Symmetry Protected Topological Criticality: Decorated Defect Construction, Signatures and Stability

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Symmetry protected topological (SPT) phases are one of the simplest, yet nontrivial, gapped systems that go beyond the Landau paradigm. In this work, we study an extension of the notion of SPT for critical systems, namely, symmetry protected topological criticality (SPTC). We introduce a systematic way of constructing a large class of SPTCs using decorated defect construction, and demonstrate the concept with several concrete lattice model examples. Furthermore, we study the physical observables that characterize the nontrivial topological signatures of SPTCs, and discuss the stability under symmetric perturbations. Our formulation not only provides new perspectives on several related preceding works [T. Scaffidi et al. Phys. Rev. X **7**, 041048 (2017), R. Verresen et al., arXiv:Phys. Rev. X **11**, 041059 (2021), R. Thorngren et al., Phys. Rev. B **104**, 075132 (2021)], but also leads to discovery of previously unknown types of SPTCs.

Contents

1 Introduction and Summary

1.1 Gapped Quantum Matter

The study of topological phases of quantum matter has led to tremendous progress in understanding quantum many body systems beyond the Landau paradigm. The gapped phases are so far relatively well understood. Based on their symmetry and entanglement properties, the gapped phases can be classified into the following categories [1]:

1. **Trivially gapped phase:** There is a single ground state on an arbitrary spatial manifold, and a finite energy gap from the first excited state in the thermodynamic limit. The ground state preserves the global symmetry, and can be deformed to the trivial product state through finite depth locally-symmetric unitary transformation without closing the energy gap. Its entanglement entropy obeys area law while the subleading contributions vanish in the thermodynamic limit. The ground state is short range entangled [1].

2. **Symmetry protected topological (SPT) phase:** Similarly to the trivially gapped phase, there is still a single ground state on an arbitrary closed spatial manifold and a finite energy gap from the first excited state in the thermodynamic limit. The ground state preserves the symmetry and is short-range entangled. The global symmetry should be anomaly free. However, unlike in the trivially gapped phase, when placing the system on a spatial manifold with nontrivial boundaries, due to the nontrivial physics appearing at the boundaries, there are either multiple ground states, or the energy spectrum becomes gapless in the thermodynamic limit. There is no finite depth locally-symmetric unitary transformation that maps the ground state to a trivial product state. [1]

[1]There are also exotic phases that do not require onsite unitary symmetries, but still satisfy the above properties,

3. **Topological ordered (TO) phases and symmetry enriched topological (SET) phases:** The low energy is described by a symmetric topological quantum field theory (TQFT). The number of ground states depends on the topology of the spatial manifold. In particular when the spatial manifold is S^d there is only one ground state. The ground states also have a finite energy gap from the first excited states in the thermodynamic limit. The entanglement entropy of the ground state has a constant contribution besides the area law part, which survives in the thermodynamic limit. This is termed topological entanglement entropy [7–10]. There are also nontrivial physics (e.g. gapless edge modes, spontaneous symmetry breaking or gapped TQFT) on the boundary when the spatial manifold is open. Finally, as the line operators (worldlines of anyons) are topological, they do not obey area law, and the theory is deconfined.

4. **Symmetry breaking phases:** There are multiple ground states even when the spatial manifold is S^d, due to spontaneous breaking of the global symmetry. These phases are within the Landau paradigm. There are also phases where the Landau symmetry breaking order and SPT/TO/SET orders coexist.

From the description above, it is clear that the SPT phase is the simplest, yet nontrivial, generalization of trivially gapped phase that goes beyond the Landau paradigm. We use gapped SPT phases to emphasize that the conventional SPT phases are for gapped systems.

1.2 Symmetry Protected Topological Criticality: Universal Features

In contrast to the gapped topological phases of quantum matter which are relatively well-understood, a systematic understanding of gapless quantum systems is still under development. See [11–20] for recent developments. When there are no symmetric relevant perturbations that drive the system away from the gapless point, such a gapless system represents a gapless phase which typically appears at infrared limit of the renormalization group flow. When there is at least one symmetric relevant perturbation that drives the system away from the gapless point, such a gapless system represents a (multi) critical point. In this work, we mainly focus on the latter situation. [2]

The main goal of this work is to construct analogues of SPT phases in gapless systems that appear at the critical points/regimes, which we denote as the symmetry protected topological criticality (SPTC), and to elucidate their topological properties. The notion of SPTC has been studied in several works recently under various terminologies [11–13, 21, 22]. Most of the examples studied in these works share the following common features:

i.e. no degeneracy on closed manifolds and nontrivial boundary physics. They include Kitaev's E_8 state in $2+1$d [2,3] and w_2w_3 theory in $4+1$d [4–6]. We also consider them as SPT phases where the symmetry is the spacetime diffeomorphism.

[2] Depending on the dynamical details, in some situations, the critical point may smear to a critical region, and the second situation becomes to the first situation. One such example is the $U(1)$ Dirac spin liquid, which spans over a finite region in the phase diagram while interpolating between Neel and VBS phases. For simplicity, we will schematically use criticality to include both cases.

1. The critical system has the global symmetry Γ. Γ should be anomaly free and is not spontaneously broken by the ground state under periodic boundary conditions.

2. When placing the system on an arbitrary spatial manifold with periodic boundary conditions, the system should have a non-degenerate ground state with a finite size bulk gap which decays polynomially with respect to the system size.

3. When placing the system on a spatial manifold with nontrivial boundaries, there are degenerate ground states with a finite size splitting decaying qualitatively faster (e.g. exponentially, or polynomially with a larger decaying constant) with respect to the system size.

4. When placing the system on a closed spatial manifold where the boundary conditions are twisted by the global symmetry Γ, a.k.a. twisted boundary conditions, the ground state carries nontrivial Γ symmetry charge.

5. The criticality is confined. In particular, if the criticality has a 1-form symmetry, it should not be spontaneously broken.

All the above properties hold for gapped SPT phases except that the bulk gap decays to zero in the thermodynamical limit, hence these properties should be the most natural ones that any candidate SPTC should share.[3] We will use a more concrete working definition of the SPTC using the decorated defect construction [11, 13, 24, 25] in the following sections. We will see that the SPTC has more subtle topological properties that are not shared with the gapped SPT.

We would like to justify the name SPTC, and compare it with the existing terminologies in [11–13]. [4] In [11], [12] and [13], gapless systems satisfying the above five properties are called "gapless symmetry protected topological phases", "symmetry enriched criticality" and "intrinsically gapless topological phases", respectively. In this work, we use SPTC instead for the following two reasons.

a. We use "criticality" in SPTC because our systems arise at the critical points or critical regimes with the codimension one or larger corresponding to symmetric gapping perturbations, and do not represent a gapless phase which is stable in a codimension zero region of the phase diagram under the symmetry. Hence we would like to avoid using "phase".

b. We use "SPT" in SPTC to emphasize the trivialness in the bulk: the global symmetry is anomaly free, and there is no ground state degeneracy under periodic boundary conditions. We would like to reserve symmetry enriched topological criticality (SETC) for the gapless generalization of SET phases, where the global symmetry is allowed to be anomalous and deconfinement is allowed to exist. See [23] for an example.

[3] We would like to comment that the fifth property is not implied by the first four. One example is the second order phase transition between a $(2+1)$d topological order and a trivially gapped phase. This system does not have any 0-form global symmetry and thus trivially satisfies the first four properties. Yet, as discussed in [23], this model has an emergent 1-form symmetry which is numerically demonstrated to be spontaneously broken, hence is deconfined. The fifth property is introduced to exclude this possibility.

[4] We sincerely thank N.Jones, R.Thorngren and R.Verresen for helpful discussions and comments.

However, SPTC does not mean that the critical point/regime can be gapped out by breaking the symmetry, but instead should be understood as the gapless generalization of the gapped SPT phases at the critical point/regime. In other words, SPTC means that the symmetry protects a certain topological characteristics of the quantum criticality, and not the gaplessness of the system. We will further comment on the relation between our critical systems and those discussed in [11–13] in section 1.3.

Before proceeding, let us clarify the conventions and simplifying assumptions throughout this work. We assume: (1) the spacetime dimension to be $d+1$, and spatial dimension to be d; (2) the global symmetry Γ is finite, unitary and onsite, and is of 0-form; (3) we only discuss bosonic systems. Generalizing to continuous symmetry, higher form symmetries, and fermionic theories should be interesting, and are left to future investigations.

1.3 Decorated Defect Construction

We sketch the idea of constructing SPTC using decorated defect construction. We will spell out the details by studying concrete examples in section 2, 3 and 4.

1.3.1 Constructing Gapped SPT

The decorated defect construction was first devised to systematically construct gapped SPT phases, starting from the known lower dimensional gapped SPTs [24–26]. Suppose one would like to construct a gapped SPT system with global symmetry Γ. Assume Γ fits into the symmetry extension

$$1 \to A \to \Gamma \to G \to 1 \tag{1.1}$$

where A is the normal subgroup of Γ, and $G := \Gamma/A$. For simplicity, we assume that the extension is central, i.e. G does not act on A.[5] One starts with a phase where G is spontaneously broken, and on each codimension p G-defect one decorates a $(d+1-p)$ dimensional gapped SPT protected by symmetry A (i.e. A gapped SPT). As we would like to eventually proliferate the G-defect network to restore the entire Γ symmetry, the decorations should be consistent such that G-defect of each codimension should be free of A-anomaly, and in particular, there are no gapless modes localized on G-defects. Otherwise, if there are nontrivial gapless degrees of freedom localized on the G-defects, proliferation would not yield a gapped phase with one ground state. After defect proliferation, the resulting theory is a gapped SPT protected by the Γ symmetry. The topological action of Γ gapped SPT is given by the Γ cocycle $\mathcal{F}^{\Gamma}_{d+1}$ which is a representative element in the cohomology group [27][6]

$$[\mathcal{F}^{\Gamma}_{d+1}] \in H^{d+1}(\Gamma, U(1)). \tag{1.2}$$

[5] The decorated defect construction of gapped SPTs was first discussed [24] in the special situation where the extension (1.1) is trivial, i.e. $\Gamma = A \times G$. The construction was later generalized to non-trivial extension (1.1) in [25].

[6] If Γ is a continuous symmetry, the cohomology group should be $H^{d+1}(B\Gamma, U(1))$ where $B\Gamma$ is the classifying space of Γ.

We remark that a given Γ can fit into multiple symmetry extensions with different pairs (A, G). For a given extension (A, G), as long as we exhaust all possible ways of decorating A gapped SPT on G-defects, proliferating the G-defects exhausts all possible Γ gapped SPTs. Hence different choices of (A, G) yield the same set of Γ gapped SPTs, and one can choose the most convenient pair (A, G).

1.3.2 Constructing SPTC

Let us proceed to construct the Γ symmetric SPTC by modifying the decorated defect construction reviewed in section 1.3.1. We still assume that the global symmetry Γ fits into the symmetry extension (1.1), and start with a gapped phase where G is spontaneously broken. On each codimension p G-defect, one decorates a $(d + 1 - p)$ dimensional A gapped SPT. We finally fluctuate the G-defect network to the critical point, and define the critical point to be the SPTC.

Comparing with the decorated defect construction of the gapped SPT, the construction of the SPTC has several important new features. As we no longer demand that fulling proliferating the G-defect network leads to a gapped SPT phase, the consistency condition for the decoration can be relaxed. We classify all possible decorations into two categories, and denote the resulting SPTC as <u>weak SPTC</u> and <u>strong SPTC</u> respectively.

1. **Weak SPTC:** The A gapped SPTs decorated on the G-defects satisfy the same consistency condition as those for constructing the gapped SPT. Concretely, the G-defect of each codimension is free of A anomaly. This means that further increasing the G-defect fluctuating strength leads to a Γ gapped SPT, and the weak SPTC is the phase transition between G spontaneously broken phase and Γ gapped SPT. In particular, when the extension (1.1) is trivial, i.e. $\Gamma = A \times G$, the construction was discussed in [11] and [12] where the authors denoted the transition point as gapless symmetry protected topological order and symmetry enriched quantum criticality, respectively. See the left panel of figure 1 for the schematic phase diagram of weak SPTC.

2. **Strong SPTC:** The A gapped SPT decorated on the G-defects satisfies only a weaker, modified consistency condition. Concretely, the symmetry breaking phase we started with has a particular anomaly of a particular (non-normal) subgroup $\widehat{\Gamma}$ of Γ, where $G \subset \widehat{\Gamma}$. The choice of $\widehat{\Gamma}$ and its anomaly should be considered as part of input data of the construction. The defect decoration is constrained such that the anomaly of $\widehat{\Gamma}$ in the G symmetry breaking phase is precisely cancelled against the anomaly induced by the defect decoration.[7] After decoration, the total symmetry group Γ is anomaly free, and fluctuating the G-defect network to the critical point yields a Γ anomaly free SPTC. We denote it as Γ strong SPTC. In particular, when $\widehat{\Gamma} = G$, the construction was discussed in [13] under the name of intrinsically gapless topological phase. On the other hand, we will discuss an example of a more

[7]The phenomenon of induced anomaly also appear in the discussion of anomalous-SPT [25, 28] and symmetry extended boundary of gapped SPT [29, 30].

