A. In the present manuscript the Authors studied theoretically the transition between gapped and gapless states in s-wave superconductors driven by magnetic impurities. Two main result are presented: a reinterpretation of the gapped-gapless transition as a topological transition and the prediction of the enhancement of the quasiparticle thermoelectric effect related to this transition. The results are presented in a clear way and appear absent of noticeable error. However, I am not convinced that the results constitute a significant advance in the field. The topological invariant the Authors propose does not reflect in any observable quantized properties, and does not indicate the presence of a topological state of matter (unlike, e.g., Chern number in a p+ip 2D superconductor).

We do not agree that the topological invariant in the form of the Euler characteristic does not reflect in any observable quantized properties. We would like to recall very briefly that the introduction of the same Chern number is directly based on the Euler characteristic.
It is well-known that Chern number arises from the topology of the band structure. In turn, the Euler characteristic dictates that the flux through a Gaussian curvature is always quantized via the so-called genus g, relating to the Euler characteristic χ = 2 - 2g.
Considering the integer Hall effect R.B. Laughlin in his seminal paper [Phys. Rev. B 23, 5632 (1981)] found out that localized cyclotron orbits form a periodic lattice, which renders a gapless edge state due to skipping orbits mechanism. This periodic lattice in a 2D plane can be represented by a torus with a non-trivial Gaussian curvature. The magnetic flux through the magnetic torus is quantized by means of the Euler characteristic. Latter allows to rewrite the expression for the Hall conductivity via another topological invariant in the form of Chern number.
Later it was shown that the band topology can be compactified to the Riemann surface, which resembles n-torus (see figure below). The genius of this surface g = q—1, where q is the number of the gaps as a two-dimensional problem.

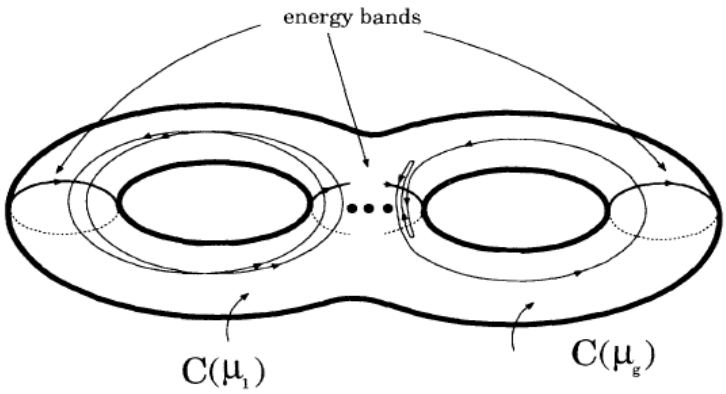


FIG. 2. Riemann surface of the Bloch function under the rational flux $\phi = p/q$. The number of the gaps, g, is the genus of the Riemann surface. $C(\mu_j)$ is a loop formed by the trace of the zero point of $\Psi_q(z)$. The energy bands are also shown by closed loops.

Figure from the paper Y. Hatsugai, Chem number and edge states in the integer quantum Hall effect. Phys. Rev. Lett. 71, 3697 (1993).

Since the genus of the surface can be defined in terms of the Euler characteristic one can say that the Euler characteristic can represent another physical observable: the number of gaps in a system under consideration.

B. The transition to the gapless phase, as pointed out by the Authors, does not reflect a new universality class and belongs to the same class as the Lifshitz transition.

The highlight of our work is that we did not reveal the subclass of Lifshitz transitions. We prove the existence of the topological transition in a s-wave conventional superconductor. It is widely accepted now that topological transitions in superconducting systems emerges under certain conditions such as a proximity effect with a topological insulator/semiconductor or the presence of the complex unconventional structure of the order parameter. Here we argue the emergence of the topological transition in a conventional superconductor with the s-wave pairing mechanism without of its destruction. Also, we would like to emphasize the additional importance of our work. This gap-gapless transition is known for more than 60 years and to the best of our knowledge there is no evidence of the topological nature of this phase transition.

C. Moreover, thermoelectric effect has been investigated in superconductors with magnetic impurities previously (Refs. 32,32) using various approximations; the Authors do not make it clear how their approach is different and if their results offer anything new except for the discussion.

In our manuscript we follow the scheme proposed by Ambegaokar and Griffin. The thermoelectric coefficient is calculated by considering the electron-impurity interaction in the ladder approximation. We do not consider other various approximations for the calculation of the thermoelectric effect; however, they do not give any new qualitatively results and do not change the general picture.

D. Finally, there are also questions regarding the generality of results (for example, whether the same results will hold if the gap is not isotropic) and the approximations used (in particular, the Born approximation appears to be implied).

