1}{4},\frac{1}{2}}$	$\frac{9}{16}$
φ_2	$\frac{1}{2}\chi_{2,2}$	$\frac{1}{2}$	φ_5	$\frac{1}{2}\theta_{\frac{1}{4},0}+\frac{1}{2}\theta_{\frac{1}{4},\frac{1}{2}}$	$\frac{1}{16}$	φ_8	$\frac{1}{2}\theta_{\frac{1}{4},0}-\frac{1}{2}\theta_{\frac{1}{4},\frac{1}{2}}$	$\frac{9}{16}$

Table 1: Primaries of an orbifold theory $U(1)_4/\mathbb{Z}_2$.

In this note, we present an example where the presence of the conserved currents of spin $\frac{3}{2}$ does not guarantee supersymmetry.

To this end, we apply the Jordan-Wigner transformation to a tensor product of two $c=1$ RCFTs, $\mathcal{B}=(U(1)_4/\mathbb{Z}_2)^2$. We show that the corresponding fermionic theory $\mathcal{F}$ has a conserved current of spin 3/2 but non-constant Ramond-Ramond partition function. Furthermore, the SUSY unitarity bound is violated in $\mathcal{F}$.

Let us start with a brief review on $U(1)_4/\mathbb{Z}_2$: It is a $\mathbb{Z}_2$ orbifold of a compact boson on S_1 with radius 2 where $\mathbb{Z}_2$ acts on the scalar field $\varphi(z)$ as $\iota:\varphi(z)\to-\varphi(z)$. The $\mathbb{Z}_2$ orbifold theory consists of nine chiral primaries summarized in table 1. Here, the character of each primary can be expressed in terms of $\chi_{k,n}$ and $\theta_{\alpha,\beta}$:

$$\chi_{k,n}=\frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}}q^{\frac{(k+2mn)^2}{4n}},\qquad \theta_{\alpha,\beta}=\frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}}q^{(m+\alpha)^2}e^{2\pi i m\beta}. \tag{1}$$

Next, we consider a tensor product theory $\mathcal{B}=(U(1)_4/\mathbb{Z}_2)^2$. We denote 81 primaries of this theory as $\varphi_{i,j}\equiv\varphi_i\otimes\varphi_j$ for $i,j=0,2,\cdots,8$ and their characters as $\chi_{\varphi_{i,j}}$. The bosonic theory $\mathcal{B}$ has a non-anomalous $\mathbb{Z}_2$ symmetry generated by the Verlinde line defect associated with the primary $\varphi_{2,1}\equiv\varphi_2\otimes\varphi_1$ of conformal weight $h=3/2$. Performing the Jordan-Wigner transformation with the above non-anomalous $\mathbb{Z}_2$ symmetry, one can compute the partition functions of $\mathcal{F}$. As a result, we show that the NS-sector partition function involves of 21 primaries and the NS vacuum character is given by $f_0^{\mathrm{NS}}=\chi_{\varphi_{0,0}}+\chi_{\varphi_{2,1}}$. This implies that $\mathcal{F}$ has spin 3/2 currents.

On the other hand, the Ramond-Ramond partition function of $\mathcal{F}$ becomes

$$\begin{aligned} Z_{\mathcal{F}}^{\widetilde{R}}(\tau,\bar\tau)=&\left|\chi_{\varphi_{0,5}}-\chi_{\varphi_{2,7}}\right|^2+\left|\chi_{\varphi_{0,6}}-\chi_{\varphi_{2,8}}\right|^2+\left|\chi_{\varphi_{0,6}}-\chi_{\varphi_{1,6}}\right|^2\\ &+\left|\chi_{\varphi_{3,5}}-\chi_{\varphi_{1,7}}\right|^2+\left|\chi_{\varphi_{0,8}}-\chi_{\varphi_{1,8}}\right|^2+\left|\chi_{\varphi_{3,6}}-\chi_{\varphi_{1,8}}\right|^2\\ &+\left|\chi_{\varphi_{0,4}}-\chi_{\varphi_{1,4}}\right|^2+\left|\chi_{\varphi_{5,5}}-\chi_{\varphi_{7,7}}\right|^2+\left|\chi_{\varphi_{5,6}}-\chi_{\varphi_{7,8}}\right|^2, \end{aligned} \tag{2}$$

where the q-expansion of each character is given as follows,

$$\begin{aligned} f_0^{\widetilde{R}}(q) =& \chi_{\varphi_{0,5}} - \chi_{\varphi_{2,7}} = \chi_{\varphi_{0,6}} - \chi_{\varphi_{2,8}} = \chi_{\varphi_{0,6}} - \chi_{\varphi_{1,6}} \\ &= q^{-1/48} + q^{95/48} + q^{143/48} + 2q^{191/48} + 2q^{239/48} + 3q^{287/48} + \cdots , \\ f_1^{\widetilde{R}}(q) =& \chi_{\varphi_{3,5}} - \chi_{\varphi_{1,7}} = \chi_{\varphi_{0,8}} - \chi_{\varphi_{1,8}} = \chi_{\varphi_{3,6}} - \chi_{\varphi_{1,8}} \\ &= q^{23/48} + q^{71/48} + q^{119/48} + q^{167/48} + 2q^{215/48} + 2q^{263/48} + \cdots , \\ f_2^{\widetilde{R}}(q) =& \chi_{\varphi_{0,4}} - \chi_{\varphi_{1,4}} = \chi_{\varphi_{5,5}} - \chi_{\varphi_{7,7}} = \chi_{\varphi_{5,6}} - \chi_{\varphi_{7,8}} \\ &= q^{1/24} + q^{25/24} + q^{49/24} + 2q^{73/24} + 2q^{97/24} + 3q^{121/24} + \cdots . \end{aligned} \tag{3}$$

We note that the three functions $f_0^{\widetilde{R}}(q)$, $f_1^{\widetilde{R}}(q)$, and $f_2^{\widetilde{R}}(q)$ are identical to the characters of the Ising model. Therefore, $\frac{1}{3}Z_{\mathcal{F}}^{\widetilde{R}}(\tau, \bar{\tau})$ is same with the partition function of the Ising model. In other words, the contribution from the excited states to $Z_{\widetilde{\mathrm{R}}}$ is not cancelled out, the fermionic theory $\mathcal{F}$ could not be supersymmetric. Furthermore, wecan see that the supersymmetry unitary bound is violated in $\mathcal{F}$.