Referee Answer: Parametric co-linear axion photon instability

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We would like to thank the referee for their very useful comments and effort to improve the present paper! Below we aim to answer the questions raised and comment on the issues mentioned in their report. For context, we quote the comment/question, as layed out in the .pdf attached to the report, in red and our answer in black below it.

1. **Is There Exponential Growth?**

On the top of page 3, the authors state that "we focus only on the positive root" for $\Gamma(k)$. How do you know that these are the roots that correspond to the physical forward-propagating modes? One way to show that there is a tachyonic instability would be to calculate the group velocity, $d\omega/dk$, and to show that the group velocity is superluminal. See for example [1]: a superluminal group velocity, combined with the requirement of causality, implies that the retarded Green's function grows exponentially in the future light cone.

In fact, Eq.(18) provides an expression for the group velocity (referred to here as a "phase velocity"), and it is not superluminal. Where, then is the evidence of any instability that would drive exponential growth? (I think that Eqs.(15–18) are not correct, however. See below.)

Thank you for this comment, indeed we should explain this in more detail. A useful analogy to draw is the growth of plasma waves (plasmons) through stimulated Raman instability. As detailed in e.g. [1] this process feeds energy into plasmons via the decay of a pump beam. The ponderomotive force onto the electrons stemming from the overlap of the pump beam and the sideband modes (at $\omega_0 \pm \omega_{\mathrm{plasmon}}$) provides the positive feedback resulting in exponential growth. In our axion case, the axion's $\mathbf{E}\cdot\mathbf{B}$ coupling plays the role of the ponderomotive force and results in the growth of the axion mode by depleting the pump. In the early stages of the instability (because of the smallness of the growth rate, this remains true for basically the entire duration) the depletion of the pump can be ignored.

Indeed, we can perform a similar analysis to [2] and find a coupled set of differential equations which do show growth.

We can find the evolution of the probe beam field by reconsidering the coupled equations (10) and(11) in the manuscript. We make the ansatz

$$\tilde{A}_\pm = \tilde{A}_\pm(t)e^{-i\omega_\pm t}, \qquad \tilde{a} = \tilde{a}(t)e^{-i\omega_a t}. \tag{1}$$

This will enable us to capture the initial growth of the axion field from $\tilde{a}(t=0)=0$. We will make use of the solution to the dispersion relation in the original manuscript to expand the resulting equations to lowest order in $g_{a\gamma\gamma}$. The hierarchies we found are $\Gamma(k) \sim \mathcal{O}(g_{a\gamma\gamma}A_0)$ and $\omega_a \sim \mathcal{O}((g_{a\gamma\gamma}A_0)^4)$. Thus, we drop terms ω_a and make the approximation $\partial_t \tilde{A}_\pm \ll \omega_\pm \tilde{A}_\pm$ to find

$$(i\partial_t - k_a)\,\tilde{A}_\pm(t) = \mp\frac{g_{a\gamma\gamma}A_0}{2}\omega_0\tilde{a}(t) \tag{2}$$

and

$$\left(\partial_t^2 + k_a^2 + m_a^2\right)\tilde{a}(t) = -\frac{g_{a\gamma\gamma}A_0}{2}\omega_0\,(i\partial_t - k_a)\left(\tilde{A}_+(t) - \tilde{A}_-(t)\right). \tag{3}$$

Note that while we can assume $\partial_t \tilde{A}_\pm \ll \omega_\pm \tilde{A}_\pm$, the situation is reversed for the axion field. Upon substitution of (2) into (3) we find, as is expected for such a system of coupled modes [2], a growing axion field

$$\tilde{a}(t) \propto \sinh\left(\omega_0\sqrt{\left(\frac{g_{a\gamma\gamma}A_0}{2}\right)^2 - \left(\frac{m_a}{\omega_0}\right)^2 - \left(\frac{k}{k_0}\right)^2}\,t\right) \tag{4}$$

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which initially grows linear from $\tilde{a}(0) = 0$ and then increases exponentially. Note that in the limit $g_{a\gamma\gamma} \to 0$ we recover a freely propagating axion mode with energy $\sqrt{k^2 + m_a^2}$.