In the paper [P. Hohenberg, Anisotropic Superconductors with Nonmagnetic Impurities, JETP 18, 834 (1964)] the effect of impurities on the density of states in anisotropic superconductors as a function of energy and direction was investigated. It was shown that the impurities lead to the transformation of the smeared DOS to the isotropic one at comparatively small concentration. Moreover, the region of smearing decreases to zero as the concentration of impurities increases.
<u>To this end we added the phrases to the text of the manuscript:</u>
For the case of superconductors with the anisotropic gap the effect of impurities on the DOS as a function of energy and the anisotropy degree has been investigated in Ref. 35 . It was shown that the impurities lead to the isotropisation of the density of states at relatively small concentration and the subsequent reduction of smearing over energy. With the increase of the

concentration of impurities the region of smearing decreases to zero. This means that the superconductor becomes effectively isotropic. A full analysis of this problem is outside of the scope of the present paper and therefore left for future studies.

(1) The Authors offer a reinterpretation of this transition in terms of a topological invariant - the Euler characteristic calculated for the DOS surface. Are any physical quantities of the system uniquely determined by this invariant?

As we mentioned above Euler characteristic can be used for the determination of the number of gaps in a system under consideration.
As such, the Euler characteristic can be applied also for the description of the well-known physically observable topological transition, which is mentioned in the manuscript, - Lifshitz transition.

FIG. 1. a – Disruption of the "neck" of a Fermi surface, b – appearance of a new-detached region.

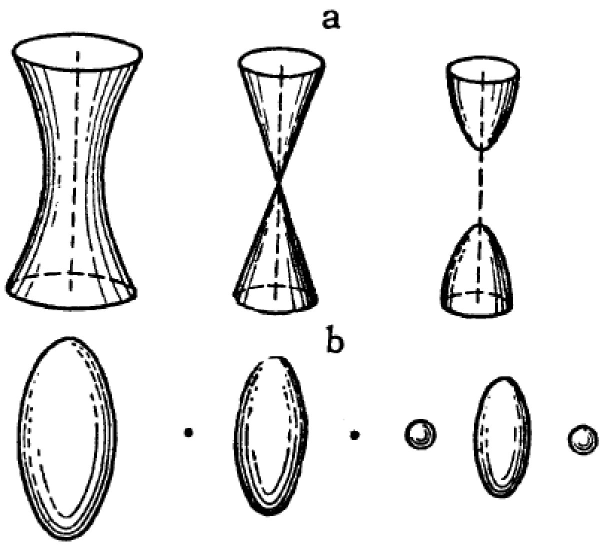
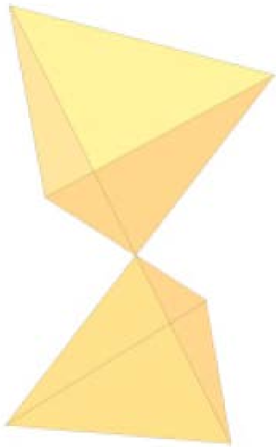


Figure from the seminal paper of Lifshitz (Ref. 3 in our manuscript).

In figure (a) a one-sheet hyperboloid (top left surface) is not compact. Its deformation retracts onto a circle, and the Euler characteristic is a homotopy invariant, so $\chi = 0$.
By tightening the neck of the hyperboloid, it is deformed into a cone (top middle surface). The cone could be simplexized into 6 singular 2-simplexes that is homotopic to the figure below. Hence, this cone has $\chi = 1$.



Simplexized cone

By detaching the pieces of the cone and smoothing it, one finds a two-sheet hyperboloid (top right surface). In this case each sheet is topologically equivalent to the disc that has χ=1. The Euler characteristic of the disjoint union of two discs is the sum of their Euler characteristics, so χ=1+1=2.
Therefore, throughout the Lifshitz transition the Euler characteristics alternates from 0 to 1 and then to 2.
The same interpretation of the Lifshits transition in term of Euler characteristic can be performed for bottom figures 1 (b). Here the Euler characteristic changes from 2 because of the sphere (bottom left) to χ= 2+1+1=4, where we consider the additive contributions from the sphere and two points (bottom middle). Finally, for the bottom right figure there three spheres and correspondingly χ = 2+2+2=6.

(●) Is there any interplay of the Euler characteristic with the known cases of topology in superconductors (e.g. p+ip state in 2D)? Being a property of DOS, I believe that it will not distinguish between trivial (s-wave) and topological (p+ip in 2D) states.

You raise a very interesting equation. However, the topic of our manuscript is devoted solely to a s-wave superconductor. Unambiguously, the complex structure of the order parameter can complicate the description of the topological gap-gapless transition in a chiral p+ip superconductor. Our study can be considered as the first milestone on the road towards the understanding of the gap-gapless topological transition in unconventional superconducting systems.
<u>We added a phrase to the text of the manuscript:</u>
It is important to note that we confine our study to the case of a s-wave isotropic superconductor and do not consider unconventional and exotic pairing symmetries.

(2) All calculations were performed for the case of an ideally isotropic gap. How will gap anisotropy affect the results? In particular, for strong anisotropy with deep minima, can the behavior of DOS and free energy near the transition change qualitatively?