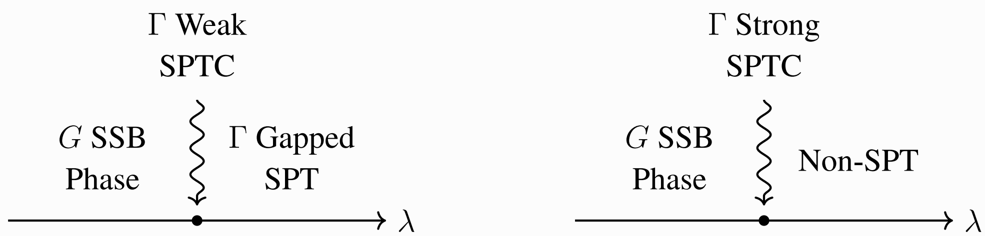

Figure 1: Phase diagram of weak and strong SPTC. The horizontal axis is the strength of G-defect fluctuation. For the weak SPTC (left panel), the G-defects can be fully proliferated and one obtains Γ gapped SPT. For the strong SPTC, one can only fluctuate the G-defects to the critical point. Further increase the fluctuation will not drive the system to Γ symmetric gapped SPT phase.

general class of strong SPTC in which $\widehat{\Gamma}$ is larger than G, in Section 4. See the right panel of figure 1 for the schematic phase diagram of strong SPTC.

It is natural to assume that the process of defect decoration and the process of G-defect fluctuation commute with each other. Then we may simplify the decorated defect construction by directly starting with a gapless critical system and decorating its G-defects. The gapless critical system is obtained by fluctuating the G-defects of the G symmetry breaking phase before decorating the A gapped SPTs, and from section 1.2 we require such critical system before decoration should have a non-degenerate ground state under periodic boundary condition, and is confined. [8] For the weak SPTC, we need to start with a critical point without any anomaly. While for strong SPTC, we need to start with a critical point with a particular $\widehat{\Gamma}$ anomaly.

As commented in section 1.3.1, for a given Γ, there can be multiple choices of the symmetry extension (1.1). We noticed that the gapped SPT can be constructed using arbitrary (A, G). However, this is no longer true for the strong SPTC. Note that one needs to specify an anomaly of $\widehat{\Gamma}$ (which includes G) as an input data of the decorated defect construction of strong SPTC. By definition, the resulting strong SPTC depends on the choice of symmetry extension (1.1), $\widehat{\Gamma}$ and the anomaly of $\widehat{\Gamma}$.

In this work, we will explore examples of weak and strong SPTCs and discuss their topological properties, and examine the stabilities under perturbations. We will find that in general, the strong SPTC is more stable than the weak SPTC as well as conventional Landau transition, in the sense that there may not be a relevant perturbation in strong SPTC that triggers the flow towards a Γ gapped SPT phase. This suggests the name strong vs weak. As this work mainly focuses on various examples, a more systematic discussion of the classification of weak and strong SPTC will be presented in a separate work [31].

[8] We will see in later sections that the defect decoration can be implemented by a unitary operation, which does not change the energy spectrum. This implies that the ground state degeneracy should be one both before and after defect decoration.

It is encouraging to note that the decorated defect construction allows us to study a large class of the “gapless SPT" in [11–13] in a systematic way, which were originally constructed via separate methods. As commented above, the decorated defect construction also points toward new examples that have not been covered before. See the example in section 4. We would also like to emphasize that [12] also discusses exotic critical systems (which have time reversal global symmetry and don't have gapped sectors) that go beyond the discussion of the present work.

1.4 Signatures and Stability of Weak and Strong SPTC

Given a gapless system with a non-degenerate ground state in the bulk with finite size, how can we tell whether it is a nontrivial SPTC? If it is nontrivial, how can we tell whether it is weak or strong SPTC? In this subsection, we propose two types of physical signatures that allow us to characterize the nontrivial SPTCs and to distinguish between weak and strong SPTC:

1. Degenerate ground states in the presence of boundaries.
2. Nontrivial symmetry charge of the ground state under the twisted boundary condition.[9]

It is well-known that these two signatures are useful in probing non-trivial gapped SPT phases [32–35]. The first signature is more limited in two aspects: (1) It is useful for $(1+1)$d systems [32, 33], but for higher dimensions the boundary is extensive and the degeneracy on the boundary depends on the boundary dynamics. For this reason, we restrict ourselves to $(1+1)$d systems when discussing the boundary degeneracy. (2) For a generic Hamiltonian respecting the symmetry, the ground states on a finite open chain are only quasi-degenerate with exponentially small splittings, instead of being exactly degenerate. This makes the identification of degenerate ground states subtle, especially in the gapless systems. While we can still separate the quasi-degenerate ground states with exponentially small finite-size excitation energies from gapless excitations with power-law finite-size excitation energies, the distinction can be challenging in practical numerical calculations.

On the other hand, the second signature is merely based on the global symmetry, hence (1) can be applied to arbitrary spacetime dimension, and (2) is expected to be stable and exact for a generic Hamiltonian in the given SPTC phase. This stability is also helpful for numerical calculations, as we will see later. See [36] for an application of twisted boundary condition to Lieb-Schultz-Mattis ingappability. Although the second signature seems to give a sharper probe, we still discuss both. In fact, both of these two signatures are useful in distinguishing the nontrivial SPTC against trivial ones. The first signature is relatively well-studied in $(1+1)$d gapless SPTs and has been discussed extensively in [11–13]. The second signature is less studied for gapless theories, and we will explore them through examples in detail. [10]

[9] We expect that this property also applies to low energy states as well. We checked this property in an example in section 3. Y.Z. thanks Jie Wang for helpful discussions on this point.

[10] During the final stage of this work, the authors realized that the second signature was also discussed in the updated version of [12].

“Weakness" of Weak SPTC: The phase diagram of the weak SPTC (left panel of figure 1) suggests that if we further stack a Γ gapped SPT on top of the Γ weak SPTC, the resulting critical point becomes a Landau-like transition since the two sides of the transition are Landau symmetry breaking phase and a trivially gapped phase. We will see this explicitly in section 2. This seems to suggest that both signatures of the Γ weak SPTC (boundary degeneracy and charge of the twisted sector ground state) are the same as the Γ gapped SPT. However, this is not entirely true. We will find in section 2 that the weak SPTC exhibits interesting boundary properties that distinguish it from the gapped SPTs. Concretely, if we place a $(1+1)$d Γ weak SPTC on an open chain, the number of ground states does not match that of Γ gapped SPT. This phenomenon has been discussed in [11, 12], where the authors attributed such mismatch of the ground state degeneracy to the nontrivial tunneling effect between the edge modes through the gapless bulk. We will also present an analytical justification of this phenomenon. On the other hand, the symmetry charge of the ground state in the twisted sector is the same for both Γ gapped SPT and Γ weak SPTC. This explains that the bulk topological properties of the Γ weak SPTC inherit from the Γ gapped SPT. Indeed there is a Γ symmetric (relevant) perturbation that drives the weak SPTC to the gapped SPT, which makes such critical point unstable upon infinitesimal perturbation toward gapped SPT. This explains the “weakness" of the weak SPTC.

Stability of Strong SPTC: From the phase diagram of strong SPTC (right panel of figure 1), we find that, by construction in section 1.3.2, further increasing the fluctuation strength of the G-defects does not drive the system to a Γ gapped SPT. However, as Γ is anomaly free, there can be another perturbation $h\mathcal{O}$ that drives the system to an invertible phase (which can be either the trivially gapped phase or the Γ gapped SPT), as long as h is large enough. In section 3, by inspecting a particular example which can be shown to be definitely in a strong SPTC, we find that the perturbed system is in the trivially gapped phase only when the perturbation strength passes a certain threshold $h = h_c$. h_c is finite and positive for a fixed system size L. Assuming that h_c does not decay to zero as $L \to \infty$, this suggests that the strong SPTC is stable and exists over a finite range of the parameter h.[11] The phase transition at $h = h_c$ can be probed by measuring the symmetry charge of the ground state either under the twisted boundary condition (i.e. the second signature) or under the periodic boundary condition, which jumps at $h = h_c$. [12] From the field theory point of view, $\mathcal{O}$ is an irrelevant operator when $h < h_c$, and becomes a relevant operator when $h \gg h_c$. This should be contrasted with the weak SPTC, where the symmetry charge of the ground state under the twisted boundary condition is unchanged as the strength of the perturbation increases, and is identical to that of the gapped SPT.

Weak vs Strong SPTC: We propose to use the second signature, i.e. symmetry charge under twisted boundary condition (TBC), to determine whether a given SPTC is weak or strong. For a

[11] In section 3, we check the behavior of h_c as L increases. Up to $L = 11$, we don’t see h_c decaying monotonically.

[12] In general, the charge of the ground state under PBC and TBC jump at different values of h. h_c is the smallest value for which the charge under either boundary condition jumps.

given SPTC, one can deform the theory to a nearby gapped SPT (possibly by a finite perturbation strength). If there are multiple choices of gapped SPT, one should deform the SPTC to all of them separately. One measures the symmetry charge of the ground state under TBC both for the SPTC and gapped SPT. If they happen to be the same, the SPTC should be weak. If the symmetry charges don't match for any choice of gapped SPT, the SPTC should be strong.

Most of the properties discussed in this subsection are obtained by studying concrete examples in the body of this work. We expect (without proving) that these properties are general features commonly shared by general models smoothly connected with the SPTCs. A systematic discussion for general symmetries and dimensions will be presented separately [31].

1.5 Organization of the Paper

This paper is organized as follows. In section 2, we discuss in detail an analytically tractable example of weak SPTC, where $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2, A = \mathbb{Z}_2, G = \mathbb{Z}_2$ and the spacetime dimension is $d = 1 + 1$. In section 3, we discuss in detail an analytically tractable example of strong SPTC, where $\Gamma = Z_4, A = \mathbb{Z}_2, G = \widehat{\Gamma} = \mathbb{Z}_2$ and $d = 1 + 1$. We discuss a more realistic spin-1 model in section 4, which hosts both weak and strong SPTC simultaneously. There are several appendices. Appendix A shows the stability of boundary degeneracy of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gapped SPT. Appendices B, C and D are devoted to further detailed discussions in section 3. Appendix E discusses an example of strong SPTC which involves time reversal symmetry.

2 Weak SPTC: $(1 + 1)$d Spin Chains With $\mathbb{Z}_2 \times \mathbb{Z}_2$ Symmetry

In this section, we study a concrete lattice model of weak SPTC: $(1 + 1)$d spin chain with global symmetry $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. We let $A = \mathbb{Z}_2, G = \mathbb{Z}_2$, and the symmetry extension in (1.1) is trivial. For clarity, we use the superscript A and G to label the two $\mathbb{Z}_2$'s.

2.1 Spin Chain Construction

We construct the $1 + 1$d spin chain with $\Gamma = \mathbb{Z}_2^A \times \mathbb{Z}_2^G$ global symmetry. Since there are two $\mathbb{Z}_2$ symmetries, it is natural to assign two spin-$\frac{1}{2}$'s per unit cell: the spin-$\frac{1}{2}$'s living on the sites are charged under $\mathbb{Z}_2^G$ while those living in between the sites are charged under $\mathbb{Z}_2^A$. The symmetry operators are defined to be

$$U_A = \prod_{i=1}^{L} \tau^x_{i+\frac{1}{2}}, \qquad U_G = \prod_{i=1}^{L} \sigma^x_i \tag{2.1}$$

where σ^a_i and $\tau^a_{i+\frac{1}{2}}$, $a = x, y, z$, are Pauli matrices acting on the two spin-$\frac{1}{2}$'s, and L is the number of unit cells. Both symmetry operators are on-site[13] and therefore Γ is anomaly free. As

[13] A symmetry operator is on-site if it can be written as a product of local operators on mutually adjacent but un-overlapping patches, $U = \prod_i U_i$, where i labels the patches.

explained in the introduction, we would like to start with a $\mathbb{Z}_2^G$ spontaneously broken phase, with the Hamiltonian

$$H_0 = -\sum_{i=1}^{L} \tau^x_{i+\frac{1}{2}} + \sigma^z_i \sigma^z_{i+1}. \tag{2.2}$$

It has two ground states

$$|\pm\rangle = \sum_{\{\tau^z_{i+\frac{1}{2}}\}} |\{\tau^z_{i+\frac{1}{2}}\}, \{\sigma^z_i = \pm 1\}\rangle. \tag{2.3}$$

Each of them spontaneously breaks $\mathbb{Z}_2^G$ but preserves $\mathbb{Z}_2^A$.

2.1.1 Domain Wall Decoration

To construct a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC, we decorate each $\mathbb{Z}_2^G$ domain wall by a $0+1$d $\mathbb{Z}_2^A$ SPT in a consistent way.[14] Each $\mathbb{Z}_2^G$ domain wall is associated with a $\mathbb{Z}_2^G$ group element g. $g = 0, 1$ means the domain wall is trivial/nontrivial, i.e. the adjacent σ^z spin configurations are the same/opposite, respectively. We present the domain wall configuration using both the spacetime picture and the Hamiltonian picture.

