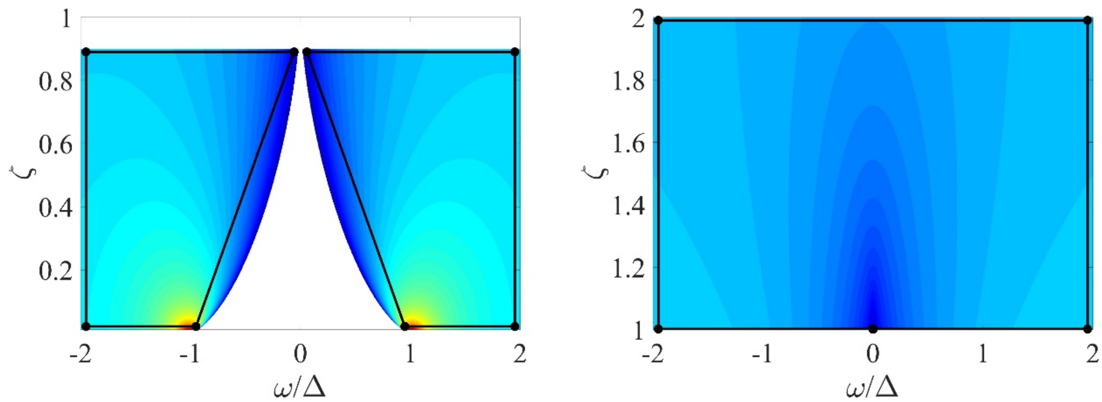
The key message of the paper is that the density of states exhibits a topological transition when the gap closes, signified by a change in the Euler characteristic $\chi$ of the surface $N(\omega,\zeta)$ for $\zeta<1$ and $\zeta>1$.

The claim is not substantiated in any way, it is merely stated that $\chi=2$ for $\zeta<1$ and $\chi=1$ for $\zeta>1$. Since it is a central claim, there should be no ambiguity. For starters, I am not even sure that the Euler characteristic is well defined on the surface $N(\omega,\zeta)$, because the cuspidal edge at $\zeta=1$ has a divergent Gaussian curvature. The poles at $\zeta=0$ are also worrisome. If these poles can be regularised, then I would think that both the surfaces at $\zeta<1$ and $\zeta>1$ can be contracted to a point, which would imply that $\chi=1$ for both surfaces and there is no topological transition.

The mechanism of polygonization of DOS surfaces is illustrated in Figure below. One can see that in the case of the gap state (left figure) the Euler characteristic of disjoint (due to the presence of the gap) polygons is the sum of their Euler characteristics, therefore $\chi_{gap} = \chi_{left} + \chi_{right} = 1+1=2$. Here $\chi_{left}$=V-E+F=4-4+1=1 is the Euler characteristic of the left sheet of the DOS surface, which coincides with the Euler characteristic for the solid rectangular. The same evaluation can be performed for the right sheet $\chi_{right}$ =V-E+F=4-4+1=1.




Contour plot of DOS surfaces shown in Figure 2 in the manuscript and their polygonization for the gap state (left) and for the gapless state (right).

The Euler characteristic of the DOS surface for the gapless state (right figure) can be calculated in the same manner $\chi_{gapless}$ =V-E+F=5-5+1=1. The emergence of the additional vertex and edges during the polygonization of the surface relates to the ``suspicious'' point at $\omega$=0, $\zeta$=1. According to the Gauss–Bonnet theorem the Euler characteristic can be determined also by the integral of the Gaussian curvature over the whole surface. One can check that the Gaussian curvature for this point of the surface $N(\omega, \zeta)$ is equal to zero and therefore does not contribute to the value of integral (see supplemental material and the chapter “The Euler characteristic of DOS surfaces and the alternative representation of the topological transition”). In the same manner one can argue for the zero curvature of the surface in the gap state when $\zeta$=0. Hence, our approach of the polygonization remains applicable for $\zeta \geq 1$.

*We added this explanation to the supplemental material as a separate chapter "The Euler characteristic of DOS surfaces and the alternative representation of the topological transition" as well as Ref. 22 where one can find movies with rotated DOS surfaces for the gap state and gapless state.*

To illustrate how the concept of the Euler characteristic can be applied to other topological transitions we consider below the well-known case of the Lifshitz transition, which is mentioned in the manuscript.

FIG. 1. a – Disruption of the "neck" of a Fermi surface, b – appearance of a new-detached region.

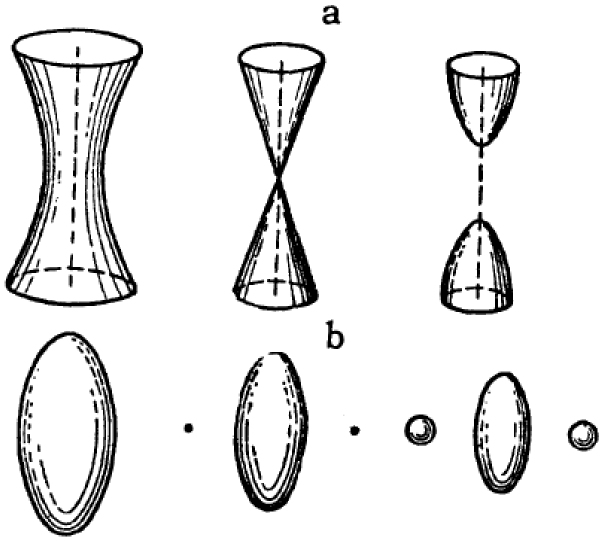
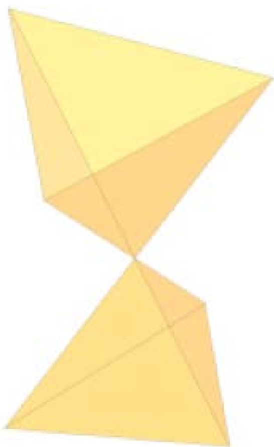



Figure from the seminal paper of Lifshitz (Ref. 3 in our manuscript).

In figure (a) a one-sheet hyperboloid (top left surface) is not compact. Its deformation retracts onto a circle, and the Euler characteristic is a homotopy invariant, so $\chi = 0$.
By tightening the neck of the hyperboloid, it is deformed into a cone (top middle surface). The cone could be simplexized into 6 singular 2-simplexes that is homotopic to the figure below. Hence, this cone has $\chi = 1$.



Simplexized cone

By detaching the pieces of the cone and smoothing it, one finds a two-sheet hyperboloid (top right surface). In this case each sheet is topologically equivalent to the disc that has $\chi=1$. The Euler characteristic of the disjoint union of two discs is the sum of their Euler characteristics, so $\chi=1+1=2$.
Therefore, throughout the Lifshitz transition the Euler characteristics alternates from 0 to 1 and then to 2.
The same interpretation of the Lifshits transition in term of Euler characteristic can be performed for bottom figures 1 (b). Here the Euler characteristic changes from 2 because of the sphere (bottom left) to $\chi= 2+1+1=4$, where we consider the additive contributions from the sphere and two points (bottom middle). Finally, for the bottom right figure there three spheres and correspondingly $\chi = 2+2+2=6$.

To conclude we thank the Referee for the criticism which as we hope resulted in a significantly improved of manuscript.