Dear Editor,

we would like to thank the referees for their report on the manuscript, their positive judgment and their constructive remarks. We reply below to their comments and questions. A redlined version of the manuscript that highlights the changes is provided.

Sincerely Yours,

Nicolas Dupuis and Romain Daviet

Report 1

1-As mentioned in the introduction and shown in Appendix B, the fact that the superfluid stiffness does not renormalize despite the presence of a periodic potential in the partially bosonized scheme without flowing bosonization, is a consequence of gauge invariance. It is also claimed that this statement holds independent of the approximation scheme used. I did not understand the latter statement. It seems to imply that it holds to arbitrary orders in the derivative expansion, while the actual calculations performed in Appendix B appear to make explicit use of the second-order truncation of the derivative expansion. Can the authors please elucidate this point?

We thank the referee for drawing our attention to this issue. In fact Appendix B does not rely on a truncation of the effective action to second order in the derivative expansion but only makes use of the most general derivative expansion of the two-point vertex $\Gamma^{(2)}$ as given in Eq. (79). We have changed the text above and below (79) and modified Eqs. (80) to clarify this point.

2-It might be useful to repeat the definition of the RG time t in the captions of Fig. 1, as these are used in the labels of the plots.

We thank the referee for this suggestion.

Report 2

1- Page 6: the authors write "We construct [...] $R_k(Q)$ in the usual way." I believe that here a citation to other works where this type of regulator has been used is needed.

We have slightly modified the sentence and added two references.

2- Page 6: "[...] we take $r(y) = \alpha/(e^y - 1)$ with α of order unity." I think a brief discussion on how α is chosen is necessary.

In the case of a precision calculation (e.g. when determining the critical exponents at a second-order phase transition), α is fixed by using the principle of minimum sensitivity. In the sine-Gordon model, the precise value of α is unimportant (see Ref. [2]). In the present study, it is therefore sufficient to take α of order unity. We have added a comment in the text.

3- Page 10, below Eq.(37): I believe, for clarity sake, that the authors should stress at this stage that $\tilde{K}_k$ is obtained from the average of $Z_{1k}(\phi)$, while K_k from its value at zero field.

We thank the referee for this suggestion.

4- Sec. 3.1.1: In my opinion it is more sound to write Eq.(38) as $\theta(Q) = -i\omega\alpha_k(Q)\phi(Q) + \beta_k(Q)\bar{\theta}(Q)$ as it is clear that θ must be a combination of $\bar{\theta}$ and $\partial_\tau\phi$. In this way it is more evident that the term in the second line of (39) is a $(\partial_\tau\phi)^2$ term and it has to vanish when summed with $Z_{1\tau,k}(\phi = 0)(\partial_\tau\phi)^2$.

Actually it is $\partial_x\theta$ which is a combination of $\partial_x\bar{\theta}$ and $\partial_\tau\phi$, so that in the end $\alpha_k(Q) \sim i\omega/q$ as obtained in Eq. (41). But we agree with the referee that we could write (38) as $\theta(Q) = -i(\omega/q)\alpha_k(Q)\phi(Q)+\beta_k(Q)\bar{\theta}(Q)$. On the other hand our notations are well suited to the calculations of Appendix E since they give a simple expression of the matrix $M_k(Q)$. For this reason we prefer keep Eq. (38) as it is now.

5- Page 13: The meaning of the paragraph from "It was pointed out in Ref. [3] ... " to "... preventing the superfluid stiffness to vanish when $k \to 0$." is unclear to me. I would ask the authors to clarify.

What we want to explain here is the fact that the convergence of $\eta_{1,k} = -\partial_t \ln Z_{1,k}$ towards 2 makes the regulator of order $Z_{1,k}k^2 \sim k^{2-\eta_k}$ for $|q|, |\omega|$ of order k. Thus the convergence of $\eta_{1,k}$ towards 2 must be extremely slow, which is not realized in practice, for the regulator function R_k to vanish in the infrared. While this issue is irrelevant for most physical quantities, which rapidly converge when k becomes smaller than the mass scale m_k/v, the non-vanishing of R_k may artificially stop the flow of K_k thus preventing the superfluid stiffness to vanish when $k \to 0$. We have improved the discussion of this issue in the manuscript.