The Spacetime Picture: It is useful to first discuss the domain wall in the spacetime picture. The spacetime is triangulated into 2-simplices. See figure 2 for an illustration. Each site i is assigned a $\mathbb{Z}_2$ group element $s_i = 0, 1$, which corresponds to $\sigma^z_i = (-1)^{s_i}$ in the Hamiltonian picture. Each link is assigned a $\mathbb{Z}_2$ 1-cochain $g_{ij} = s_j - s_i$. The g_{ij} is understood as a flat background field for the $\mathbb{Z}_2^G$ symmetry, and it measures the local domain wall excitation on the link. The locus where $g_{ij} = 1$ form a closed loop $[g]$ in the dual spacetime lattice, representing the worldline of the domain wall, a.k.a. the $\mathbb{Z}_2^G$ symmetry defect line. Decorating the $\mathbb{Z}_2^G$ domain wall by a 1d $\mathbb{Z}_2^A$ SPT [11, 24] means that we insert a $\mathbb{Z}_2^A$ Wilson line, a.k.a. 1d $\mathbb{Z}_2^A$ SPT, supported on $[g]$

$$\exp\left(i\pi \int_{[g]} a\right) = \exp\left(i\pi \int_{M_2} a \cup g\right) \tag{2.4}$$

in the path integral. The flatness of the $\mathbb{Z}_2^A$ background field a ensures that the decoration is consistent: the domain wall junctions do not have $\mathbb{Z}_2^A$ anomaly. This fits into the construction of weak SPTC mentioned in section 1.3.2. The equality in (2.4) used the Poincare duality to transform the integral on $[g]$ into the integral over the entire 2d spacetime M_2. The topological term on the right hand side of (2.4) is precisely the effective action of $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT.

[14] In $(1+1)$d, we only have codimension 1 defects, i.e. the domain walls. For this reason, the decorated defect construction is more conventionally called the decorated domain wall construction.

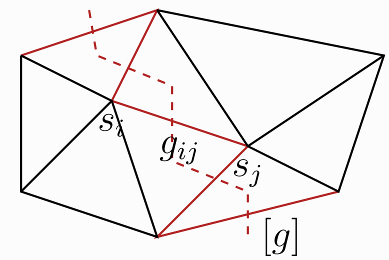

Figure 2: Triangulation of 2d spacetime. The black and red solid links are where the background field $g_{ij} = 0, 1$ respectively. The red dashed line in the dual lattice is the spacetime trajectory of the $\mathbb{Z}_2$ domain wall $[g]$, i.e. $\mathbb{Z}_2$ symmetry defect line. Flatness of g ensures that $[g]$ forms loops.

The Hamiltonian Picture: In the Hamiltonian picture, domain wall decoration is implemented as follows [11]. We first identify the configuration representing the $\mathbb{Z}_2^G$ domain wall, i.e. $\sigma_i^z \sigma_{i+1}^z = -1$. Then on the link $(i, i+1)$, we stack a $\mathbb{Z}_2^A$ SPT (2.4), which assigns the wavefunction a minus sign if $\tau^z_{i+\frac{1}{2}} = -1$ (i.e. $a_{i,i+1} = 1$ in the spacetime picture) on the wall. Combining the two steps, one assigns a minus sign to the two configurations $(\sigma_i^z, \tau^z_{i+\frac{1}{2}}, \sigma^z_{i+1}) = (1, -1, -1), (-1, -1, 1)$ and leaves the wavefunction unchanged for other configurations. This operation can be realized by acting the unitary operator

$$U_{DW} = \prod_{i=1}^{L} \exp\left[\frac{\pi i}{4}(1 - \sigma_i^z)(1 - \tau^z_{i+\frac{1}{2}})\right] \exp\left[\frac{\pi i}{4}(1 - \sigma^z_{i+1})(1 - \tau^z_{i+\frac{1}{2}})\right] \tag{2.5}$$

on the states (2.3) [11]. In terms of the Hamiltonian, domain wall decoration just amounts to conjugating the original Hamiltonian (2.2) by U_{DW}, yielding

$$H_1 := U_{DW} H_0 U_{DW}^\dagger = -\sum_{i=1}^{L} (\sigma_i^z \tau^x_{i+\frac{1}{2}} \sigma^z_{i+1} + \sigma_i^z \sigma^z_{i+1}). \tag{2.6}$$

The ground states of H_1 are still (2.3), but the first excited states associated with the domain wall excitations are decorated.

2.1.2 $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Weak SPTC

The next step is to fluctuate the decorated domain walls. It is helpful to discuss the fluctuation without decoration first. The fluctuation is well-known to be achieved by adding a transverse field $\Delta H = -\lambda \sum_{i=1}^{L} \sigma_i^x$, so that the $\mathbb{Z}_2^G$ spontaneously broken ferromagnetic phase of the Ising model (when $\lambda < 1$) is driven to the $\mathbb{Z}_2^G$ preserving paramagnetic phase (when $\lambda > 1$) where the domain walls are fully proliferated. The transition happens at $\lambda = 1$, which is of second order, and is described by a critical Ising CFT.

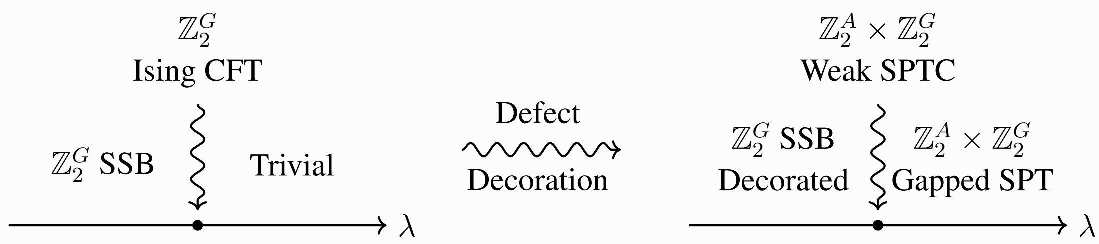

Figure 3: Phase diagram of $\mathbb{Z}_2^G$ Ising CFT (before decoration) and $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC (after decoration). The horizontal axis represents the transverse field λ.

After domain wall decoration, the fluctuation should be realized by adding a decorated transverse field $U_{DW} \Delta H U_{DW}^\dagger = -\lambda \sum_{i=1}^{L} \tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}}$. As the unitary transformation U_{DW} does not change the energy spectrum, the critical point also takes place at $\lambda = 1$. The decorated model $U_{DW}(H_0 + \Delta H)U_{DW}^\dagger$ at $\lambda = 1$, is the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC [11] (see also section 1.3.2 for the definition of weak SPTC)

$$H_{\text{WeakSPTC}} = -\sum_{i=1}^{L} \left(\sigma^z_i \tau^x_{i+\frac{1}{2}} \sigma^z_{i+1} + \sigma^z_i \sigma^z_{i+1} + \tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}} \right). \tag{2.7}$$

When $\lambda > 1$, the domain wall is fully proliferated, yielding a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT described by the well-known cluster model [37–39]

$$H_{\text{SPT}} = -\sum_{i=1}^{L} \left(\sigma^z_i \tau^x_{i+\frac{1}{2}} \sigma^z_{i+1} + \tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}} \right). \tag{2.8}$$

See figure 3 for the phase diagram before and after decoration.

As commented in section 1.3.2, we can simplify the above construction of weak SPTC by directly starting with the $\mathbb{Z}_2^G$ Ising CFT (whose Hamiltonian is given by $H_0 - \sum_{i=1}^{L} \sigma^x_i$), and conjugate it by U_{DW}. This simplification will be useful in section 3.

2.1.3 More On U_{DW}

We make a remark on the unitary operator U_{DW}. Although H_{WeakSPTC} and $H_0 - \sum_{i=1}^{L} \sigma^x_i$ are related through a unitary transformation U_{DW}, they are actually not equivalent as the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ symmetric Hamiltonians. Recall that two Γ symmetric Hamiltonians H_1, H_2 are considered equivalent if there is a locally-symmetric unitary transformation $U = \exp(i \int_{t_0}^{t_1} dt V(t))$ where $V(t)$ is a sum of local operators satisfying $[V(t), \Gamma] = 0$, such that $U H_1 U^\dagger = H_2$ [1, 27]. Since U_{DW} is a product of local unitary operators and each of them only acts on one or two unit cells, U_{DW} is a local unitary transformation. Moreover, U_{DW} on a closed chain with the periodic boundary condition is symmetric in the sense that $[U_{DW}, \Gamma] = 0$. Nevertheless, as each local operator $\exp(\frac{\pi i}{4}(1 - \sigma^z_i)(1 - \tau^z_{i+\frac{1}{2}}))$ does not commute with U_A and U_G, U_{DW} is not a locally-symmetric unitary transformation. As an indication, U_{DW} does not commute with the symmetry generator Γ on an open chain, in contrast to the closed chain discussed above. In summary, H_{SPTC} and $H_0 - \sum_{i=1}^{L} \sigma_i^x$ are not related by $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ locally-symmetric unitary transformation, hence they are not equivalent as $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ symmetric systems. This also justifies that the SPTC is protected by the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$.

It is interesting to compare U_{DW} with the Kennedy-Tasaki (KT) transformation [40–42] introduced for integer-spin chains. Although the KT transformation is also implemented by a unitary operator U_{KT}, there are several differences. First, U_{KT} is non-local, unlike U_{DW} which is as discussed above a product of local unitary operators. Second, the KT transformation is useful for an open chain rather than for a closed chain, which is mapped to a non-local Hamiltonian by U_{KT}. Lastly, it maps a gapped SPT phase (on an open chain) to an SSB phase, while U_{DW} maps a gapped SPT phase to a trivially gapped phase. The KT transformation will be relevant for the discussion in Section 4.

2.2 Trivializability Upon Stacking Gapped SPTs

We show that upon stacking a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT, the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC is equivalent to $\mathbb{Z}_2^G$ Ising criticality via a symmetric local unitary transformation. This partially justifies the adjective weak in its name.

Let us consider two decoupled systems. The first system is a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC given by (2.7). The second system is a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT given by (2.8). Since two systems are decoupled, the two Hamiltonians act on decoupled Hilbert spaces. We use the Pauli operators $\{\sigma_i^a, \tau_{i+\frac{1}{2}}^a\}$ for the first system, and $\{\widetilde{\sigma}_i^a, \widetilde{\tau}_{i+\frac{1}{2}}^a\}$ for the second system. The Hamiltonian for the entire system is the sum

$$H_{\text{WeakSPTC}} + H_{\text{SPT}} = -\sum_{i=1}^{L} \left(\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \sigma_i^z \sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z + \widetilde{\sigma}_i^z \widetilde{\tau}_{i+\frac{1}{2}}^x \widetilde{\sigma}_{i+1}^z + \widetilde{\tau}_{i-\frac{1}{2}}^z \widetilde{\sigma}_i^x \widetilde{\sigma}_{i+\frac{1}{2}}^z \right). \tag{2.9}$$

The decoupled system has enlarged global symmetry $(\mathbb{Z}_2^A \times \mathbb{Z}_2^G) \times (\widetilde{\mathbb{Z}}_2^A \times \widetilde{\mathbb{Z}}_2^G)$, whose generators are

$$U_A = \prod_{i=1}^{L} \tau_{i+\frac{1}{2}}^x, \quad U_G = \prod_{i=1}^{L} \sigma_i^x, \quad \widetilde{U}_A = \prod_{i=1}^{L} \widetilde{\tau}_{i+\frac{1}{2}}^x, \quad \widetilde{U}_G = \prod_{i=1}^{L} \widetilde{\sigma}_i^x. \tag{2.10}$$

There exists a symmetric local unitary transformation[15]

$$U_{\text{diag}} = \prod_{i=1}^{L} \exp\left(\frac{i\pi}{4}(1 - \sigma_i^z \widetilde{\sigma}_{i+1}^z)(1 - \tau_{i+\frac{1}{2}}^z \widetilde{\tau}_{i+\frac{1}{2}}^z)\right) \exp\left(\frac{i\pi}{4}(1 - \sigma_{i+1}^z \widetilde{\sigma}_{i+1}^z)(1 - \tau_{i+\frac{1}{2}}^z \widetilde{\tau}_{i+\frac{3}{2}}^z)\right) \tag{2.11}$$

[15]Without multiplying over i, each exponent in U_{diag} commutes with the diagonal symmetries $U_A \widetilde{U}_A$ as well as $U_G \widetilde{U}_G$. As discussed in section 2.1.3, this implies that U_{diag} is a symmetric local unitary transformation, which establishes the equivalence between different systems.

which (locally) preserves the diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$, where two $\mathbb{Z}_2$'s are generated by $U_A \widetilde{U}_A$ and $U_G \widetilde{U}_G$ respectively. It is straightforward to check that

$$U_{\text{diag}}(H_{\text{WeakSPTC}} + H_{\text{SPT}})U_{\text{diag}}^{\dagger} = -\sum_{i=1}^{L} \left(\tau^x_{i+\frac{1}{2}} + \sigma^z_i \sigma^z_{i+1} + \sigma^x_i + \widetilde{\tau}^x_{i+\frac{1}{2}} + \widetilde{\sigma}^x_i \right) \tag{2.12}$$

which is simply the Hamiltonian of the Ising CFT, a.k.a. the $\mathbb{Z}_2^G$ Landau transition, stacked with some trivially gapped degrees of freedom. In summary, we have shown that upon stacking a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT, the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC (2.7) is related to an ordinary $\mathbb{Z}_2^G$ Landau transition by a symmetric local unitary transformation. The above equivalence can be schematically represented as

$$\mathbb{Z}_2^A \times \mathbb{Z}_2^G \text{ weak SPTC } \oplus \mathbb{Z}_2^A \times \mathbb{Z}_2^G \text{ gapped SPT } \longleftrightarrow \mathbb{Z}_2^G \text{ Landau Transition.} \tag{2.13}$$

This implies that the nontrivial topological properties of the weak SPTC in the bulk (such as nontrivial charge of the ground state under the twisted boundary condition, see section 2.3.2) are basically inherited from the gapped SPT sector. However, we will find in section 2.3.3 that the boundary properties of the weak SPTC differ from those of the gapped SPT.