2. If there is growth in the axion field, where does the energy come from? Depleting the laser pulse? If so, please explain why this is an example of positive rather than negative feedback.

   Exactly, the energy comes from depleting the initial pump pulse. The axion field grows (see point 1), hence, we have positive feedback of the axion field.

3. What assumptions have been made about the axion mass in Section III? More precisely, why is the quantity Γ of Eq.(14) and Eq.(16) assumed to be real? In the limit of small couplings and photon intensities, $g_{a\gamma\gamma} A_0 \to 0$, the "growth rate" $\Gamma(k)$ of Eq.(16) is strictly imaginary. Γ is supposed to be real-valued, and its imaginary component should be included in ω_a.

   Thank you for pointing out that we have indeed not stressed this point clearly enough. We solve the dispersion relation (13) in the manuscript under the assumption that ω_a and Γ are real valued quantities. Expanding $\omega = \omega_a + i\Gamma$ is always possible. Hence the solution we find in (14) and (16) are only consistent as long as they are real valued. Once they become complex, our assumption of two linearly independent equations for the real and imaginary part of the dispersion relation no longer apply and the solution is not a solution to the original equation for ω. Indeed, once the growth rate Γ (or frequency ω_a) is no longer real, there is also no instability present in the numerical solution. Thus, we find cutoffs for this instability at large masses $(m_a/\omega_0)^2 > (g_{a\gamma\gamma} A_0)^2$ and at large momenta $k/k_0 > (g_{a\gamma\gamma} A_0)^2 - (m_a/\omega_0)^2$ (see equation (17) and explanation thereafter). These match the numerical solution. Analytic treatment is still possible in this case, but the only solution to the dispersion relation which satisfies $\omega_a,\, \Gamma \in \mathbb{R}$ in this case has $\Gamma = 0$.

4. **Axion production and Eq.(2):**

   According to the text above Eq.(2), a population of axions is being created from the electromagnetic fields by the "decay...of one photon ...into an axion and secondary photon." Please elaborate: what is the underlying assumption about the background electromagnetic fields? A laser fired into a vacuum does not decay in this way, the $\gamma \to \gamma + a$ decay process is kinematically forbidden.

   The PVLAS and light-shining-through-walls experiments rely on strong external magnetic fields. Other proposed experiments rely on a pre-existing background axion field (ALP dark matter) to modify the propagation of light. An external static $\mathbf{B}$ (or $\mathbf{E}$) field is needed to produce (massive) axions from a beam of (massless) photons: the photons associated with the external field are not on-shell, but come instead from some distribution $A_\mu(k)$ that is not constrained by $\omega_2 = k_2$.

   In this paper, on the other hand, the "seed pulse" and the "weaker probe" are both linearly polarized radiation. So, the axion production mechanism here at the particle level must be light-by-light scattering, $\gamma + \gamma \to a$, which is kinematically allowed only if the frequencies of the two lasers differ by an amount tuned to the axion mass. If this is the picture in mind, then Eq.(2) should be written as

   $$\omega_0 - \omega_\gamma = \omega_a, \qquad k_0 - k_\gamma = k_a \tag{5}$$

   to more clearly show the "before" and "after" pictures.

   Taken at face value, this paper appears to claim that freely propagating photons are liable to decay into an axion and a redshifted photon, and that this photon decay occurs in the absence of any external $\mathbf{E}$, $\mathbf{B}$, or axion fields. The paper should be substantially revised so as not to give the reader this impression.

   Thank you for raising this point. It is a very important point to stress and explain more clearly. First, we fully agree with the assessment and explanation of the necessity of a magnetic or electric field in light-shining-through-wall searches. While we do indeed claim our instability proceeds via the $\gamma \to \gamma + a$ process, we *do not* claim that it is kinematically allowed in vacuum and apologise should the paper give this impression. Our set-up consists of two strong laser pulses which provide the necessary initial photons and background fields for the aforementioned process and indeed are both necessary for it to be allowed.

