We thank Referee 4 for their comments and critique. Below, we provide a reply to those, including the changes we applied to the manuscript.

> Unfortunately, the authors suggest no general conclusions based on their numerical results, which could emphasize the relevance of the work to the field. The statements like "... the fixed point is compatible with the one of short-range Clifford circuits" and "... consistent with the one reported in Ref. [51], suggesting that ... the measurement induced transition is governed by the same underlying ... theory despite the protocols being different" are too specific to the models addressed in the paper. Apparently, these models were chosen to demonstrate some more general properties of the corresponding classes of systems. For this purpose, a more extended discussion of the numerical observations in analytical terms is necessary.

We agree with the Referee that analytical considerations are fundamental to the theory of measurement-induced criticality in variable range systems. However, we also stress that the analytical arguments presented so far in hybrid random circuits with long-range interactions are phenomenological and based on arguments of plausibility (see e.g. Ref.[7,51]), and some of those might be better suited for free-fermion theories and Haar circuits (in particular, their applicability to Clifford circuits is unclear to us).

A possible path to overcome these arguments and tackle analytically the full problem is to consider the exact mapping from random circuits to classical statistical mechanics models (as in, e.g., Ref.[59,60,90]): a recent work (2110.02988) has laid down the foundation on how to carry this out for the case of Clifford unitary circuits and short-range interactions. Starting from there, one should work out an approximation that overcomes the difficulties arising from the convoluted geometry of the resulting lattice (a problem that, in some form at least, also exists when considering path integral descriptions of quantum statistical mechanics models).

In the text, we added a discussion in Sec. 5 on the above point. We believe however that, due to its highly non-trivial nature (even the short-range case was not analytically understood until 2110.02988) and importance, this analysis goes well beyond the purpose of our work - our numerical exploration, together with the other works on similar Clifford models, would then constitute a solid background to benchmark and check analytical findings.

> The description of the models and the protocol should be modified. The main problem is that the unitary gate defined in Eq. (3) does not seem to define the CHRC as described in Fig. 1. Indeed, if one naturally assumes that Eq. (3) describes each of U-blocks in Fig. 1a, i.e., the time variable t corresponds to the whole block there, the "fine structure" (internal architecture) of the large U-block shown in Fig. 1b (left) is not actually defined in Eq. (3). Indeed, the pattern of the elementary small boxes in Fig. 1b (left) is very specific, suggesting the introduction of a certain "time-ordering" inside the large U-block, which is missing in the formal equation for U(t). ("The unitary gates U(t) are laid out in a manner that mimics a soft-shoulder potential extending over M sites" – is this layout crucial for the CHRC model, or one can use an arbitrary ordering?). If, however, the time step t refers to a horizontal slice in Fig. 1b(left) rather than in Fig. 1a, Eq. (3) does not describe Fig. 1b(left) since all sites i are involved in the product in Eq. (3), while there are empty sites in each elementary slice in Fig. 1b(left). To avoid

> confusion, the authors should define their models (and notation) in a much more careful way!

We have clarified the descriptions of the CHRC model, including further mathematical definitions for the quantities of interest. We updated the caption of Fig. 1 and the text in Sec. 2 to spell out additional details of the models.

> The randomness in the gates, as well as the definition of averaging over this type of randomness, should also be described explicitly in analytical terms.

We have added the required analytical descriptions.

> In addition to this major point, there are also related minor issues:
>
> i The circuit realization Km in Eq. (1) requires an explicit analytical representation in the form Km=... (may be given after defining the elements of the hybrid circuit)
>
> ii It would be nice to add the label (+-) to the projectors in Eq. (2)
>
> iii The caption of Fig. 1 says: "M = 4 in the above illustration". However, this is not what is seen in the figure, where the dashed links between the largest pairs of the smallest blocks have different lengths (in particular, the top left block obviously has M>4)
>
> iv The different colors used for pairs in Fig. 1b(right) should be explained in the figure caption

We have added the required specifications for (i),(ii), and (iv).

Regarding (iii), in the previous version of the paper, we amplified the size of the gates passing through the boundary condition to guide the reader's eye. Since, as the Referee points out, this graphical choice lead to lack of clarity, we have updated the figure in such a way that the length of all the blocks in Fig 1(b-Left) is the same. In addition, we have reminded that we are utilizing periodic boundary conditions, and also specified this in the caption, thus avoiding possible ambiguities regarding the topology of the system depicted in the CHRC panel.

> Now, turning to the results, it would be nice to have the data points for M=3 and M=5 in Fig. 4, especially in the upper right panel for nu (by the way, the panels are not labeled in contrast to what the figure caption says). Is it clear why M=6 is the "magic cluster range" beyond which the area-law phase disappears?

We have added the labels. As pointed out in the text, we did not treat the case of $M = 3, 5, 7$ and others to avoid commensurability problems. In fact, these odd values would require a change of parity also in L.

About the case of larger $M = 8$, it is important to note that we cannot conclude that, for that case, the area-law disappears: we can only say that, within our error bars, it is impossible to discriminate whether a transition occurs or not. The same holds true for larger M. The only statement we can really make is that, if such phase exist, one would require much larger system sizes ($O(1000)$) to verify its existence. This information was already present in the text, in the "Critical exponent" paragraph of Sec. 4.1. We have now added a reminder on this specific point in the conclusions.

> Figure 7: It is not clear what the numerical evidence for the vertical dashed-dotted line at alpha=2 is provided. Is it at all possible, with available system sizes and accuracy shown by error bars, to distinguish in such a phase diagram between the "algebraic phase" and a crossover region separating the area-law and volume-law phases?

We have added a discussion on how the dashed line (crossover line) arises. This combines phenomenological (semi-analytical) arguments and our finite-size scaling analysis of the entanglement negativity. From the results of the negativity, the trend seems clear: it scaling exponent is a monotonous function of α, and, most importantly, we do not observe any crossover-type signature (such as, e.g., a non-monotonous behavior of the negativity as a function of L, or plateaus after which the derivative of the negativity versus size changes sign). The quality of the agreement between exact numerical and phenomenological argument over the full regime $1 < \alpha < 3$ further supports the picture we draw. The only caveat is, if such crossover region is extremely tiny in parameter space, we could have missed that given that we only take a finite grid in α: while we cannot rigorously exclude this scenario, we find it unlikely, also in view of the fact that we are not aware of any other instance where long-range interactions induce crossover behavior in entanglement negativity between separate regions.

Lastly, we have corrected the highlighted typos and included 2111.08018 as an additional bibliographic entry for our discussion in Sec. 5.