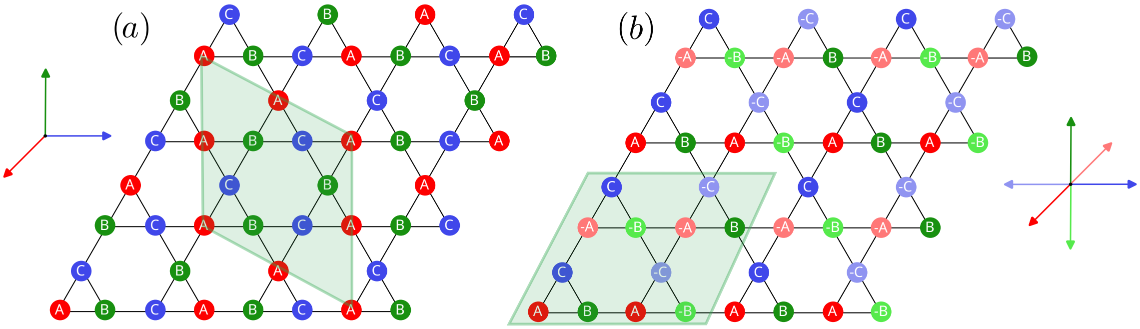


Figure 1: (a) Example of a three-color triaxial state. The directions A, B, and C form an orthogonal basis. (b) Variant of the octahedral phase with AFM chains along the diagonals of the hexagons. The axis are illustrative and not unique. The unit cell is represented by the parallelograms.

## A Classical ground state degeneracy near the chiral point

In section 3 we encountered an extended disordered region in the classical model with dominant three-spin interactions. Generically, a classically disordered phase can be connected to classically degenerate states and often appears at the boundaries between two ordered phases. Its extended nature in the present problem may be traced back to the frustrated geometry of the kagome lattice combined with the effects of the three-spin interaction.

The extensive degeneracy of the classical $J_1$-$J_\chi$ model was discussed in Ref. [1] for $J_1 > 0$ and both uniform and staggered chirality. Let us take the pure chiral model, $J_1 = J_d = 0$, as our reference point. The energy of the staggered chiral interaction is minimized imposing that the spins in the up (down) triangles form a right (left)-handed orthogonal basis. An important subset of theses states are the triaxial states [1], shown in Fig. 1 (a), in which each spin is collinear with one of three directions represented by three colors. Triaxial states in the pure chiral model have an extensive degeneracy which scales as $2^{N/6}$ and is associated with local $\mathbb{Z}_2$ degrees of freedom. The latter is reminiscent of the degeneracy of coplanar states for the antiferromagnetic Heisenberg model on the kagome lattice [2]. In the presence of nonzero $J_1$, the triaxial states can be generalized by considering three directions which are no longer orthogonal. In terms of the angle $\theta$ that the spins form with the space diagonal, the energy for a single triangle is $\mathcal{E}(\theta) = \frac{3}{4}[J_1 S^2(1+3\cos 2\theta) - \sqrt{3} J_\chi S^3 \sin\theta \sin 2\theta]$. For $J_1 < 0$, the angle $\theta_0$ that minimizes the energy decreases with $|J_1|$ until we reach the critical value $J_{1c} = -S/\sqrt{3}$. For $J_1 < J_{1c}$, we obtain $\theta_0 = 0$, corresponding to the ferromagnetic state. On the other hand, for $J_1 > J_{1c}$ the classical ground state remains massively degenerate because one can construct a subextensive set of states which are degenerate with a given three-color state, as explained in Ref. [1].