   The process $\gamma \to \gamma + a$ never enters into a perturbative calculation, which is an expansion around the vacuum state, because in vacuum it is forbidden by energy-momentum conservation. In the presence of strong background fields, however, the process can take place and non-perturbative calculations reveal that, provided the particles mass is small enough, massless particles can decay into massive ones, see [3]. The presence of strong background fields alters the dispersion relation of the coupled waves. Therefore, a microscopic interpretation in terms of the vacuum photon and axion particles is not applicable in this set-up.

5. **Axion production and Eq.(19):**

My interpretation of Eq.(8) is that the "main" pulse is a monochromatic plane wave with frequency ω_0, and that at $t = 0$ the axion energy density, $\rho_a \sim m_a^2 \tilde{a}^2$, is zero everywhere. The functions $\tilde{A}$ and $\tilde{a}$ keep track of the induced fields, which are small compared to A_0. This is fine. But, the modified Maxwell's equations with $a = 0$ are simply SM electrodynamics, so for anything interesting to happen we need to create some axions. To find out how axions are produced from a laser beam, I look to Section IV, which adds "a weaker probe with polarisation in $\hat{y}$."

Nothing is said about the frequency of the weaker probe. Is it the same as the "pump pulse"? Does it propagate in the same direction? Equation (19) should provide an answer, but does not. Taken literally, Eq.(19) is insensible:

$$\frac{\mathbf{A} \cdot \hat{x} - \mathbf{A} \cdot \hat{y}}{A_0} = \frac{\mathbf{A} \cdot (\hat{x} - \hat{y})}{A_0} \tag{6}$$

This is just linearly polarized light, along some new $\hat{x} - \hat{y}$ axis. Even if the two instances of $\mathbf{A}$ are supposed to differ in magnitude, this initial setup is still just linearly polarized light along $(\hat{x} - \delta\hat{y})$ propagating in a vacuum. In order to produce any axions through light-by-light scattering, the frequencies of the two pulses must differ by a precise amount tuned to the axion mass. Given that there is no discussion of any of this in the text, I am forced to conclude that there is never any axion particle production in the setup suggested in Section IV, and that all equations from Eq.(20) onward are incorrect.

Thank you for the question, we have not explained the set-up sufficiently. The pump pulse has a frequency of ω_0. A parametric decay instability can start operating when all except for one of the coupled modes are occupied. In this case the empty mode starts growing. This is completely analogous to stimulated Raman scattering. To be independent of the dark matter composition, we take the axion field to be initially unoccupied everywhere. For the process to commence we need to provide a strong pump and a weaker probe such that the sideband modes are populated. Hence, the weaker probe is tuned to be the $\tilde{\mathbf{A}}_\pm$ fields. The weaker probe beam, in principle, has $\omega_\pm$, however, because the axion frequency ω_a is smaller than the spectral width of real laser systems, it is sufficient to have both beams at the same frequency.

It is true that in the co-linear limit, as long as the axion field is not populated, the fields of the two beams have $\mathbf{E} \cdot \mathbf{B} = 0$ always. As soon as the axion field is populated this is no longer true. To produce the first axions, we make use of the fact that any realistic laser pulse will be focused. Hence, the momenta of two photons colliding will have a small, but non-zero angle. For two such plane waves, $\mathbf{E} \cdot \mathbf{B} \neq 0$. Our equations still apply once an initial population is produced, we only describe the part of the axion field which is co-linear. We appreciate that this should have been stated more clearly in the paper and have added a corresponding explanation to the revised document.

If the axion field is populated, the equations for $\mathbf{E}$ and $\mathbf{B}$ are no longer linear and the instability proceeds as layed out in the paper.

6. **Colinearity:**

Disregarding my previous objections, let us assume that some axions are indeed produced by the laser, following Eq.(2). For there to be positive feedback, this population of axions must stay put on the timescales associated with the laser pulse length, τ.

In Section II, an assumption is made: the calculation will be done in a "colinear limit where all momenta, are parallel." For some parts of the calculation this is fine: the experimentalist can ensure, for example, that the laser pulses will all be parallel. For the axion production described in Eq.(2), however, this is suspicious. If the axions do indeed come from "photon decay," why must $\mathbf{k}_a$ be parallel to $\mathbf{k}_0$? Unfortunately, this assumption seems to be important for the posited feedback loop: without a population of axions, there is no positive feedback on the photon source of axion production. Instead, the posited photon-to-axion conversion merely depletes the photon source.