6- Sec. 3.2, below Eq.(57): the authors write "The coefficients α and β are given by (41)." It is not clear whether, within an active frame perspective, one has to assume the coefficients to be given by Eq.(41) or this can be somewhat derived in a similar way to what is done for the passive frame approach.

This is a consequence of the change of variables being linear as shown in Ref. [21]. We have slightly changed the text after Eq. (57) and added a reference to [21].

7- In the conclusion the authors refer to the possibility of studying the Bose fluid by directly using the bosonic fields ψ and ψ^*. I believe this discussion deserves to be at least briefly mentioned in the introduction as well, as it seems to be a little decontextualized.

We have followed the referee's suggestion and added a paragraph in the introduction.

8- I believe that the authors should improve their conclusion by further stressing (if possible) the potential applications of their method.

We have slightly expanded the discussion on the two potential applications of our method in the conclusion: the Bose-glass phase of one-dimensional disordered bosons and the systems of weakly coupled one-dimensional chains.

9- Appendix A.2: I think, for completeness sake, that if the authors do not want to show the explicit form of their flow equations, they should at least refer to other works where they appear.

The flow equations did not appear elsewhere as they are specific to the two-field formalism used in the manuscript. We could show explicitly the flow equations in the Appendix (it would take at least two pages) or provide a mathematica file but we are not sure that this would be very illuminating. If the referee thinks otherwise, of course we will be happy to provide the file as a supplemental material. The flow equations are shown in the following two pages.

Flow equations

Threshold functions

$$L_{n,m,l,p,b,i}\left(\tilde{q},k,r\right)=\frac{K_k}{4\pi}\int_{\mathbb{R}}d\tilde{q}\int_{\mathbb{R}}d\tilde{\omega}\,\tilde{\omega}^p\tilde{q}^b\tilde{\partial}_t\tilde{G}_i\left(\tilde{q},\tilde{\omega}\right)\tilde{G}_0\left(\tilde{q},\tilde{\omega}\right)^n\tilde{G}_1\left(\tilde{q},\tilde{\omega}\right)^m\tilde{G}_2\left(\tilde{q},\tilde{\omega}\right)^l\tilde{G}_r{}^{(k,0)}\left(\tilde{q},\tilde{\omega}\right)$$

$$L_{n,m,l,p,b,i}\left(\tilde{\omega},k,r\right)=\frac{K_k}{4\pi}\int_{\mathbb{R}}d\tilde{q}\int_{\mathbb{R}}d\tilde{\omega}\,\tilde{\omega}^p\tilde{q}^b\tilde{\partial}_t\tilde{G}_i\left(\tilde{q},\tilde{\omega}\right)\tilde{G}_0\left(\tilde{q},\tilde{\omega}\right)^n\tilde{G}_1\left(\tilde{q},\tilde{\omega}\right)^m\tilde{G}_2\left(\tilde{q},\tilde{\omega}\right)^l\tilde{G}_r{}^{(0,k)}\left(\tilde{q},\tilde{\omega}\right)$$

$$L_{n,m,l,p,b,i}=\frac{K_k}{4\pi}\int_{\mathbb{R}}d\tilde{q}\int_{\mathbb{R}}d\tilde{\omega}\,\tilde{\omega}^p\tilde{q}^b\tilde{\partial}_t\tilde{G}_i\left(\tilde{q},\tilde{\omega}\right)\tilde{G}_0\left(\tilde{q},\tilde{\omega}\right)^n\tilde{G}_1\left(\tilde{q},\tilde{\omega}\right)^m\tilde{G}_2\left(\tilde{q},\tilde{\omega}\right)^l$$

with

$$\tilde{G}_0=\tilde{G}_{\phi\phi},\,\tilde{G}_1=\tilde{G}_{\theta\theta},\,\tilde{G}_2=\frac{1}{i}\tilde{G}_{\phi\theta}$$

and

$$\tilde{\partial}_t\tilde{G}_0=\left(\tilde{G}.\partial_t\tilde{R}_k.\tilde{G}\right)_{\phi\phi},\,\tilde{\partial}_t\tilde{G}_1=\left(\tilde{G}.\partial_t\tilde{R}_k.\tilde{G}\right)_{\theta\theta},\,\tilde{\partial}_t\tilde{G}_2=\frac{1}{i}\left(\tilde{G}.\partial_t\tilde{R}_k.\tilde{G}\right)_{\phi\theta}$$