2.3 Signatures of $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Weak SPTC

We discuss the physical signatures of the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC (2.7) that allow one to distinguish trivial vs nontrivial weak SPTCs. As motivated in the introduction (see section 1.4), we will consider the ground state degeneracy under open boundary condition (OBC), as well as the symmetry charge of the ground state under twisted boundary condition (TBC). We summarize the main properties in table 1.

2.3.1 Periodic Boundary Condition

On a finite chain with periodic boundary condition (PBC), the ground state of the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC is non-degenerate. To see this, we first consider the Ising CFT described by the Hamiltonian $H_0 - \sum_{i=1}^{L} \sigma^x_i$. It is well-known that the critical Ising model has only one ground state on a finite chain, and the first excited state is separated from the ground state by a finite size gap decaying polynomially with respect to the system size. The non-degenerate ground state preserves the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ global symmetry. Moreover, as noted in section 2.1.2, H_{WeakSPTC} and the Ising CFT have exactly the same energy eigenvalues because they are related via a unitary transformation U_{DW}, which implies that H_{WeakSPTC} also has a non-degenerate ground state on a finite closed chain, with a finite size gap, and is $\mathbb{Z}_2^G \times \mathbb{Z}_2^A$ symmetric under PBC.

2.3.2 Twisted Boundary Condition

We show that on a closed chain with boundary condition twisted by $\mathbb{Z}_2^A$ (or $\mathbb{Z}_2^G$), the ground state of the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC carries nontrivial symmetry charges under $\mathbb{Z}_2^G$ (or $\mathbb{Z}_2^A$) respectively.

		$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Weak SPTC	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Landau Transition	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Gapped SPT
PBC:	GSD	1	1	1
	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Charge	$(0,0)$	$(0,0)$	$(0,0)$
$\mathbb{Z}_2^A$-TBC:	GSD	1	1	1
	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Charge	$(0,1)$	$(0,0)$	$(0,1)$
$\mathbb{Z}_2^G$-TBC:	GSD	1	1	1
	$\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Charge	$(1,0)$	$(0,0)$	$(1,0)$
OBC:	GSD	$4 \to 2$	1	4

Table 1: Ground state degeneracy and symmetry charges of the ground state under PBC, TBC and OBC. $\mathbb{Z}_2^A$ (or $\mathbb{Z}_2^G$)-TBC means the boundary condition is twisted by $\mathbb{Z}_2^A$ (or $\mathbb{Z}_2^G$). We compare these properties between weak SPTC, Landau transition and gapped SPT, all with the same global symmetry $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$. The $4 \to 2$ means that H_{WeakSPTC} has four ground states under OBC, but two of them are lifted under a symmetric perturbation localized on the boundary.

The same idea has been widely used to characterize nontrivial gapped SPT order [34, 35, 43–49], and here we used it to characterize the weak SPTC (and also strong SPTC in section 3).

Twist By $\mathbb{Z}_2^A$: We first twist the boundary condition using the $\mathbb{Z}_2^A$ symmetry (labeled by $\mathbb{Z}_2^A$-TBC), and measure the $\mathbb{Z}_2^G$ charge of the ground state. Twisting the boundary condition by $\mathbb{Z}_2^A$ means imposing a $\mathbb{Z}_2^A$ domain wall between sites $L-\frac{1}{2}$ and $L+\frac{1}{2}$ by changing the sign of the term $\tau^z_{L-\frac{1}{2}}\sigma^x_L\tau^z_{L+\frac{1}{2}}$. The weak SPTC Hamiltonian (2.7) becomes

$$H^{\mathbb{Z}_2^A}_{\text{WeakSPTC}} = -\sum_{i=1}^{L-1}\left(\sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1} + \sigma^z_i\sigma^z_{i+1} + \tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}}\right) - \sigma^z_L\tau^x_{L+\frac{1}{2}}\sigma^z_1 - \sigma^z_L\sigma^z_1 + \tau^z_{L-\frac{1}{2}}\sigma^x_L\tau^z_{L+\frac{1}{2}}. \tag{2.14}$$

It is useful to note that the twisted and untwisted SPTC Hamiltonian are related by a unitary transformation $H^{\mathbb{Z}_2^A}_{\text{WeakSPTC}} = \sigma^z_L H_{\text{WeakSPTC}}\sigma^z_L$, hence the ground state of $H^{\mathbb{Z}_2^A}_{\text{WeakSPTC}}$ is also non-degenerate. Denote the ground state under PBC as $|\text{GS}\rangle$, and that under $\mathbb{Z}_2^A$-TBC as $|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}}$. We have

$$|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} = \sigma^z_L\,|\text{GS}\rangle\,. \tag{2.15}$$

It follows that

$$U_G\,|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} = U_G\sigma^z_L U_G^\dagger U_G\,|\text{GS}\rangle = -\sigma^z_L\,|\text{GS}\rangle = -\,|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} \tag{2.16}$$

which shows that $|\text{GS}\rangle^{\mathbb{Z}_2^G}_{\text{tw}}$ has $\mathbb{Z}_2^G$ charge 1. [16]

[16] We used the fact that the ground state under PBC is neutral under $\mathbb{Z}_2^G$. More precisely, (2.16) only shows the relative charge, i.e. the $\mathbb{Z}_2^G$ charge of the ground state under TBC minus that under PBC, is one. The relative charge will be useful in section 3.

Twist By $\mathbb{Z}_2^G$: We can alternatively twist the boundary condition using $\mathbb{Z}_2^G$ symmetry (labeled by $\mathbb{Z}_2^G$-TBC), and measure the $\mathbb{Z}_2^A$ charge of the ground state. Twisting the boundary condition by $\mathbb{Z}_2^G$ means imposing a $\mathbb{Z}_2^G$ domain wall on the link between L-th and 1st sites, by changing the sign of the terms $\sigma_L^z\sigma_1^z$ and $\sigma_L^z\tau_{L+\frac{1}{2}}^x\sigma_1^z$. The weak SPTC Hamiltonian (2.7) becomes

$$H_{\text{WeakSPTC}}^{\mathbb{Z}_2^G} = -\sum_{i=1}^{L-1}\left(\sigma_i^z\tau_{i+\frac{1}{2}}^x\sigma_{i+1}^z + \sigma_i^z\sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^z\sigma_i^x\tau_{i+\frac{1}{2}}^z\right) + \sigma_L^z\tau_{L+\frac{1}{2}}^x\sigma_1^z + \sigma_L^z\sigma_1^z - \tau_{L-\frac{1}{2}}^z\sigma_L^x\tau_{L+\frac{1}{2}}^z. \tag{2.17}$$

Note that $\sigma_i^z\tau_{i+\frac{1}{2}}^x\sigma_{i+1}^z$ commutes with every term in $H_{\text{WeakSPTC}}^{\mathbb{Z}_2^G}$, the ground state $|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G}$ should be its eigen-vector

$$\sigma_i^z\tau_{i+\frac{1}{2}}^x\sigma_{i+1}^z\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G} = U_{DW}^\dagger\tau_{i+\frac{1}{2}}^x U_{DW}\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G} = \begin{cases} |\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G}, & i = 1, ..., L-1 \\ -\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G}, & i = L. \end{cases} \tag{2.18}$$

Consequently, the ground state has $\mathbb{Z}_2^A$ charge 1:

$$U_A\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G} = \prod_{i=1}^{L}\tau_{i+\frac{1}{2}}^x\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G} = -\prod_{i=1}^{L}(\sigma_i^z\sigma_{i+1}^z)\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G} = -\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^G}. \tag{2.19}$$

In summary, we find that when we use $\mathbb{Z}_2^{A,G}$ to twist the boundary condition on a closed chain, the ground state of the twisted Hamiltonian has nontrivial $\mathbb{Z}_2^{G,A}$ charge. This is the property distinguished from the $\mathbb{Z}_2^A\times\mathbb{Z}_2^G$ Landau transition, where its ground state under the twisted boundary conditions does not carry any nontrivial symmetry charge. This tells us that we can use the symmetry charge of the ground state in the twisted sector as a topological invariant to distinguish the nontrivial weak SPTC from trivial weak SPTC (e.g. second order Landau transition). On the other hand, the symmetry charges under TBC coincide with those of the gapped SPT. We summarize the results in table 1.

2.3.3 Open Boundary Condition

As the nontrivial boundary modes protected by the global symmetry is a signature of gapped SPT, we will find that same is true for the weak SPTC. We use the symmetry to analytically show that the ground states of H_{WeakSPTC} have to be exactly degenerate under OBC, but the number of degeneracy differs from the gapped SPT. However, it should be noted that the exact degeneracy of the ground states in a finite open chain is a special property of the particular Hamiltonian H_{WeakSPTC}. In section 2.4, we further discuss perturbations in the bulk, and we show that the exact degeneracy is lifted, with an exponentially small splitting. This implies that the ground states of a generic Hamiltonian which belong to the weak SPTC are only quasi-degenerate in a finite open chain.

We place the spin system on an open chain. The left most spin is the σ spin, and the right most spin is the τ spin. The σ spins are supported on $i = 1, ..., L$, and the τ spins are supported on

$i+\frac{1}{2}=\frac{3}{2},...,L+\frac{1}{2}$. We first choose the OBC such that only the interactions completely supported on the chain are kept. The Hamiltonian is

$$H^{\text{OBC}}_{\text{WeakSPTC}} = -\sum_{i=1}^{L-1}\left(\sigma_i^z\tau_{i+\frac{1}{2}}^x\sigma_{i+1}^z + \sigma_i^z\sigma_{i+1}^z\right) - \sum_{i=2}^{L}\tau_{i-\frac{1}{2}}^z\sigma_i^x\tau_{i+\frac{1}{2}}^z \tag{2.20}$$

and the symmetry operators are

$$U_A = \prod_{i=1}^{L}\tau_{i+\frac{1}{2}}^x, \qquad U_G = \prod_{i=1}^{L}\sigma_i^x. \tag{2.21}$$

We find that the set of operators $\{\sigma_1^z, \tau_{L+\frac{1}{2}}^z, \sigma_L^z\tau_{L+\frac{1}{2}}^x, U_A, U_G\}$ all commute with the Hamiltonian, hence the ground state degeneracy must be at least the dimension of its irreducible representation. To find the representation, we choose the maximally commuting subset of operators as $\{\sigma_1^z, \tau_{L+\frac{1}{2}}^z\}$, and denote their eigenvalue of a particular ground state $|\psi\rangle$ by (a, b), where $a, b = \pm 1$. It is then possible to generate other ground states with different quantum numbers as follows:

	σ_1^z	$\tau_{L+\frac{1}{2}}^z$
$\|\psi\rangle$	a	b
$U_A\|\psi\rangle$	a	$-b$
$U_G\|\psi\rangle$	$-a$	b
$U_AU_G\|\psi\rangle$	$-a$	$-b$

(2.22)

This shows that there must be at least four exactly degenerate ground states of $H^{\text{OBC}}_{\text{WeakSPTC}}$ of four different sets of quantum numbers. Numerical exact diagonalization confirms that the ground state degeneracy is exactly four.

However, symmetry does not forbid us to perturb (2.20) by adding symmetric boundary terms. We can add a boundary interaction

$$\Delta H^{\text{OBC}}_{\text{WeakSPTC}} = -\tau_{L+\frac{1}{2}}^x \tag{2.23}$$

which changes the original OBC to a new OBC. This interaction does not commute with $\tau_{L+\frac{1}{2}}^z$, so the set of operators commuting with the Hamiltonian $H^{\text{OBC}}_{\text{WeakSPTC}} + \Delta H^{\text{OBC}}_{\text{WeakSPTC}}$ reduces to $\{\sigma_1^z, \sigma_L^z\tau_{L+\frac{1}{2}}^x, U_A, U_G\}$. As a consequence, the dimension of irreducible representation reduces from four to two. Indeed, numerical exact diagonalization confirms that there are only two exactly degenerate ground states under the new OBC. This degeneracy splitting was already noted in [11, 12]. Here, we provide a simple analytical argument of this splitting by finding the representation. In appendix A, we show that arbitrary finite range perturbation does not lift the 4-fold exact degeneracy of the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT.

2.4 Stability of Weak SPTC

In this section, we discuss the stability of the weak SPTC by examining the signatures found in section 2.3 under the perturbations in the bulk as well as on the boundary.