The colinearity is indeed ensured by the initial set-up we chose. The pump and probe beam are set-up in such a way that they are colinearly propagating. The axions are produced at the difference frequency of the two beams, hence through stimulated decay of pump photons. A pump photon with momentum along, without loss of generality, $\hat{z}$ is stimulated by a photon of the probe beam which has momentum also along $\hat{z}$. In this case the momentum of the emitted axion must also be parallel to the same axis. We fully agree that, a hypothetical spontaneous decay of a pump photon to an axion, would leave the axion with random momentum direction, however this process is kinematically disallowed and would be suppressed by a factor of the occupation number of the probe beam compared to the stimulated process. This photon number is smaller than the pump beam photon number but still significantly larger than 1.

7. **Spatial homogeneity:**

   There seems to be an unspoken further assumption of spatial homogeneity: or at least, no comment is made about the spatial distributions of either $\mathbf{A}(\mathbf{x},t)$ or $a(\mathbf{x},t)$ after Eq.(9), except to say that the laser pulse is of finite duration. A lack of spatial gradients is a problem: the modified Maxwell equations depend entirely on the spatial and temporal derivatives of $a(\mathbf{x},t)$. Without some nonzero $\partial_t a$ or ∇a, the laser will propagate trivially through the vacuum.

   We are working in the limit that the envelope varies little over the beams extend and therefore take the beam to be a plane wave. Thus, $\nabla\tilde{a} = i\mathbf{k}_a\tilde{a}$. If the axion momentum $\mathbf{k}_a \propto \mathbf{k}_0/k_0$, as was argued in point 6., the axions remain within the overlap of the pump and probe beam for a significant time and the axion field grows in time, homogeneous over the laser's spatial extend.

8. **What happens to the produced axions?**

   If the axions are produced with some nonzero $\mathbf{k}$, then they will tend to leave the path of the laser. If the axions are produced at relativistic speeds, as suggested in Section III, then this depletion of the axion density will be prompt (assuming that the laser beam is not several meters wide).

   Axions produced by this laser pulse would not be spatially homogeneous along the direction of photon propagation, and yet there is no discussion of what $\tilde{a}(x,t)$ is supposed to look like. The spatial profile of $\tilde{a}$ is very important, if there is to be any positive feedback loop: the presence of axions is supposed to be altering the behavior of the parts of the laser pulse that pass through the space after the axions have been created. Without any gradients in $\tilde{a}(x,t)$, there is no modification to Maxwell's equations.

   Thank you very much for asking for clarification on this point. As we have argued under point 6., the axion momentum is indeed colinear with the laser's propagation direction. Therefore, the axions do not instantly leave the overlap of the pump and probe beam but rather copropagate.

   It is also worth pointing out, that the axion's velocity, as given in equation (18) is not fast. It is $\mathcal{O}((g_{a\gamma\gamma}A_0)^4) \ll 1$.

9. **Coherence of the axion field:**

   To apply Eq.(3) to this problem, we need to treat the axions as a coherent classical field. This is well motivated if axions are the dark matter, because the velocity dispersion $v \leq 10^{-3}$ is small, so the axions remain coherent on relatively long length and time scales, large enough to encompass the experimental setup if the axion is sufficiently light.

   It is not clear that the axion production rate here is large enough to justify treating the axions as a coherently oscillating field. What is the axion energy density $\rho_a(\mathbf{x},t)$ in this scenario (for the axions created by a strong laser pulse)? For what values of the axion mass is the number density ρ_a/m_a large enough to take the classical limit?

   Thank you for the comment. The applicability of equation (3) does in fact not depend on the axion field being a coherent classical field. The argument is similar to the argument in [4]. The source of the axion field $\mathbf{E}\cdot\mathbf{B}$ is a classical source. Therefore, also the axion field is classical.

We hope the above answers and revised document are sufficient to resolve the concerns raised by the referees.

Sincerely,
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