Equation for $\tilde{U}_k$

$$\begin{aligned}\partial_t\tilde{U}_k(\phi)=&\left(-2+\eta_{1,k}\right)\tilde{U}_k(\phi)-\frac{1}{2}L_{0,0,0,0,2,1}\tilde{Z}'_{2,k}(\phi)+L_{0,0,0,1,1,2}\tilde{Z}'_{3,k}(\phi)\\&-\frac{1}{2}L_{0,0,0,0,2,0}\tilde{Z}'_{1x,k}(\phi)-\frac{1}{2}L_{0,0,0,2,0,0}\tilde{Z}'_{1\tau,k}(\phi)-\frac{1}{2}L_{0,0,0,0,0,0}\tilde{U}_k{}^{(3)}(\phi)\end{aligned}$$

Equation for $\tilde{Z}_{3,k}$

$$\begin{aligned}\partial_t\tilde{Z}_{3,k}(\phi)=&\left(-1+z_k+\frac{1}{2}\left(\eta_{1,k}+\eta_{2,k}\right)\right)\tilde{Z}_{3,k}(\phi)-L_{0,0,0,0,3,1}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{2,k}(\phi)^2\\&+L_{0,0,0,0,3,2}\left(\tilde{\omega},1,1\right)\tilde{Z}'_{2,k}(\phi)^2+L_{0,1,0,0,2,0}\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi)+L_{0,0,0,1,2,0}\left(\tilde{\omega},1,1\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi)\\&-L_{0,0,0,1,2,1}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi)-L_{0,0,1,1,1,0}\tilde{Z}'_{3,k}(\phi)^2-L_{1,0,0,1,1,2}\tilde{Z}'_{3,k}(\phi)^2\\&+L_{0,0,0,0,3,0}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1x,k}(\phi)-L_{0,0,0,0,3,2}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1x,k}(\phi)+L_{1,0,0,0,2,0}\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi)\\&+L_{0,0,1,1,1,0}\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi)-L_{1,0,0,1,1,2}\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi)+L_{0,0,0,2,1,0}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi)\\&-L_{0,0,0,2,1,2}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi)+L_{1,0,0,2,0,0}\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi)-\frac{1}{2}L_{0,0,0,0,0,0}\tilde{Z}''_{3,k}(\phi)\\&+L_{0,0,0,0,1,0}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{U}_k{}^{(3)}(\phi)-L_{0,0,0,0,1,2}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{2,k}(\phi)\tilde{U}_k{}^{(3)}(\phi)+L_{1,0,0,0,0,0}\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi)\end{aligned}$$

Equation for $\tilde{Z}_{2,k}$

$$\partial_t\tilde{Z}_{2,k}(\phi)=\eta_{2,k}\tilde{Z}_{2,k}(\phi)+2L_{0,0,1,0,2,2}\tilde{Z}'_{2,k}(\phi)^2+L_{0,1,0,0,2,0}\tilde{Z}'_{2,k}(\phi)^2+L_{1,0,0,0,2,1}\tilde{Z}'_{2,k}(\phi)^2-\frac{1}{2}L_{0,0,0,0,0,0}\tilde{Z}''_{2,k}(\phi)$$