2.4.1 Open Boundary Condition: Exact Degeneracy Lifted

In section 2.3.3, we found that the Hamiltonian (2.20) together with the boundary perturbation (2.23) has two exact degenerate ground states under OBC. However, the question remains if such an exact degeneracy is a generic feature of the (weak) SPTC. To clarify the issue, let us add a $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ symmetric perturbation in the bulk

$$V = -h \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x. \tag{2.24}$$

where for definiteness we take $h > 0$. After turning on V, one can not find local boundary operators which commute with the Hamiltonian, and the discussion in section 2.3.3 does not imply exact degeneracy. Indeed, by exact diagonalization, we find that the exact two fold degeneracy is lifted. It is important to observe that the splitting due to the perturbation decays exponentially with respect to L. One can see this from the degenerate perturbation theory. Denote the two degenerate ground states in the unperturbed theory as $|+\rangle$ and $|-\rangle$, which satisfy $\langle +| \prod_{i=1}^L \sigma_i^x |-\rangle = 1$. The degenerate perturbation theory tells us that the splitting between the two lowest states is of the order of the matrix element

$$\langle +| V^p |-\rangle \tag{2.25}$$

where p is the order of the perturbation. We will consider the smallest p for which the matrix element is non-vanishing. Since $\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z |\pm\rangle = |\pm\rangle$ for $i = 1, ..., L$, the matrix element vanishes unless V^p contains the symmetry operator $\prod_{i=1}^L \sigma_i^x$. This is possible only when $p = L/2$, which implies that (2.25) is of order $h^{L/2}$.[17] When $h \ll 1$, the energy split decays exponentially as $\sim e^{-L\log(1/h)/2}$. This makes it possible to distinguish the quasi-degenerate ground states from gapless excitations in the finite open chains, by looking at the scaling of their energy eigenvalues measured from the finite-size ground states.

2.4.2 Twisted Boundary Condition

Does the lift of exact two fold degeneracy found in section 2.4.1 imply instability of the weak SPTC under the perturbation V, or that exact boundary degeneracy is not a good criteria of weak SPTC but weak SPTC is still stable? We use the TBC to determine whether the weak SPTC is stable under V.

[17] For instance, when $p = 1$, we have a term $\langle +|\sigma_i^x \sigma_{i+1}^x |-\rangle$ where $i = 1, ..., L-1$. By using $\sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z |\pm\rangle = |\pm\rangle$, we find $\langle +|\sigma_i^x \sigma_{i+1}^x |-\rangle = \langle +|\sigma_{i-1}^z \tau_{i-\frac{1}{2}}^x \sigma_i^z \sigma_i^x \sigma_{i+1}^x |-\rangle = -\langle +|\sigma_i^x \sigma_{i+1}^x \sigma_{i-1}^z \tau_{i-\frac{1}{2}}^x \sigma_i^z |-\rangle = -\langle +|\sigma_i^x \sigma_{i+1}^x |-\rangle$ which shows that $\langle +|\sigma_i^x \sigma_{i+1}^x |-\rangle = 0$, and hence $\langle +|V|-\rangle = 0$. Similar argument also applies $p < L/2$. When $p = L/2$, there is a nonvanishing term $\langle +| \prod_{i=1}^L \sigma_i^x |-\rangle = 1$ in the expansion of (2.25), hence the degeneracy is lifted at $p = L/2$-th order perturbation. When L is odd, one can modify the perturbation (2.24) by further adding $-h\sigma_L^x$, and one can repeat the same discussion.

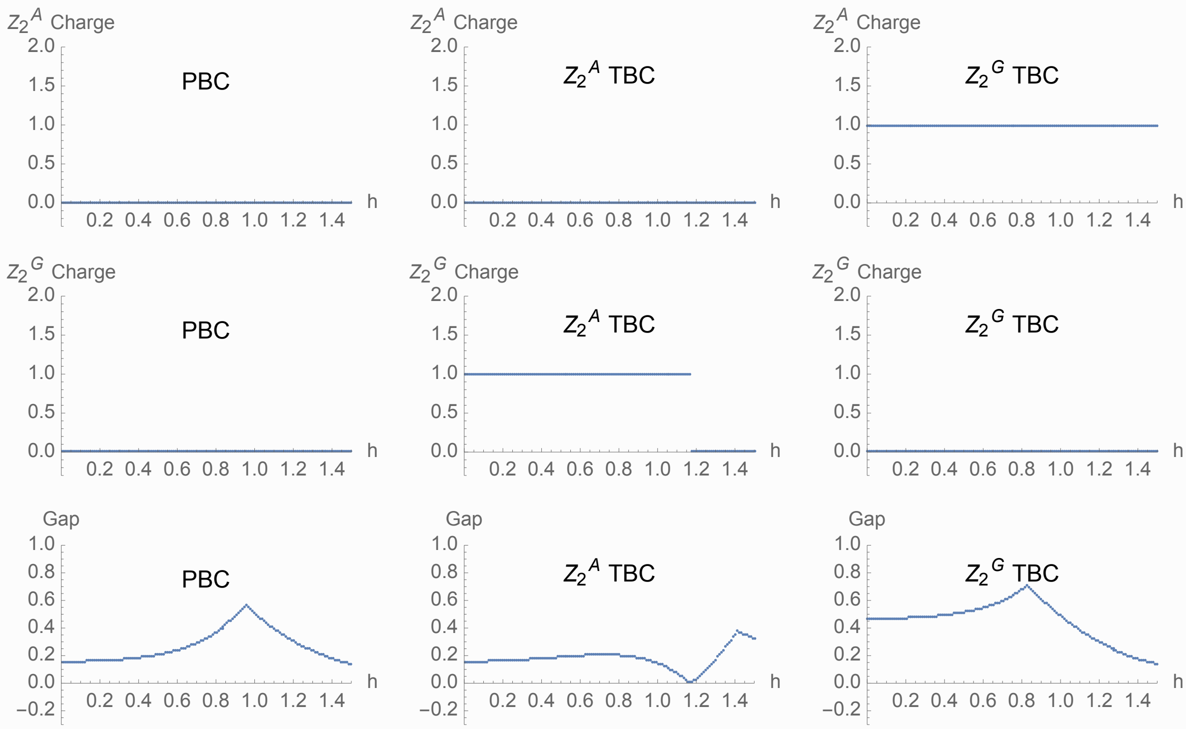

Figure 4: $\mathbb{Z}_2^A$ charge, $\mathbb{Z}_2^G$ charges of the ground state and the finite size gap between the ground state and first excited state under PBC, $\mathbb{Z}_2^A$-TBC and $\mathbb{Z}_2^G$ TBC, as a function of the perturbation strength h. The system size is $L = 10$.

The perturbed Hamiltonian under $\mathbb{Z}_2^A$-TBC and $\mathbb{Z}_2^G$-TBC are

$$H^{\mathbb{Z}_2^A \text{pert}}_{\text{WeakSPTC}} = H^{\mathbb{Z}_2^A}_{\text{WeakSPTC}} - h \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x, \tag{2.26}$$

and

$$H^{\mathbb{Z}_2^G \text{pert}}_{\text{WeakSPTC}} = H^{\mathbb{Z}_2^G}_{\text{WeakSPTC}} - h \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x, \tag{2.27}$$

respectively. $H^{\mathbb{Z}_2^{A,G}}_{\text{WeakSPTC}}$ are defined in (2.14) and (2.17). However, the method used in section 2.3.2 to show nontrivial ground state symmetry charge of $H^{\mathbb{Z}_2^{A,G}}_{\text{WeakSPTC}}$ no longer applies to $H^{\mathbb{Z}_2^{A,G}\text{pert}}_{\text{WeakSPTC}}$. But by exact diagonalization, we found that the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ charge is unchanged under any boundary condition as long as $h \le h_c$, where $h_c \simeq 1.19$ for $L = 10$. See figure 4 for numerical calculation of the charge and gap under various boundary conditions. h_c in general depends on L, which we do not study in this paper. When $h = h_c$, the $\mathbb{Z}_2^G$ charge under $\mathbb{Z}_2^A$-TBC jumps, and at the same time the finite size gap of the Hamiltonian under $\mathbb{Z}_2^A$-TBC closes. Note that the symmetry charge remains unchanged and the finite size gap remains open under PBC at $h = h_c$. When $h \gg h_c$, the theory is in a gapped phase where $\mathbb{Z}_2^A$ is spontaneously broken. These results suggest that there is a topological phase transition at $h = h_c$, between the gapless $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC region and $\mathbb{Z}_2^A$ SSB phase. [18]

We conclude that the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC exhibited by H_{WeakSPTC} is stable against perturbation V as long as $h < h_c$. We expect that the lesson we learnt here is generally valid: charge of the twisted sector is a robust feature against small enough perturbation, and can be used as a topological invariant for the weak SPTC. The weak SPTC can be also characterized by the quasi-degeneracy of the ground states due to the edge states in open chains. However, this is a less sharp probe in practice, as we need to distinguish the quasi-degenerate ground states with an exponentially decaying gap from gapless excitations whose energies are scaled by powers of the system size. Moreover, as commented in section 1.4, another advantage of symmetry charge under TBC is that it is generalizable to higher dimensions.

3 Strong SPTC: $(1 + 1)$d Spin Chain With $\mathbb{Z}_4$ Symmetry

In this section, we study a concrete lattice model of strong SPTC: $(1 + 1)$d spin chain with $\mathbb{Z}_4$ global symmetry. We let $A = \mathbb{Z}_2, G = \mathbb{Z}_2$, and the symmetry extension (1.1) is now nontrivial. We still use superscripts A and G to label the two $\mathbb{Z}_2$'s, and use superscript Γ to label $\mathbb{Z}_4$.

The strong SPTC has been studied in [13], under the name of "intrinsically gapless topological phases". As emphasized in the introduction, the novelty here is that we discuss the strong vs weak SPTC in the same framework through decorated defect construction, and present a simpler and more analytically tractable model than that in [13]. The signature of boundary degeneracy as well as the string order parameter have been extensively discussed in [13]. Here, we will instead emphasize the usefulness of TBC as a probe of nontrivial SPTCs and its stability upon perturbation to the trivially gapped phase. See also [50] for the discussions of a boson-fermion coupled model which is related to our spin model by Jordan-Wigner transformation.

3.1 Spin Chain Construction

3.1.1 Domain Wall Decoration and Induced Anomaly

Domain Wall Decoration: We construct the $(1 + 1)$d spin chain with $\mathbb{Z}_4^\Gamma$ global symmetry, by applying the decorated defect construction reviewed in section 1.3.2. Concretely, we start with a $\mathbb{Z}_2^G$ symmetry spontaneously broken phase with a nontrivial anomaly of $\mathbb{Z}_2^G$, and then decorate the $\mathbb{Z}_2^G$ domain wall by $\mathbb{Z}_2^A$ SPT. We will show below that the domain wall decoration induces a nontrivial $\mathbb{Z}_2^G$ anomaly due to the nontrivial extension (1.1), and two $\mathbb{Z}_2^G$ anomalies are designed to cancel against each other. Thus the entire $\mathbb{Z}_4^\Gamma$ symmetry is anomaly free. We further proliferate

[18]This means that SPTC is stable under perturbing to $\mathbb{Z}_2^A$ SSB phase, at least for a given finite system size. However, by construction, it is unstable under perturbing to $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT phase. Note that from table 1 the symmetry quantum numbers of weak SPTC are the same as those of the gapped SPT.

the decorated $\mathbb{Z}_2^G$ domain wall, and fine tune the system to the critical point. The resulting critical point is the $\mathbb{Z}_4^\Gamma$ strong SPTC.

Induced Anomaly: We explain why the domain wall decoration induces nontrivial $\mathbb{Z}_2^G$ anomaly. Let us denote the background fields of $\mathbb{Z}_2^G$ and $\mathbb{Z}_2^A$ as g and a respectively, both of which are 1-cochains. The $\mathbb{Z}_4^\Gamma$ background field is $2a - \tilde{g}$, where $\widetilde{g}$ is a lift of g to a $\mathbb{Z}_4^\Gamma$ valued cochain, i.e. $g = \widetilde{g} \mod 2$. By requiring the $\mathbb{Z}_4^\Gamma$ background to be flat, we find

$$\delta(2a - \widetilde{g}) = 2\delta a - \delta\widetilde{g} = 0 \mod 4 \tag{3.1}$$

which implies

$$\delta a = \text{Bock}(g) := \frac{1}{2}\delta\widetilde{g} \mod 2, \qquad \delta g = 0 \mod 2. \tag{3.2}$$

$\text{Bock}(g)$ is the Bockstein of g, which is defined as in (3.2). As (2.4), decorating the $\mathbb{Z}_2^G$ domain wall by a 1d $\mathbb{Z}_2^A$ SPT means stacking a $\mathbb{Z}_2^A$ Wilson line to the worldline of $\mathbb{Z}_2^G$ domain wall. However, due to the nontrivial bundle constraint (3.2), the domain wall decoration is not gauge invariant, and equivalently it induces a nontrivial dependence on the extension to the 3d bulk M_3,

$$\exp\left(i\pi\int_{[g]} a\right) = \exp\left(i\pi\int_{M_2} a\cup g\right) = \exp\left(i\pi\int_{M_3} g\cup \text{Bock}(g)\right). \tag{3.3}$$

In the second equality, we applied total derivative to promote the 2d integral to the 3d integral and used (3.2). A physical interpretation of (3.3) is that domain wall decoration induces a $\mathbb{Z}_2^G$ anomaly. We will denote this anomaly as the induced anomaly.