The construction of classically degenerate ground states for the $J_1$-$J_\chi$ model does not hold once we add the exchange coupling $J_d$. In fact, starting from the pure chiral point, a small $J_d > 0$ has an immediate impact: it selects triaxial states with AFM chains along the diagonals of the hexagons, shown in Fig. 1(b). This state minimizes both the $J_\chi$ and $J_d$ terms. As a result, the extensive ground state degeneracy is lifted at first order in $J_d > 0$ and we enter the AFMd phase, see Fig. 2 of section 3. The situation for $J_d < 0$ is distinct because FM chains running along the diagonals of the hexagons are incompatible with triaxial states. Within the set of triaxial states, those in which spins across the diagonals point in perpendicular directions, as in Fig. 1(a), have lower energy than the AFMd state in Fig. 1(b). However, the criterion of triaxial states with orthogonal spins across the diagonals still leaves an extensive residual degeneracy due to the $\mathbb{Z}_2$ degrees of freedom. On the other hand, in the states obtained by the

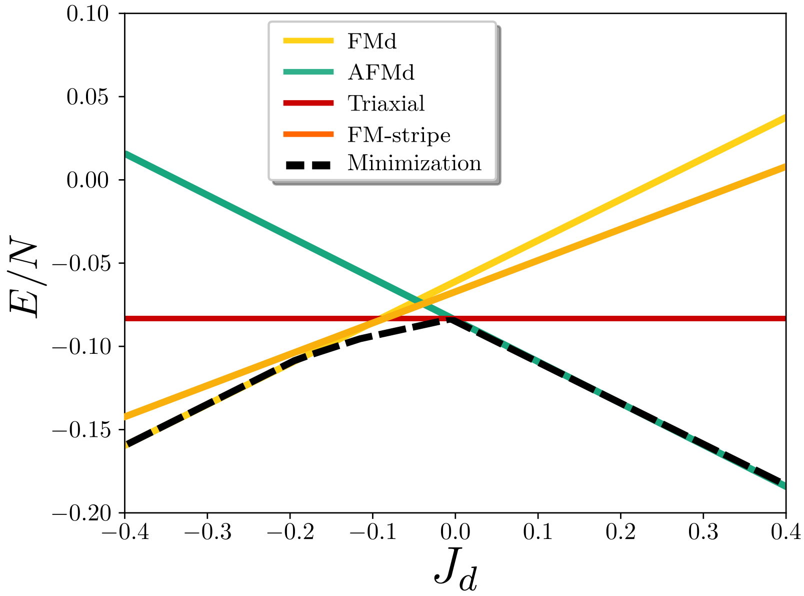


Figure 2: Ground state energy per site for the classically ordered phases: FMd, AFMd and FM-stripe for $J_1 \approx -0.02$ and $J_\chi = 1$. The energy of the triaxial state corresponds to minimizing the chiral term. The energy coming from the gradient descent (GD) minimization, Sec. 3, is also shown for comparison. Same color code as in Fig. 2

gradient descent minimization algorithm for small $J_d < 0$, such as the one illustrated in Fig. 3(f), the spins within the same unit cell remain approximately orthogonal to each other, but the directions of the axes vary in an apparently disordered fashion between different unit cells. Thus, they are not obviously related to triaxial states. While we have not been able to identify the local transformations that may connect these ground states, our numerical results strongly suggest that a massive classical ground state degeneracy persists in the regime of small $J_1$ and $J_d < 0$ up to some critical values beyond which the system enters the FMd or FM-stripe ordered phases.

We can make this argument more quantitative by calculating the classical ground-state energy per site. In Fig. 2 we show the energy for the AFMd, FMd, FM-stripe, and triaxial states—the latter for the pure chiral point—comparing them with the energy of the gradient descent minimization algorithm. In accordance with our qualitative analysis, for $J_d > 0$ the AFMd order is immediately selected out of the set of triaxial states. For $J_d < 0$, on the other hand, the ground state energy remains close to the energy of the triaxial state, and the FMd state is reached only at $J_d \approx -0.15$. Moreover, the energy difference between the FMd and FM-stripe phases is rather small in this region. In the interval $-0.15 \lesssim J_d < 0$, the structure factor displays no Bragg peaks and we interpret this disordered region as partially inheriting the extensive ground state degeneracy of the triaxial state.

## References

[1] J. Pitts, F. L. Buessen, R. Moessner, S. Trebst and K. Shtengel, *Order by disorder in classical kagome antiferromagnets with chiral interactions*, doi:10.48550/arXiv.2110.11427 (2021).

[2] A. B. Harris, C. Kallin and A. J. Berlinsky, *Possible néel orderings of the kagomé antiferromagnet*, Phys. Rev. B **45**, 2899 (1992), doi:10.1103/PhysRevB.45.2899.