To The SciPost Physics Team.

Thank you for your message of 16 Jan, regarding our submitted manuscript scipost-202110-00013v1. We also thank the referee efforts. However, we vehemently disagree with the referee's views, and below we make our reasons clear:

1) In the report (on 2022-1-16) the referee states that:

The authors have improved the presentation of their paper substantially and addressed most of the points I had raised in my initial report (or instead provided some explanation about them in their response letter). However, my main objection is still valid; I do not see how the developments reported in this paper help solve or shed light on an outstanding physics problem. As far as I can see, the authors' findings do not lead to a specific prediction of a new physical effect nor do they relate to a known physical effect. I am also unable to think of a potential physical application for them. Therefore, I cannot qualify them as "groundbreaking results." They are theoretical developments that should be of interest for experts working on the formal aspects of non-Hermitian Hamiltonians. I suggest the authors to seek publication of their work in a more specialized theoretical/mathematical physics journal.

Indeed we have not presented a "new physical effect", but we have taken a significant step in the construction of the quantum mechanics of time-dependent (TD) pseudo-Hermitian Hamiltonians and metric operators. More specifically, with the aim of exploring more general symmetries than $\mathcal{PT}$ in the general scenario of TD non-Hermitian Hamiltonians, we have provided, for the first time in the literature, a method for the derivation of TD symmetries associated with these TD Hamiltonians which applies indistinctly to *linear* or *antilinear*, *unitary* or *nonunitary symmetries*.

As a way of proving the validity of our developments, we have then assumed the symmetry to be an *antilinear* operator to retrieve the results by Mostafazadeh [3] and Bender-Berry-Mandilara [22] in the particular case of a TI scenario, i.e., TI non-Hermitian Hamiltonians, metrics and symmetries. Indeed, the above mentioned theorem by Mostafazadeh [3] is retrieved when considering an *antilinear symmetry* while the result by Bender-Berry-Mandilara is retrieved when considering a *unitary antilinear* or *antiunitary symmetry*.

In the case where the TD symmetry is assumed to be a *linear* (instead of antilinear) transformation, we obtain the equation which defines a dynamical invariant for a non-Hermitian Hamiltonian, showing, again for the first time in the literature, that a TD (linear) symmetry operator is the dynamical invariant for a non-Hermitian Hamiltonian. (And here we note that we must make a small change in the manuscript, modifying the sentence below Eq. (10) to its correct form: "**which defines a linear dynamical invariant [29-31] for a non-Hermitian Hamiltonian $H(t)$.)**

In addition, guided by the results in Refs. [2,22], we have also explored the connection between *antilinear symmetries* and *metrics*. We have derived a relation between the TD symmetry and a pair of TD metric operators [$I(t) = \Xi^{-1}(t)\rho(t)$], one linear [$\rho(t)$] and the other antilinear [$\Xi(t)$], which is analogous to the Mostafazadeh's relation [3] ($I = \Xi^{-1}\rho$) for the TI scenario. This connection between symmetry and metric is explored a little further, leading us to propose a relation between symmetry (I) and metric (ρ) operators [$I(t) = \eta(t)\mathcal{T}$]. Then, this symmetry-metric relation is put in perspective with the TD antilinear symmetry $I(t) = \Xi^{-1}(t)\rho(t)$ we have derived, allowing us to finally compute the $\Xi(t)$-anti-pseudo-Hermitian operator.

In short, following the path opened by Mostafazadeh in 2002, and in an attempt to adapt his formalism to the scenario of TD non-Hermitian Hamiltonians and metric operators, we have thus presented solid advances that can contribute to the debate of TD pseudo-Hermitian quantum mechanics, which we believe, do make our manuscripts suitable for publication in SciPost.