However, the SPTC by definition should be free of $\mathbb{Z}_4^\Gamma$ anomaly, and the system should be independent of the extension to M_3. This demands that the $\mathbb{Z}_2^G$ spontaneously broken system before domain wall decoration should already exhibit an opposite anomaly of $\mathbb{Z}_2^G$, which is given by the same inflow action

$$\exp\left(i\pi\int_{M_3} g\cup \text{Bock}(g)\right). \tag{3.4}$$

After domain wall decoration, the anomaly (3.4) from the low energy cancels against the induced anomaly (3.3) from the domain wall decoration, and the total system is anomaly free.

As commented at the end of section 2.1.2, one can simplify the discussion by directly starting with a critical system with a non-degenerate ground state and a $\mathbb{Z}_2^G$ anomaly (3.4). A standard candidate is the critical boundary theory of $(2+1)$d $\mathbb{Z}_2^G$ SPT, known as the Levin-Gu model [35]. We then decorate the $\mathbb{Z}_2^G$ domain walls (via conjugating by the unitary operator U_{DW} in (2.5)). We will take this simplified strategy of domain wall decoration below.

3.1.2 The Model

We still let the σ spins (living on integer sites) to be charged under $\mathbb{Z}_2^G$, and τ spins (living on half integer sites) to be charged under $\mathbb{Z}_2^A$. The $\mathbb{Z}_4^\Gamma$ symmetry is generated by

$$U_\Gamma = \prod_{i=1}^{L} \sigma_i^x \exp\left(\frac{i\pi}{4}(1-\tau_{i+\frac{1}{2}}^x)\right) \tag{3.5}$$

under which $\sigma_i^x \to \sigma_i^x, \sigma_i^{y,z} \to -\sigma_i^{y,z}$, $\tau_{i+\frac{1}{2}}^x \to \tau_{i+\frac{1}{2}}^x$, $\tau_{i+\frac{1}{2}}^y \to \tau_{i+\frac{1}{2}}^z$, $\tau_{i+\frac{1}{2}}^z \to -\tau_{i+\frac{1}{2}}^y$. Since U_Γ is on-site, $\mathbb{Z}_4^\Gamma$ is anomaly free. The normal subgroup $\mathbb{Z}_2^A$ is also generated by an on-site operator

$$U_A = U_\Gamma^2 = \prod_{i=1}^{L} \tau_{i+\frac{1}{2}}^x. \tag{3.6}$$

We propose the Hamiltonian for the $\mathbb{Z}_4^\Gamma$ strong SPTC, and justify that it comes from the construction at the end of section 3.1.1. The Hamiltonian is

$$H_{\text{StrongSPTC}} = -\sum_{i=1}^{L}\left(\tau_{i-\frac{1}{2}}^z\sigma_i^x\tau_{i+\frac{1}{2}}^z + \tau_{i-\frac{1}{2}}^y\sigma_i^x\tau_{i+\frac{1}{2}}^y + \sigma_{i-1}^z\tau_{i-\frac{1}{2}}^x\sigma_i^z\right). \tag{3.7}$$

To justify whether (3.7) comes from the Levin-Gu model with an anomalous $\mathbb{Z}_2^G$ symmetry by domain wall decoration, we conjugate the same U_{DW} as (2.5) on (3.7) to obtain a pre-decorated model

$$U_{DW}H_{\text{StrongSPTC}}U_{DW}^\dagger = -\sum_{i=1}^{L}\left(\sigma_i^x - \sigma_{i-1}^z\tau_{i-\frac{1}{2}}^x\sigma_i^x\tau_{i+\frac{1}{2}}^x\sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^x\right) \tag{3.8}$$

and the pre-decorated $\mathbb{Z}_4^\Gamma$ symmetry operator becomes

$$U_{DW}U_\Gamma U_{DW}^\dagger = \prod_{i=1}^{L}\sigma_i^x \prod_{i=1}^{L}\exp\left(\frac{i\pi}{4}(1-\sigma_i^z\tau_{i+\frac{1}{2}}^x\sigma_{i+1}^z)\right). \tag{3.9}$$

As the last term in (3.8) commutes with the rest of the terms, the ground state should be the eigenstate of $\tau_{i-\frac{1}{2}}^x$ with eigenvalue 1. See appendix C for a more detailed discussion on this point.[19] In the low energy sector, we simply substitute $\tau_{i-\frac{1}{2}}^x = 1$ in (3.8), and obtain the low energy effective Hamiltonian

$$U_{DW}H_{\text{StrongSPTC}}U_{DW}^\dagger|_{\text{low}} = -\sum_{i=1}^{L}\left(\sigma_i^x - \sigma_{i-1}^z\sigma_i^x\sigma_{i+1}^z\right) \tag{3.10}$$

which is precisely the Levin-Gu Hamiltonian [35]. The $\mathbb{Z}_2^A$ normal subgroup decouples from the low energy. Only $\mathbb{Z}_2^G$ acts nontrivially on the low energy degrees of freedom

$$U_{DW}U_\Gamma U_{DW}^\dagger|_{\text{low}} = \prod_{i=1}^{L}\sigma_i^x\prod_{i=1}^{L}\exp\left(\frac{i\pi}{4}(1-\sigma_i^z\sigma_{i+1}^z)\right). \tag{3.11}$$

[19] In fact, all the low energy states with energy $E - E_{\text{GS}} \ll 1$ satisfy $\tau_{i-\frac{1}{2}}^x = 1$.

		$\mathbb{Z}_4^\Gamma$ Strong SPTC	$\mathbb{Z}_4^\Gamma$ Landau Transition
PBC:	GSD	1	1
$\mathbb{Z}_2^A$-TBC:	GSD	1	1
	Relative $\mathbb{Z}_2^A$ Charge	0	0
	Relative $\mathbb{Z}_4^\Gamma$ Charge	2	0
$\mathbb{Z}_4^\Gamma$-TBC:	GSD	$L = \text{odd} : 2;\quad L = \text{even} : 4$	1
	Relative $\mathbb{Z}_2^A$ Charge	1	0
	Relative $\mathbb{Z}_4^\Gamma$ Charge	1 or 3	0
OBC:	GSD	≥ 2	1

Table 2: Ground state degeneracy and symmetry charges of the ground state under PBC, TBC and OBC. We focus on the system size $L = 0, 1, 3, 4, 5, 7 \mod 8$ to ensure trivial ground state degeneracy. Relative $\mathbb{Z}_2^A$ (or $\mathbb{Z}_4^\Gamma$) charge means the difference between the corresponding charge under the TBC and that under the PBC. We compare these properties between the strong SPTC and Landau transition, both with the same global symmetry $\mathbb{Z}_4^\Gamma$.

The $\mathbb{Z}_2^G$ symmetry operator is realized in a non-on-site way, which is demanded by the $\mathbb{Z}_2^G$ anomaly (3.4) at the low energy. This justifies that the proposed Hamiltonian (3.7) comes from the prescribed construction in section 3.1.1.

3.2 Signatures of $\mathbb{Z}_4^\Gamma$ Strong SPTC

We discuss the physical signatures of the $\mathbb{Z}_4^\Gamma$ strong SPTC (3.7). An immediate fact to realize is that there is no $\mathbb{Z}_4^\Gamma$ gapped SPT in $(1+1)$d.[20] Thus it is not possible to stack a gapped SPT to unitarily connect it to another possibly more trivial SPTC. For this reason, the origin of the nontrivial SPT order at the critical point here is less obvious, in contrast to the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC. This motivates us to use strong vs weak to distinguish these two SPTCs. In this subsection, we discuss its properties under various boundary conditions. We summarize the main results in table 2.

3.2.1 Periodic Boundary Condition

We have motivated in section 1.2 that any SPTC should have one non-degenerate ground state, with a finite size splitting with the first excited state. Thus we would like to check the ground state degeneracy of (3.7) under PBC to be one.

As we find in section 3.1.2, the number of ground states of the $\mathbb{Z}_4^\Gamma$ strong SPTC is identical to that of the Levin-Gu model (3.10). In appendix B.1, we show, by Jordan-Wigner transformation,

[20]The $(1+1)$d bosonic SPT with a discrete symmetry G is classified by $H^2(G, U(1))$. In our case, $G = \mathbb{Z}_4$, and it is well-known [27] that $H^2(\mathbb{Z}_4, U(1)) = 0$ is trivial, hence there is no nontrivial $\mathbb{Z}_4$ SPT phase in $(1+1)$d.

that the number of ground states of the Levin-Gu model depends on $L \mod 4$ and is given as

$$\text{GSD}_L = \begin{cases} 2, & L = 2 \mod 4 \\ 1, & \text{otherwise.} \end{cases} \tag{3.12}$$

Thus the number of ground state of the $\mathbb{Z}_4^\Gamma$ strong SPT criticality under periodic boundary condition is also given by (3.12).

Let us further discuss the $\mathbb{Z}_4$ charge of the ground state. Denote the ground states of (3.10), (3.8) and the (3.7) as $|\text{GS}\rangle_{\text{LG}}$, $|\text{GS}\rangle_{\text{pre}}$ and $|\text{GS}\rangle$ respectively. Suppose the $\mathbb{Z}_2^G$ charge of $|\text{GS}\rangle_{\text{LG}}$ in the Levin-Gu model (3.10) is q_{LG}, then by definition we have

$$U_{DW} U_\Gamma U_{DW}^\dagger|_{\text{low}} |\text{GS}\rangle_{\text{LG}} = (-1)^{q_{\text{LG}}} |\text{GS}\rangle_{\text{LG}} . \tag{3.13}$$

As $\mathbb{Z}_2^A$ decouples from the low energy, we also have $U_{DW} U_\Gamma U_{DW}^\dagger |\text{GS}\rangle_{\text{pre}} = (-1)^{q_{\text{LG}}} |\text{GS}\rangle_{\text{pre}}$. Since $|\text{GS}\rangle_{\text{pre}} = U_{DW} |\text{GS}\rangle$, we can then compute the $\mathbb{Z}_4^\Gamma$ charge of $|\text{GS}\rangle$ via,

$$U_\Gamma |\text{GS}\rangle = U_{DW}^\dagger (U_{DW} U_\Gamma U_{DW}^\dagger) |\text{GS}\rangle_{\text{pre}} = (-1)^{q_{\text{LG}}} U_{DW}^\dagger |\text{GS}\rangle_{\text{pre}} = e^{i\frac{\pi}{2}\cdot 2q_{\text{LG}}} |\text{GS}\rangle . \tag{3.14}$$

So the $\mathbb{Z}_4^\Gamma$ charge q of the ground state $|\text{GS}\rangle$ is related to the $\mathbb{Z}_2^G$ charge of $|\text{GS}\rangle_{\text{LG}}$ via $q = 2q_{\text{LG}} \mod 4$.

We are left to determine the symmetry charge of the Levin-Gu model, q_{LG}. While the ground-state degeneracy was obtained exactly in Eq. (3.12) by the Jordan-Wigner transformation as discussed in Appendix B.1, we could not find q_{LG} from the Jordan-Wigner transformation. Nevertheless, we can utilize an alternative mapping to the XX chain as discussed in Appendix B.2, to determine q_{LG} for even L's. The analytical result for even L's was confirmed by exact numerical diagonalization for small L's, which also gives q_{LG} for odd L's. As a result, extending the $L \mod 4$ dependence of the ground-state degeneracy (3.12), we find that the symmetry charge of the Levin-Gu model q_{LG} depends on $\alpha = L \mod 8$: $q_{\text{LG}} = 0$ for $\alpha = 0, 1, 7$, while $q_{\text{LG}} = 1$ for $\alpha = 3, 4, 5$. As presented in Eq. (3.12), for $\alpha = 2, 6$, the ground states are two fold degenerate. We find that, each of the two degenerate ground states has $q_{\text{LG}} = 0$ and $q_{\text{LG}} = 1$.

We conclude that the $\mathbb{Z}_4^\Gamma$ charge q of ground state of (3.7) is

$$U_\Gamma |\text{GS}\rangle = e^{i\pi q/2} |\text{GS}\rangle , \quad q = 2q_{\text{LG}} = \begin{cases} 0, & \alpha = 0, 1, 7 \\ 2, & \alpha = 3, 4, 5 \\ 0\&2, & \alpha = 2, 6. \end{cases} \tag{3.15}$$

From the above result, it appears that the ground state degeneracy is not well defined in the limit $L \to \infty$. While we do not completely understand the physical mechanism behind the periodic dependence of the ground-state degeneracy on the system size, the ground-state degeneracy for $\alpha = 2, 6$ might be interpreted as a consequence of an effective twist [51]. The effective twist can be seen by mapping the Levin-Gu model to an XX chain. In appendix B, we showed that under a unitary transformation, the Levin-Gu model with PBC can be mapped to an XX chain with PBC and one ground state when $L \in 4\mathbb{Z}$, and XX chain with the twisted boundary condition and two degenerate ground states when $L \in 4\mathbb{Z}+2$. This is analogous to the phenomenon that an antiferromagnetic chain of odd length is effectively subject to a twisted boundary condition. Here we simply consider the sequence of systems only with $\alpha \in \{0,1,3,4,5,7\}$. This would be reasonable if the ground-state degeneracy for $\alpha = 2,6$ is indeed due to an effective twist; we just consider the sequence of effectively untwisted systems.[21] Then the ground state degeneracy in the thermodynamic limit is regarded as one, consistently with our definition of SPTC.