As it was mentioned above the effect of impurities on the density of states in anisotropic superconductors has been studied in the paper of P. Hohenberg, Anisotropic Superconductors with Nonmagnetic Impurities, JETP 18, 834 (1964). It was revealed that the transformation of the smeared DOS to the isotropic one at comparatively small concentration occurs. With the increase of the concentration of impurities the region of smearing decreases to zero and the superconductor becomes effectively isotropic.
We added a paragraph with the explanation of this effect before Conclusions.

(3) The Authors refer to Abrikosov-Gor'kov theory for the disordered superconductivity throughout the text - does that imply that Born approximation for scattering is used? In particular, does the justification for the stability of the mean-field description given in "Smearing of the transition due to spacial fluctuations of the magnetic impurities concentration", rely on Born approximation? How will rare region effects and the presence of impurity bound states affect the argument?

We perform our calculation in the assumption of validity of the weak enough scattering and applicability of the Born approximation. Let us stress, that the main results in superconducting alloys including the effect of magnetic impurities were obtained namely in these approximations. The same concerns the theory of Lifshitz transitions. The origin of the giant Seebeck coefficient close to the transition point is ***the small, but energy dependent*** contribution to the relaxation time, which is perfectly seen already in the case of a weak scattering, amendable to the Born approximation consideration. What concerns the rare regions and impurities bound states this is the exotics lying far away from the scope of our consideration.

(4) The relation to previous works on thermoelectric coefficients in superconductors with magnetic impurities has to be discussed in more detail. In Ref. 33, multiple scattering effects were considered beyond the Born approximation, so it appears that the results of Ref. 33 are more general. In Ref. 32, on the other hand, results in Born approximation were reported and Eq. (11) and (12) there are indeed quite similar to Eq. (13) and (8) of the current work. However, the denominators in Eq. (13) and Eq. (11) of Ref. 32 appear different - could the Authors explain this difference?

Despite the more general consideration in Ref. 34 from figure 3 of this paper one can see the same strong enhancement of a thermoelectric coefficient in a dependence of the concentration impurities which is the equivalent to the parameter zeta in our manuscript. The only difference that the authors of that paper did not realize the amplification of the thermoelectric coefficient as the hallmark of the topological gap-gapless phase transition.
At the same time in Ref. 33 it was assumed that the exchange scattering is much smaller than the nonexchange scattering. This yields that the first and last terms in the denominator of Eq. (11) in the present manuscript can be neglected. Thus, we obtain Eq. (11) of Ref. 33.

<u>To clarify this point we added phrases</u>:

We perform our calculation in the assumption of validity of the weak enough scattering and applicability of the Born approximation.

Although different assumptions have been used in Refs. 33 and 34 for the calculation of the thermoelectric coefficient the effect of its enhancement remain the same on the qualitative level.

(●) Since results at finite T are reported in Fig. 3, was Δ(ζ) calculated self-consistently for finite T or were the zero-temperature expressions used? If the latter is true, this has to be mentioned and justified.

For the calculations of Δ(ζ) we used approximations for the low-temperature behaviour of the order parameter. We added these expressions for the gap and the gapless regimes to the supplemental material.

Minor comments/suggestions:

--- Fig. 1 misses a color scale; the drastic change from purple to blue seems to suggest a jump, which can be confusing to readers, since the transition is actually continuous.

To clarify this point we added sentences in the legend of Figure 1:
Purple and cyan colors in the background of the plot illustrate separation between gap and gapless state respectively. Weak blur near zeta=1 represents smearing of the transition due to spacial fluctuations of the magnetic impurities' concentration (see the corresponding chapter of the paper).

--- Mentions of the applications to s++/s± transition and color superconductivity in QCD and string theory are not really substantiated or discussed. The Authors should either provide a discussion of what new physics can their approach reveal in those systems or refrain from stating the connection (at least in the abstract and conclusion).

We extended the part regarding the applications to s++/s± transition and color superconductivity in QCD:

In the case of a dirty multi-band superconductor with increasing of the nonmagnetic impurities concentration, one of the gaps is seen to close, leading to a finite residual DOS, followed by a reopening of the gap. Such a behavior allows to speculate about the topological nature of s++/s± transition. For a color superconductor it was shown that, at zero temperature and small values of the strange quark mass, the ground state of neutral quark matter corresponds to the so-called color-flavor-locked phase. At some critical value of the strange quark mass, there is a transition to the gapless color-flavor-locked phase, where the energy gap in the quasiparticle spectrum is not mandatory [38, 39]. As in the case of multi-band *superconductivity one can again speculate about the emergence of topological phase transition in the phase diagram of the neutral quark matter.*

--- Some links in citations are not working (e.g. 17,18); Ref. 17 links to the same URL as Ref. 18; Ref. 18 is missing the journal information.

We corrected these links.

To conclude we thank the Referee for the very careful reading and for the detailed criticism which as we hope resulted in a significantly improved of manuscript.