Regarding the referee's claim that "**I do not see how the developments reported in this paper help solve or shed light on an outstanding physics problem",** we reply that when treating a complex physical problem involving a TD non-Hermitian Hamiltonian, we now know from the developments presented in our manuscript, that it is possible to compute a general TD symmetry $I(t)$ from equation $i\partial_t I(t) + H(-t)I(t) - I(t)H(t) = 0$. Then we automatically compute the Dyson map $\eta(t)$ using the symmetry-metric relation $I(t) = \eta(t)\mathcal{T}$, and consequently the $\rho(t)$-pseudo-Hermitian metric operator $\rho(t) = \eta^\dagger(t)\eta(t)$ and the $\Xi(t)$-anti-pseudo-Hermitian operator $\Xi(t) = \eta^\dagger(t)\mathcal{T}$. All these proposed equations were tested with two examples presented in detail: the processes of linear and parametric amplifications of a cavity mode.

The referee then claims that **"As far as I can see, the authors' findings do not lead to a specific prediction of a new physical effect nor do they relate to a known physical effect. I am also unable to think of a potential physical application for them".** Indeed, we are not looking for a new physical effect in our manuscript, we are exploring the quantum mechanics of TD pseudo-Hermitian Hamiltonians beyond $\mathcal{PT}$ symmetries. Regarding the potential physical application for them, we reinforce, as we did in our previous answer to the referee, that with the advances in radiation-matter interaction platforms, we have reasons to believe that we will soon be engineering time-dependent processes of linear and parametric amplification, as those described in our manuscripts, or even time-dependent Josephson-type coupling in two-mode Bose-Einstein condensates and the consequences of this time-dependent coupling for the associated phases transitions. Moreover, non-Hermitian Hamiltonians are becoming a topic of great interest in all areas of physics. We mention, for example, the works on Majorana bound states and pairing in non-Hermitian superconducting, both driven by non-Hermiticity.

Next, we address the referee's comment that the results in our manuscript **"are theoretical developments that should be of interest for experts working on the formal aspects of non-Hermitian Hamiltonians. I suggest the authors to seek publication of their work in a more specialized theoretical/mathematical physics journal".** In view of all our new results mentioned above (a method for the derivation of the general TD symmetry, the finding that the TD linear symmetry is a dynamic invariant associated with the non-Hermitian Hamiltonian, the generalization of the results of Mostafazadeh [3] and Bender-Berry-Mandilara [22], and the new equation relating symmetry and metric), we have no doubt that our manuscript is perfectly suitable for SciPost Physics.

We would also like to comment on the points that the referee indicated as weaknesses in our paper:

1) It lacks discussion on the physical aspects of the results.

2) It involves a number of formal assumptions with no physical justification.

3) The lenghtly manipulation performed to deal with the specific toy models considered in the paper do not lead to any physical results.

Regarding the first point, we do have presented a lenghtly discussion of the physical results. For example, all the physical discussion presented in this letter is entirely contained

in the manuscript. Regarding the point 2), the only formal assumption we have made concerns the symmetry-metric relation $I(t) = \eta(t)\mathcal{T}$. There is no other formal assumption in the manuscript. Finally, regarding point 3), we first mention that there are no toy models in our manuscript; both of our examples are realistic physical systems deeply studied in the quantum optics literature, both theoretically and experimentally. Then, our lenghtly manipulation performed to deal with the linear and parametric amplification cases, lead to our proposed symmetry-metric relation and all the consequences we have discussed that follows from this relation presented for the first time in the literature.

Finally, it is important to emphasize that in his/her second letter, the referee did not contest our answers that demanded formal knowledge in the field of quantum mechanics of pseudo-Hermitian Hamiltonians. In fact, the referee did not contest our answers to questions 3 to 9, as posed in our first response, showing that the referee was wrong in all these 7 points.

We are completely sure of the importance and correctness of the work we have done, and we cordially request the SciPost team to make a decision based on the results we have presented.

Best regards,

L. F. A. da Silva, R. A. Dorado, and M. H. Y. Moussa.