There still remains the periodic dependence of the $\mathbb{Z}_4^\Gamma$ charge in the ground state on the system size: for $\alpha = 0,1,7$, the ground state is neutral under $\mathbb{Z}_4^\Gamma$, while $\alpha = 3,4,5$, the ground state gets a minus sign under the $\mathbb{Z}_4^\Gamma$ transformation. However, this minus sign can always be absorbed by suitably modifying the definition of U_Γ in (3.5). In fact, in the following sections, we will only be interested in the relative charge of the ground state between the periodic and twisted boundary conditions, which turns out to be system-size independent.

3.2.2 Twisted Boundary Condition

We further discuss the charge of the ground state under the TBC. We can either twist by $\mathbb{Z}_4^\Gamma$, or its normal subgroup $\mathbb{Z}_2^A$.

Twist by $\mathbb{Z}_2^A$: We twist the boundary condition by $\mathbb{Z}_2^A$ (labeled by $\mathbb{Z}_2^A$ TBC). The Hamiltonian is

$$\begin{aligned} H^{\mathbb{Z}_2^A}_{\text{StrongSPTC}} &= -\sum_{i=1}^{L-1}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + \sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1}\right) + \tau^z_{L-\frac{1}{2}}\sigma^x_L\tau^z_{\frac{1}{2}} + \tau^y_{L-\frac{1}{2}}\sigma^x_L\tau^y_{\frac{1}{2}} - \sigma^z_L\tau^x_{\frac{1}{2}}\sigma^z_1 \\ &= \sigma^z_L H_{\text{StrongSPTC}}\sigma^z_L \end{aligned} \tag{3.16}$$

where $H_{\text{StrongSPTC}}$ is (3.7). We have already encountered the same algebra below (2.14). Denote the ground state of $H_{\text{StrongSPTC}}$ and $H^{\mathbb{Z}_2^A}_{\text{StrongSPTC}}$ as $|\text{GS}\rangle$ and $|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}}$, respectively. Then we have $|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} = \sigma^z_L|\text{GS}\rangle$. As $U_\Gamma\sigma^z_L U_\Gamma^\dagger = -\sigma^z_L$, we find

$$U_\Gamma|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} = -e^{i\pi q/2}|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} = e^{i\pi(q+2)/2}|\text{GS}\rangle^{\mathbb{Z}_2^A}_{\text{tw}} \tag{3.17}$$

where q is the $\mathbb{Z}_4$ charge of $|\text{GS}\rangle$ under PBC, given by (3.15). (3.17) means that the $\mathbb{Z}_4^\Gamma$ charge of the ground state with the $\mathbb{Z}_2^A$ twisted boundary condition differs from that with the periodic boundary condition by two. We thus define the difference between the $\mathbb{Z}_4^\Gamma$ charge under $\mathbb{Z}_2^A$-TBC and that under PBC to be the relative $\mathbb{Z}_4^\Gamma$ charge, which is two. Relative charge is more physical since there are ambiguilities in defining the absolute charge as we noticed in the previous subsection. The nontrivial relative $\mathbb{Z}_2^A$ charge shows that the strong SPTC we constructed in (3.7) is topologically nontrivial. We also note that the proof applies to all the states.

[21] See also [52] for the system size dependent ground state degeneracy in the $(1+1)$d Luttinger liquids.

Twist by $\mathbb{Z}_4^\Gamma$: We further use the $\mathbb{Z}_4^\Gamma$ symmetry to twist the boundary condition (labeled by $\mathbb{Z}_4^\Gamma$ TBC). The Hamiltonian is

$$H_{\text{StrongSPTC}}^{\mathbb{Z}_4^\Gamma} = -\sum_{i=1}^{L-1}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + \sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1}\right) - \tau^z_{L-\frac{1}{2}}\sigma^x_L\tau^y_{\frac{1}{2}} + \tau^y_{L-\frac{1}{2}}\sigma^x_L\tau^z_{\frac{1}{2}} + \sigma^z_L\tau^x_{\frac{1}{2}}\sigma^z_1. \tag{3.18}$$

The ground state $|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma}$ satisfies

$$\tau^x_{i+\frac{1}{2}}|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma} = \sigma^z_i\sigma^z_{i+1}|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma} \quad (1\le i\le L-1), \qquad \tau^x_{\frac{1}{2}}|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma} = -\sigma^z_L\sigma^z_1|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma}. \tag{3.19}$$

We then measure the $\mathbb{Z}_2^A$ charge using U_A in (3.6),

$$U_A|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma} = -\prod_{i=1}^{L-1}\left(\sigma^z_i\sigma^z_{i+1}\right)\sigma^z_L\sigma^z_1|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma} = -|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma} \tag{3.20}$$

which means that the ground state carries $\mathbb{Z}_2^A$ charge 1. This also implies that if $|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_4^\Gamma}$ is an eigenstate of U_Γ, then it should carry $\mathbb{Z}_4^\Gamma$ charge 1 mod 4 or 3 mod 4.

In fact, by exact numerical diagonalization, we find that there are two degenerate ground states if L is odd and four if L is even. If we organize them into eigenstates of $\mathbb{Z}_4^\Gamma$, half of them have charge 1 mod 4 and the other half have charge 3 mod 4. Since there are different charges, an arbitrary linear combination of them is generically not an $\mathbb{Z}_4^\Gamma$ eigenstate. However, as all of the ground states have $\mathbb{Z}_2^A$ charge 1, an arbitrary linear combination of them also has $\mathbb{Z}_2^A$ charge 1.

From (3.15), the $\mathbb{Z}_2^A$ charge of the ground state under PBC is always trivial, independent of the system size. Moreover, as we find in (3.20) the $\mathbb{Z}_2^A$ charge of the ground state under $\mathbb{Z}_4^\Gamma$ TBC is one, independent of the system size. We thus found that the relative $\mathbb{Z}_2^A$ charge is size-independent, and it shows that the strong SPTC we constructed in (3.7) is topologically nontrivial. Since (3.19) also holds for all low energy states with energy $E - E_{\text{GS}} \ll 1$, the above proof of nontrivial $\mathbb{Z}_2^A$ charge of the ground state also applies to low energy states.

In summary, we have checked that using either $\mathbb{Z}_2^A$-TBC or $\mathbb{Z}_4^\Gamma$-TBC one can probe the topological nontriviality of the $\mathbb{Z}_4^\Gamma$ strong SPTC.

3.2.3 Open Boundary Condition

We proceed to discuss the ground state degeneracy under the OBC. When placing the $\mathbb{Z}_4$ strong SPTC on an open chain, as in section 2.3.3, we let the left most spin to be σ spin, and right most spin to be τ spin. The Hamiltonian is

$$H_{\text{StrongSPTC}}^{\text{OBC}} = -\sum_{i=2}^{L}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}}\right) - \sum_{i=1}^{L-1}\sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1} \tag{3.21}$$

and the symmetry operator is

$$U_\Gamma = \prod_{i=1}^{L}\sigma^x_i\prod_{i=1}^{L}\exp\left(\frac{i\pi}{4}(1-\tau^x_{i+\frac{1}{2}})\right). \tag{3.22}$$

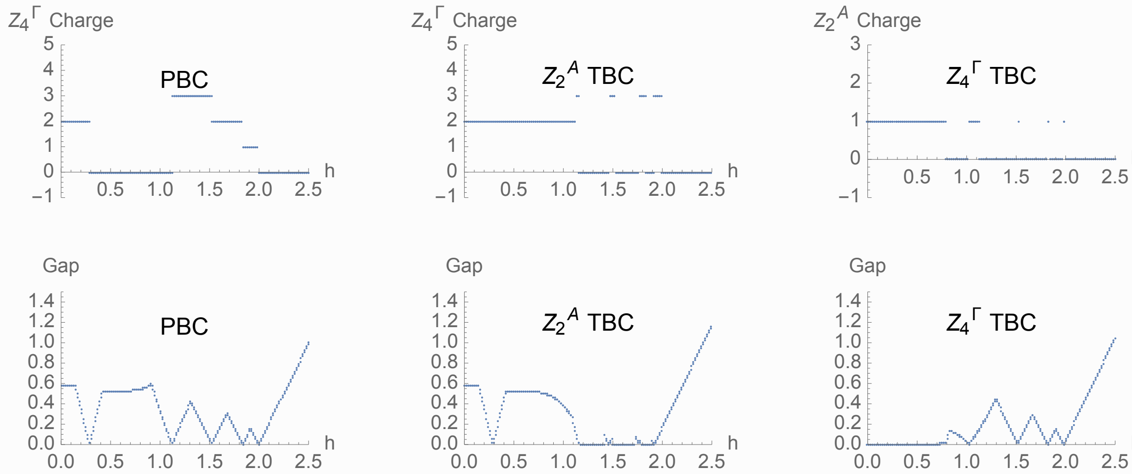

Figure 5: $\mathbb{Z}_4^\Gamma$ charge of the ground state under PBC, relative $\mathbb{Z}_4^\Gamma$ charge of the ground state under $\mathbb{Z}_2^A$ TBC, relative $\mathbb{Z}_2^A$ charges of the ground state under $\mathbb{Z}_4^\Gamma$ TBC, and the gap between the ground state and first excited state under PBC and two TBC's. The horizontal axis is the perturbation strength (3.23). The system size is $L = 11$.

We find that the set of operators $\{\sigma_1^z, \sigma_L^z \tau_{L+\frac{1}{2}}^x, U_\Gamma\}$ commute with the Hamiltonian (3.21). The irreducible representation of the above algebra is two, hence the ground states of (3.21) are at least two fold degenerate. In appendix D.2, we show that the ground state degeneracy is four for $L \in 2\mathbb{Z}+1$, and two for $L \in 2\mathbb{Z}$. We emphasize again that the exact degeneracy can be lifted exponentially by a generic symmetric perturbation, as in section 2.4.1.

3.3 Stability of Strong SPTC

As discussed in section 2, the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ weak SPTC is unstable upon perturbation towards the gapped SPT phase. It immediately enters the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT phase when transverse field λ passes the critical value $\lambda_c = 1$. How about the stability of the $\mathbb{Z}_4^\Gamma$ strong SPTC against perturbation into a gapped phase with a unique ground state?

First of all, since $\mathbb{Z}_4^\Gamma$ is non-anomalous, in principle, there is no obstruction to deform the system to $\mathbb{Z}_4$ symmetric gapped phase with a unique ground state [53]. Secondly, since there is no $\mathbb{Z}_4^\Gamma$ gapped SPT, the only gapped phase with a non-degenerate ground state is the trivially gapped phase. In this subsection, we will examine the most obvious $\mathbb{Z}_4^\Gamma$ symmetric perturbation that can drive the strong SPTC into a trivially gapped phase,

$$-h \sum_{i=1}^{L} \left(\sigma_i^x + \tau_{i+\frac{1}{2}}^x \right) \tag{3.23}$$

where $h > 0$. When $h \gg 1$, as $\tau_{i+\frac{1}{2}}^x$ anticommutes with the first and second term of the Hamiltonian (3.7), and σ_i^x anticommutes with the third term, only (3.23) survives and it is in the trivially

L	$\mathbb{Z}_4^\Gamma$ Charge under PBC	$\mathbb{Z}_4^\Gamma$ Charge under $\mathbb{Z}_2^A$-TBC	$\mathbb{Z}_2^A$ Charge under $\mathbb{Z}_4^\Gamma$-TBC
4	1.01	1.01	1.01
5	1.30	1.30	0.50
7	0.44	1.32	0.98
8	0.70	0.70	0.70
9	0.86	0.86	0.86
11	0.28	1.12	1.01

Table 3: Lowest h where the symmetry charge of the ground state under three boundary conditions jumps, for $L = 4, 5, 7, 8, 9, 11$.

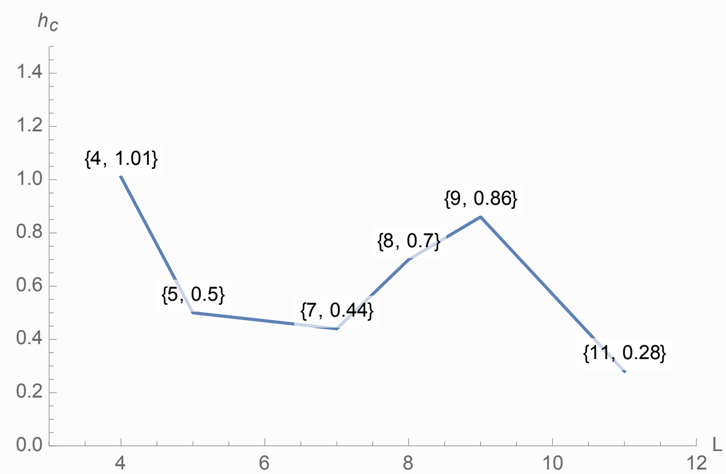

Figure 6: System size L dependence of the first transition out of the $\mathbb{Z}_4^\Gamma$ strong SPTC.

gapped phase. This means that there must be at least one phase transition as h increases from zero where either the $\mathbb{Z}_4^\Gamma$ charge under PBC, or the relative $\mathbb{Z}_2^A$ charge under $\mathbb{Z}_4^\Gamma$-TBC or relative $\mathbb{Z}_4^\Gamma$ charge under $\mathbb{Z}_2^A$-TBC jumps. We perform the exact diagonalization numerically, and record the lowest h where the charges jump in table 3. We also plot the charges and the gaps under various boundary conditions for $L = 11$ in figure 5.

