Reply to Report 1

In this manuscript, the Authors study the Harper-Hofstadter model in the strongly interacting regime. This system is expected to host bosonic fractional Chern insulator (FCI) states. They compute several experimentally measurable quantities which could serve as signatures of the FCI state, such as incompressible bulk density and vanishing particle currents. They also apply local pinning potentials and investigate the formation of anyonic excitations with fractional charge. Moreover, they determine the necessary conditions for observing the signatures of the FCI state and the parameter regimes where this state is robust by varying the lattice size, particle number, magnetic flux per plaquette, tunneling anisotropy and other parameters.

Although the idea of realizing bosonic FCI states in optical lattices with artificial gauge fields and strong interaction is not new and has already been extensively studied, this manuscript provides some new insights. The Authors perform numerical simulations based on DMRG, which allows them to study larger system sizes and gives them an advantage over some previous works which were based on exact diagonalization. Significant progress in engineering artificial gauge fields in optical lattices has been made in recent years, as well as in development of quantum gas microscopy, which gives hope that bosonic FCI states could be indeed experimentally realized and probed. This work could provide guidance for such future experiments.

To summarize, I recommend the publication of this paper in SciPost Physics. The manuscript is well written and will surely be of interest to other researchers in this field. All claims are supported by clear numerical evidence. However, I would like the Authors to address several minor issues before publication.

Thank the referee very much for the comments and suggestions. Below we give a point-to-point response with the original comments cited in bold.

1- Introduction, second paragraph: "Various realistic protocols were proposed for both the detections of the FCT states..." Do you mean FCI instead of FCT? If not, the acronym FCT needs to be defined.

Yes. FCT was a typo and is now corrected to be FCI.

2- Discussion of Fig.1(b): It is said that the FCI state breaks down when N≥17, but it is not shown what happens when N<14. What is the lower limit for the particle number needed to obtain the FCI state for this system size?

Good point. We also found noticeable oscillations for N≤10 in the system of size 8x17. Since we aimed to emphasize the difference between N_phi and N’_phi, and to avoid messy presentation in Fig. 1(b), we decided not to show all the plots. Now, in the revised version, we point out the lower limit by replacing ‘Only for N≥17, …’ with ‘For N≥17 or N≤10, …’ in the 2nd paragraph of Section 3.

3- Page 5: "the current at the third row away from the boundary is opposite to the current at the boundary." Is it the third or the second row? Fig.1(c) suggests that this happens for n=2. If n=0 is the boundary, then n=1 is the first row away from the boundary and n=2 is the second row.

Thank the referee for the clarification. Indeed, by saying the third row we mean n=2. Now we adapt the referee's suggestion and revise it to be 'the current at the second row (n=2) away from the boundary (n=0) is…'

4- Fig.2(a): There is no colour scale. Is it the same as in Fig.1(a)?

Thank the referee for pointing this out. As the density distribution depends on the finite size, we add a colour bar in Fig. 2(a) normalized by the maximal value of the density.

5- Fig.3: What is the reason for using two different shapes of pinning potentials (2x2 and 3x1)? Can you comment on the differences between these two cases? Figs.3(b) and (e) suggest that the 3x1 potential is less sensitive to different values of r and N (at least for r≥3 and N≥6), most likely due to its smaller extent in the y-direction, but the 2x2 potential is used in later figures.

We thank the referee for the good questions. We choose different shapes mainly for two reasons. (1) The results show that it is a robust way to create quasi-particles/holes by applying pinning potentials, which would be useful in future experiments even with possible finite spatial resolution of on-site pinning potentials; (2) The shape of topological excitations could be tailored by designing the geometry of pinning potentials. In the revised version, we include the above comment in the 3rd paragraph in page 8. We choose the shape of 2x2 in later discussion for no particular reason, as we mainly focus on large enough systems with Ly=10 in the following and there is almost no difference between these two shapes of pinning potentials.

Additionally, Figs.3(c) and (f) are referred to in the main text before all other panels in this figure. Maybe it would make sense to move these two panels to the left and change their labels to (a) and (d)?

Yes. But considering that a detailed discussion of Figs. 3(c, f) is followed by discussing Figs. 3(a-d), now we keep Fig. 3 as it is, but modify the 2nd paragraph in page 8 by putting 'Figs. 3(c, f)' into a square bracket.

6- Inset of Fig.4(a): What is the difference between the squares and the circles? Is the value of V lower for the circles (it seems to be below the threshold)?

Thank the referee for pointing this out. The circles and squares respectively represent the charges in the left and right regions as defined in Fig. 3(c). Now we make it clearer in the caption of Fig. 4.

7- Fig.6(c): Are the y-labels supposed to be $\Delta N(t)$ in the main figure and $\Delta N(\tau)$ (charge after the ramp) in the inset? If not, to what time t do the data points in the inset correspond? Same comment for Fig.6(h). Also, Figs.6(h-k) are not referred to at all in the main text.

Yes, they are. We thank the referee for the useful comment. We have revised the labels in Fig. (6) and modified the caption accordingly in the revised manuscript.

We now also add the reference of Figs. 6(h-k) in the 2nd paragraph of Section 5.

8- Figs.7(e) and (f) are not mentioned at all in the text of Appendix A. Do we learn anything new from this figures that was not clear from Fig.1(b)? Maybe it could be said that the FCI signatures are more pronounced (flatter density and current distributions) for this system size than what was shown in Fig.1(b)?

We thank the referee for the useful comment and suggestion. We have included it in the Appendix A in the revised manuscript.

9- Suggestion: It would be interesting to see a more detailed study of the effects of finite interactions. The main text focuses on hard-core bosons and the current Appendix B is very short. This is an important point since the Authors wish to simulate realistic experimental conditions. If this system is experimentally realized, the interaction strength in the experiment will of course be finite, and it might be preferable not to increase the interaction strength more than is necessary as it could lead to unwanted heating. It would be useful to show the effects of interactions on some other quantities, for example Fig.1(b) with different interaction strengths instead of different particle numbers.

We appreciate the referee's suggestion. The results shown here represents a typical example to see the effect of finite interaction, which allows for an approximation of the hard-core limit. Similar behavior is expected for other simulations. In order to present the results for finite interactions more prominently, now we modify the Appendix B a bit and transform it into a new section 5 in the main text, with an additional plot of charge distributions at finite interaction U=20 in Fig. 6(c).

There are also several small typos:
1- In the References, many titles are not properly capitalized and should be corrected. For example, "rydberg atoms" → "Rydberg atoms", "haldane model" → "Haldane model", "bose-einstein condensates" → "Bose-Einstein condensates", and many others.
2- Caption of Fig.1(c): "black circle are used" → "black circle is used"
3- Page 5: "typical currents ... is plotted" → "typical currents ... are plotted"
4- Two different notations (In and In) are used for the density imbalance in the paragraph around Eq.(4). Please make the notation consistent.
5- Caption of Fig.3(c): "red circle are used" → "red circles are used"

We thank the referee for careful reading. The above typos have been fixed in the revised manuscript, except for the first one. We have checked that the titles are of the correct form in our bib file. The typos might be due to the SciPost Phys template. We will consult the Editor for more information.