
Report of the Second Referee

The present work deals with applications of the entropy method to calculate the ground state energy and specific heat c(T) for spatially anisotropic triangular spin lattices. The results are also compared with measurements [37,38] on the perovskite Ba8CoNb6O24, which is considered to be a good realization of the isotropic triangular lattice.
The entropy method [41] interpolates between the high-temperature behavior (given by the power series expansion in the inverse temperature) and the low-temperature behavior of the specific heat, which must usually be assumed to be known. However, the latter is often a problem if, for example, the ground state energy is only known imprecisely. In this situation, the authors offer a captivating solution approach, which has also been used in previous work [44-46], but is the focus of the present paper: The ground state energy chosen is the one for which the largest majority of consistent Padé approximants of different order (m,n) for c(T) can be found. This approach is compared with the known values for certain limiting cases (isotropic triangular lattice, square lattice, system of disconnected chains) and proves to be surprisingly accurate.
After calculating c(T) based on these results, a comparison with the experimental data yields the somewhat sobering result that the interpolation does not provide a significant improvement over the usual Padé approximations published in [7,37,38]. These approximations already reproduce the broad maximum of c(T), which agrees with the experiments. For low temperatures the interpolated c(T) correctly goes to 0, as expected based on the approach, but deviates from the experimental results. Analogous limitations arise for the spatially anisotropic lattice in the transition ranges between J=J'=1 and $J' \to 0$ on the one hand and $J' \to \infty$ on the other: in the intermediate range, the present method fails at certain points for reasons that do not become entirely clear, and one can only say, "At least the method is able to detect something unusual."
Nevertheless, this is a beautiful work that seems worthy of publication in SciPost Physics. After all, it is questionable to appreciate only positive results. The presentation of the method and the discussion of the state of research and the physical significance of the results are very well done.
Before the final decision, I still
recommend the following minor points to be considered in a possibly revised manuscript:

We thank the Second Referee for considering our work as beautiful and to be worthy of publication. However, we disagree with the statement that the previous approximations of c(T) already reproduce the broad maximum. In the mentioned articles it appears to be like that because a linear T axis is used. In log scale, these Padé Approximants do not reproduce the linear part of c(T) between T/J=0.2 and 0.6. To make this point clearer, we have added the curves from Ref. [7] to our figure 4. Also we would like to stress out that in Ref. [7] only 3 Padé Approximants are picked out of the 27 that exist at orders 12 and 13. Even though this is a common practice, it is a rather arbitrary choice. On the other hand, our method provides a clear indication of when the extrapolation is working, for which case we obtain between 70 and 90% of all the Padé Approximants that are the same down to the lowest temperature.
We address below all of the Referee suggestions and questions, along with the corresponding modifications of the revised manuscript.

1. In the Introduction it is not quite clear what TLHAF means: the general anisotropic or the isotropic triangular lattice. It should also be made clear that Ba8CoNb6O24 is always understood as an isotropic triangular lattice, notwithstanding the remarks in section 4.4.

Indeed TLHAF refers exclusively to the isotropic triangular lattice Heisenberg antiferromagnet. For the anisotropic triangular lattice Heisenberg antiferromagnet we use “anisotropic TLHAF”. We have made this clearer in the introduction of the revised manuscript.

2. The last sentence before (5) “When the ground state is a gapped spin liquid, we have c_v(T)~T^2 exp(-T_0/T)...” is insofar misleading as the restriction to α=2 of the general power term T^α is only due to the entropy method, not to the physics.

The Referee is right about this point. We have corrected this issue by changing the sentence in the revised manuscript (before equation (5)): “We also consider a gapped ground state with a low-temperature behavior as c_v(T)~T^2 exp(-T_0/T), where T_0 is the gap.”

3. In Section 4.1, one would like to find reasons why groups of 3 orders are formed for the pCPA. Obviously, this leads to a smoothing and to an improvement of the approximation. Why?

As the values of the ground-state energy are obtained from the peaks in pCPA, it is useful to pick groups of two or three orders. This implies having a higher number of Padé Approximants (which implies a smother pCPA(e_0) curve). If s(e) does not change with n, then pCPA(e_0,[n-2,n]) taking into account the 3 highest orders is just the sum_i=0,2 pCPA(e_0,n-i). If the results are badly converged, taking groups of 3 orders will end up in pCPA(e_0) with several peaks or wide peaks. To clarified this point, we have added in the revised manuscript: “Using three consecutive orders is a way to measure the convergence with n. If converged with n, pCPA from 3 consecutive orders will be close to the individual pCPA of a single order. Otherwise, it will be much smaller.”

4. Padé approximants are rational functions in T which may have poles on the positive T axis. Usually, these cases will have to be discarded (but see Figure 3 in [7]). This problem is not mentioned in the paper. Do no poles occur, or are these cases discarded and simply do not belong to the majority of consistent approximants?

In our case, we are working in the microcanonical ensemble and the Padé Approximants are functions of the energy. In this case, all Padé Approximants with poles or roots in the energy range [e_0, 0] are non physical and discarded directly. We have made this point clearer in the manuscript.

We hope that we have answered properly all the questions presented by the Referee and that they consider that the present version is suitable for publication in SciPost Physics.