Equation for $\tilde{Z}_{1x,k}$

$$
\begin{aligned}
\partial_t \tilde{Z}_{1x,k}(\phi) = {} & \eta_{1,k}\tilde{Z}_{1x,k}(\phi) + L_{0,1,0,0,2,1}\tilde{Z}'_{2,k}(\phi)^2 + 2L_{0,0,0,0,3,1}\left(\tilde{q},1,1\right)\tilde{Z}'_{2,k}(\phi)^2 + \frac{1}{2}L_{0,0,0,0,4,1}\left(\tilde{q},2,1\right)\tilde{Z}'_{2,k}(\phi)^2 \\
& - 2L_{0,1,0,1,1,2}\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) - 2L_{0,0,0,1,2,1}\left(\tilde{q},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) - 4L_{0,0,0,1,2,2}\left(\tilde{q},1,1\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) \\
& - L_{0,0,0,1,3,1}\left(\tilde{q},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) - L_{0,0,0,1,3,2}\left(\tilde{q},2,1\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) - L_{0,1,0,2,0,0}\tilde{Z}'_{3,k}(\phi)^2 \\
& - 2L_{0,0,0,2,1,0}\left(\tilde{q},1,1\right)\tilde{Z}'_{3,k}(\phi)^2 + 2L_{0,0,0,2,1,2}\left(\tilde{q},1,2\right)\tilde{Z}'_{3,k}(\phi)^2 - \frac{1}{2}L_{0,0,0,2,2,0}\left(\tilde{q},2,1\right)\tilde{Z}'_{3,k}(\phi)^2 \\
& - \frac{1}{2}L_{0,0,0,2,2,1}\left(\tilde{q},2,0\right)\tilde{Z}'_{3,k}(\phi)^2 + L_{0,0,0,2,2,2}\left(\tilde{q},2,2\right)\tilde{Z}'_{3,k}(\phi)^2 - 4L_{0,0,1,0,2,2}\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1x,k}(\phi) \\
& - 4L_{0,0,0,0,3,2}\left(\tilde{q},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1x,k}(\phi) - L_{0,0,0,0,4,2}\left(\tilde{q},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1x,k}(\phi) - 4L_{0,0,1,1,1,0}\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) \\
& - 2L_{1,0,0,1,1,2}\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) - 4L_{0,0,0,1,2,0}\left(\tilde{q},1,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) - 2L_{0,0,0,1,2,2}\left(\tilde{q},1,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) \\
& - L_{0,0,0,1,3,0}\left(\tilde{q},2,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) - L_{0,0,0,1,3,2}\left(\tilde{q},2,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) + 3L_{1,0,0,0,2,0}\tilde{Z}'_{1x,k}(\phi)^2 \\
& + 2L_{0,0,0,0,3,0}\left(\tilde{q},1,0\right)\tilde{Z}'_{1x,k}(\phi)^2 + \frac{1}{2}L_{0,0,0,0,4,0}\left(\tilde{q},2,0\right)\tilde{Z}'_{1x,k}(\phi)^2 - 2L_{0,0,0,2,1,2}\left(\tilde{q},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& - L_{0,0,0,2,2,2}\left(\tilde{q},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - 2L_{0,0,0,3,0,0}\left(\tilde{q},1,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& - L_{0,0,0,3,1,0}\left(\tilde{q},2,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - L_{0,0,0,3,1,2}\left(\tilde{q},2,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) + 2L_{1,0,0,2,0,0}\tilde{Z}'_{1x,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& + 2L_{0,0,0,2,1,0}\left(\tilde{q},1,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) + L_{0,0,0,2,2,0}\left(\tilde{q},2,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& + \frac{1}{2}L_{0,0,0,4,0,0}\left(\tilde{q},2,0\right)\tilde{Z}'_{1\tau,k}(\phi)^2 - \frac{1}{2}L_{0,0,0,0,0,0}\tilde{Z}''_{1x,k}(\phi) - 2L_{0,0,0,0,1,2}\left(\tilde{q},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& - L_{0,0,0,0,2,2}\left(\tilde{q},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) - 2L_{0,0,0,1,0,0}\left(\tilde{q},1,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& - L_{0,0,0,1,1,0}\left(\tilde{q},2,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) - L_{0,0,0,1,1,2}\left(\tilde{q},2,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) + 2L_{1,0,0,0,0,0}\tilde{Z}'_{1x,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& + 2L_{0,0,0,0,1,0}\left(\tilde{q},1,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) + L_{0,0,0,0,2,0}\left(\tilde{q},2,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& + L_{0,0,0,2,0,0}\left(\tilde{q},2,0\right)\tilde{Z}'_{1\tau,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) + \frac{1}{2}L_{0,0,0,0,0,0}\left(\tilde{q},2,0\right)\tilde{U}_k{}^{(3)}(\phi)^2
\end{aligned}
$$

Equation for $\tilde{Z}_{1\tau,k}$

$$
\begin{aligned}
\partial_t \tilde{Z}_{1\tau,k}(\phi) = {} & \left(-2+2z_k+\eta_{1,k}\right)\tilde{Z}_{1\tau,k}(\phi) + \frac{1}{2}L_{0,0,0,0,2,1}\left(\tilde{\omega},2,1\right)\tilde{Z}'_{2,k}(\phi)^2 - 2L_{0,0,0,0,2,2}\left(\tilde{\omega},1,1\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) \\
& - L_{0,0,0,1,1,1}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) - L_{0,0,0,1,2,2}\left(\tilde{\omega},2,1\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{3,k}(\phi) - L_{0,1,0,0,2,0}\tilde{Z}'_{3,k}(\phi)^2 \\
& + 2L_{0,0,0,1,1,2}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{3,k}(\phi)^2 - 2L_{0,0,0,1,2,0}\left(\tilde{\omega},1,1\right)\tilde{Z}'_{3,k}(\phi)^2 - \frac{1}{2}L_{0,0,0,2,0,1}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{3,k}(\phi)^2 \\
& + L_{0,0,0,2,1,2}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{3,k}(\phi)^2 - \frac{1}{2}L_{0,0,0,2,2,0}\left(\tilde{\omega},2,1\right)\tilde{Z}'_{3,k}(\phi)^2 - L_{0,0,0,0,3,2}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1x,k}(\phi) \\
& - 2L_{0,0,0,0,3,0}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) - L_{0,0,0,1,2,2}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) \\
& - L_{0,0,0,1,3,0}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1x,k}(\phi) + \frac{1}{2}L_{0,0,0,0,4,0}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{1x,k}(\phi)^2 - 2L_{0,0,1,0,1,2}\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& - 2L_{0,0,0,1,1,2}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - L_{0,0,0,2,1,2}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - 4L_{0,0,1,1,1,0}\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& - 2L_{1,0,0,1,0,2}\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - 2L_{0,0,0,2,0,2}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - 4L_{0,0,0,2,1,0}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& - L_{0,0,0,3,0,2}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) - L_{0,0,0,3,1,0}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& + 2L_{1,0,0,0,2,0}\tilde{Z}'_{1x,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) + 2L_{0,0,0,1,2,0}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) \\
& + L_{0,0,0,2,2,0}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{Z}'_{1\tau,k}(\phi) + 3L_{1,0,0,2,0,0}\tilde{Z}'_{1\tau,k}(\phi)^2 + 2L_{0,0,0,3,0,0}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{1\tau,k}(\phi)^2 \\
& + \frac{1}{2}L_{0,0,0,4,0,0}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{1\tau,k}(\phi)^2 - \frac{1}{2}L_{0,0,0,0,0,0}\tilde{Z}''_{1\tau,k}(\phi) - L_{0,0,0,0,1,2}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{2,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& - 2L_{0,0,0,0,1,0}\left(\tilde{\omega},1,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) - L_{0,0,0,1,0,2}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& - L_{0,0,0,1,1,0}\left(\tilde{\omega},2,2\right)\tilde{Z}'_{3,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) + L_{0,0,0,0,2,0}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{1x,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& + 2L_{1,0,0,0,0,0}\tilde{Z}'_{1\tau,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) + 2L_{0,0,0,1,0,0}\left(\tilde{\omega},1,0\right)\tilde{Z}'_{1\tau,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) \\
& + L_{0,0,0,2,0,0}\left(\tilde{\omega},2,0\right)\tilde{Z}'_{1\tau,k}(\phi)\tilde{U}_k{}^{(3)}(\phi) + \frac{1}{2}L_{0,0,0,0,0,0}\left(\tilde{\omega},2,0\right)\tilde{U}_k{}^{(3)}(\phi)^2
\end{aligned}
$$