From the plots in figure 5, we find that the $\mathbb{Z}_4^\Gamma$ charge under PBC and both relative charges under TBC's are unchanged until h reaches the first critical value $h_c \simeq 0.28$. This first transition is probed by the charge jump under PBC, where the finite size gap closes simultaneously. When h further passes h_c, the system goes through a sequence of transitions, some are probed by the $\mathbb{Z}_2^A$-TBC, some are probed by the $\mathbb{Z}_4^\Gamma$-TBC and the others are probed by PBC. When h is sufficiently large ($h > 2$), the system enters into a trivially gapped phase, and all charges become trivial.

For different system sizes, for instance $L = 5$ as shown in table 3, the first transition can be probed by the relative charge under TBC instead. Hence it is important to examine all the boundary conditions and find the minimal h_c where the charge jumps. We plot the minimal h_c for each L in figure 6.

The above discussion seems to suggest that strong SPTC is more stable than the weak SPTC. Let us however make a cautionary remark. As observed in figure 6, the critical perturbation strength h_c depends on the system size L. Logically, there are two possibilities:

1. $h_c \to 0$ **when** $L \to \infty$**:** This would suggest that in the thermodynamic limit, strong SPTC is unstable upon infinitesimal perturbation (3.23). However, it still requires a phase transition to go from the strong SPTC and trivially gapped phase, since the symmetry quantum numbers of the ground state under TBC don't match between the two.

2. h_c **converges to a finite value when** $L \to \infty$**:** This would suggest that strong SPTC is stable against small enough perturbation (3.23).

From figure 6, h_c does not monotonically decrease as the system size increases, and the data does not rule out either possibility. It should be interesting to study the asymptotic behavior of h_c either numerically or analytically (by understanding the CFT of the $\mathbb{Z}_4^\Gamma$ strong SPTC) in the future.[22] It would also be interesting to study more sophisticated perturbation than (3.23) which can drive the system to the trivially gapped phase, and discuss the transition for small perturbation strength.

4 Strong and Weak SPTC in the Spin-1 System

In this section, we briefly introduce a more realistic spin-1 model which hosts the strong SPTC and weak SPTC simultaneously. This model is studied in detail in [54] by one of the authors in this work (L.L.) together with Yang, Okunishi and Katsura. We briefly review the results there, and fit them into our framework.

4.1 The Model and Phase Diagram

The Hamiltonian is given by

$$H(\theta, \lambda) = (1-\lambda) H_{\text{BLBQ}} + (1+\lambda) U_{KT} H_{\text{BLBQ}} U_{KT}^\dagger \qquad -\frac{\pi}{4} < \theta < \arctan\frac{1}{2}, \tag{4.1}$$

[22] Since the system without perturbation (3.7) is decorating the Levin-Gu model (whose CFT description is a free fermion) by an anomalous SPT (which are gapped degrees of freedom), one may attempt to propose that the CFT of (3.7) in the low energy is simply the free fermion, and identify the perturbation in (3.23) as one of the free fermion operators. However, this is not entirely correct – the perturbation (3.23) can not be fully described by the free fermion operators. As the perturbation strength h increases, the gap between the two set of degrees of freedom decreases, as we explicitly see in Figure 5. Further increasing h above a certain threshold reopens a gap and thus drives the system to trivially gapped phase. This process involves the dynamics of gapped sector, and hence can not be fully described merely by the free fermion operators. An analytical understanding of h_c should require a CFT description including the gapped sector from the domain wall decoration, which we leave to the future study.

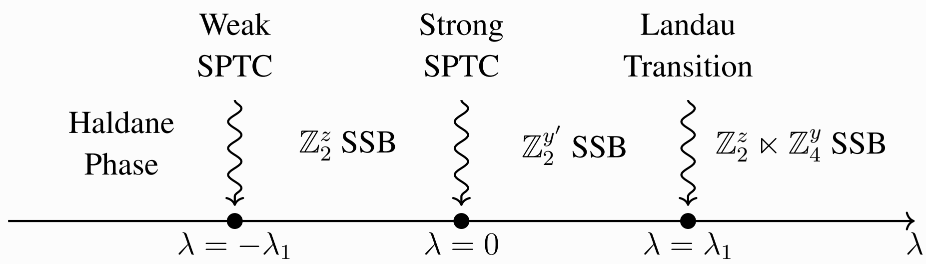

Figure 7: The phase diagram of (4.1) when θ=0.

where

$$H_{\text{BLBQ}} = \cos\theta(\vec{S}_i \cdot \vec{S}_{i+1}) + \sin\theta(\vec{S}_i \cdot \vec{S}_{i+1})^2, \tag{4.2}$$

$$U_{KT} = \prod_{\mu<\nu} \exp(i\pi S_\mu^z S_\nu^x). \tag{4.3}$$

$\vec{S}$ is spin-1 operator. U_{KT} is a non-local unitary operator implementing the Kennedy-Tasaki (KT) transformation [40–42]. Under the KT transformation, $\lambda \leftrightarrow -\lambda$, and $\lambda = 0$ is the self-dual point. For each θ and λ, the Hamiltonian (4.1) preserves three global symmetries:

1. $\mathbb{Z}_2^z$: π rotation in z direction, generated by $\prod_j e^{i\pi S_j^z}$
2. $\mathbb{Z}_4^y$: $\pi/2$ rotation in y direction, generated by $\prod_j e^{i\frac{\pi}{2}S_j^y}$
3. $\mathbb{Z}^{\mathbf{T}}$: translation symmetry.

The phase diagram of $\theta = 0$ is obtained in [54], as shown in figure 7. See [54] for the full 2d phase diagram in the (λ, θ) plane.

4.2 $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}^{\mathbf{T}}$ Strong SPTC

Let us start by discussing the self-dual point $\lambda = 0$ which we argue to be a strong SPTC. Taking the low energy limit around this point, some degrees of freedom decouple, and the 3-dimensional Hilbert space per site in the spin-1 model reduces to 2-dimensional Hilbert space per site, hence effectively becomes a spin-$\frac{1}{2}$ model. The spin-$\frac{1}{2}$ Hamiltonian turns out to be the XXZ model [54]:

$$H(\lambda \ll 1) = -(1+\lambda)\sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + (1-\lambda)\sum_{j=1}^{L} \sigma_j^y \sigma_{j+1}^y. \tag{4.4}$$

This model also has three global symmetries:

1. $\mathbb{Z}_2^{z'}$: generated by $\prod_i \sigma_i^z$

2. $\mathbb{Z}_2^{y'}$: generated by $\prod_i i\sigma_i^y$

3. $\mathbb{Z}^{\mathbf{T}}$: translation symmetry.

We use the primes to distinguish the symmetries of the spin-1/2 model from those of the spin-1 model. Denote their background fields as A'_z, A'_y and A_T. The symmetries $\mathbb{Z}_2^{z'}$, $\mathbb{Z}_2^{y'}$ and $\mathbb{Z}_2^{\mathbf{T}} \subset \mathbb{Z}^{\mathbf{T}}$ have a mixed anomaly [51, 55, 56] whose inflow action is

$$\omega_{3d} = e^{i\pi \int_{M_3} A'_y A'_z A_T}. \tag{4.5}$$

However, in the entire Hibert space of spin-1 system, the $\mathbb{Z}_2^{z'} \times \mathbb{Z}_2^{y'}$ is realized as $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y$ symmetry with the following extension:

$$Y'Z' = R_\pi^y Z'Y', \tag{4.6}$$

where $R_\pi^y = \prod_{j=1}^L \exp(i\pi S_j^y)$, $Y' = \prod_{j=1}^L \exp(i\pi S_j^y/2)$ and $Z' = \prod_{j=1}^L \exp(i\pi S_j^z)$. $\exp(i\pi S_j^y)$ has eigenvalues $\{-1, -1, 1\}$. In the low energy limit, the spin-$\frac{1}{2}$ model only acts nontrivially on the first two components of the spin-1 Hilbert space under the eigenbasis of $\exp(i\pi S_j^y)$, hence $\exp(i\pi S_j^y) = -1$ in the spin-$\frac{1}{2}$ model, Y', Z' in (4.6) reduces to the standard spin-$\frac{1}{2}$ operators $\sigma_j^z = \exp(i\pi S_j^z)$ and $i\sigma_j^y = \exp(i\pi S_j^y/2)$. In terms of the background fields, (4.6) gives us the restriction

$$dA_Y = A'_y A'_z \mod 2 \tag{4.7}$$

where A_Y is 1-cochain for $\mathbb{Z}_2^Y$ normal subgroup of $\mathbb{Z}_4^y$ symmetry. In summary, we can identify $\mathbb{Z}_2^{z'}$ and $\mathbb{Z}_2^{y'}$ in the spin-$\frac{1}{2}$ theory with the $\mathbb{Z}_2^z$ and $\mathbb{Z}_4^y/\mathbb{Z}_2^Y$ in the spin-1 theory respectively.

Besides, since $\exp(i\pi S_j^y) = -1$ for each site in the low energy sector, the ground state is stacked by a weak gapped SPT phase protected by translation and $\mathbb{Z}_2^y$ symmetry [57]. This is represented by the topological action $e^{i\pi \int_{M_2} A_Y A_T}$ and by (4.7), it depends on the extension to a 3d bulk M_3,

$$e^{i\pi \int_{M_2} A_Y A_T} = e^{i\pi \int_{M_3} A'_y A'_z A_T}. \tag{4.8}$$

This induced anomaly from stacking a weak gapped SPT phase cancels against the mixed anomaly (4.5) in the low energy. Thus the total spin-1 system is anomaly free. This shows that the spin-1 system is a strong SPTC, protected by the total symmetry $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}^{\mathbf{T}}$.

The total symmetry can be decomposed into two extensions,

$$1 \to \mathbb{Z}_2^z \times \mathbb{Z}_2^Y \times \mathbb{Z}_2^{\mathbf{T}} \to \mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}_2^{\mathbf{T}} \to \mathbb{Z}_2^{y'} \to 1, \tag{4.9}$$

and

$$1 \to \mathbb{Z}_4^y \to \mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}_2^{\mathbf{T}} \to \mathbb{Z}_2^z \times \mathbb{Z}_2^{\mathbf{T}} \to 1. \tag{4.10}$$

Note that (4.10) is still a nontrivial extension. Comparing with (1.1), we see that the $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}^{\mathbf{T}}$ strong SPTC can be constructed either by starting with $G = \mathbb{Z}_2^{y'}$ SSB phase or $G = \mathbb{Z}_2^z$ SSB phase, which exactly correspond to the regimes $\lambda > 0$ and $\lambda < 0$ in figure 7. Moreover, from (4.5), the anomalous symmetries in the low energy are $\widehat{\Gamma} = \mathbb{Z}_2^{y'} \times \mathbb{Z}_2^{z'} \times \mathbb{Z}_2^{\mathbf{T}}$. This provides an example where the SSB symmetry G is strictly smaller than the anomalous symmetry $\widehat{\Gamma}$, which generalizes the construction in [13].

4.3 $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}^{\mathbf{T}}$ Weak SPTC

Let us further consider the critical point at $\lambda = -\lambda_1$. The two phases around this critical point are $\mathbb{Z}_2^z$ SSB phase and a nontrivial gapped SPT protected by $\mathbb{Z}_2^z \times \mathbb{Z}_2^Y$, a.k.a. the Haldane phase. This fits into the phase diagram of weak SPTC in the left panel of figure 1.

Moreover, at $\lambda = -\lambda_1$, the Hamiltonian (4.1) has a unique ground state under periodic boundary condition for a finite system size but has two ground states under the open boundary condition (up to exponential splitting). There are also three string order parameters with nonzero expectation value in the Haldane phase $O_\mu = \langle S_m^\mu \prod_{m<j<n} \exp(i\pi S_j^\mu) S_n^\mu \rangle$ ($\mu = x, y, z$). When the system is turned into this critical point, only O_y remains nonzero but the other two decay to zero algebraically quickly. All these evidence suggest that the critical point at $\lambda = -\lambda_1$ is a nontrivial weak SPTC. As the system has total symmetry $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}^{\mathbf{T}}$, we name the critical point as $\mathbb{Z}_2^z \ltimes \mathbb{Z}_4^y \times \mathbb{Z}^{\mathbf{T}}$ weak SPTC, although only a subgroup $\mathbb{Z}_2^z \times \mathbb{Z}_2^Y$ protects the gapped SPT in the nearby phase.

Acknowledgements

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A Stability of Boundary Degeneracy of $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ Gapped SPT

We find in section 2.3.3 that if we suitably change OBC by adding boundary interactions, the ground state degeneracy can be lifted from four to two. In this appendix, we would like to argue that exactly degenerate ground states of the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT, which is always four, does not lift under arbitrary symmetric perturbations localized at the boundary.

Let us truncate the system in the same way as section 2.3.3. The σ spins are supported on $i = 1, ..., L$, and the τ spins are supported on $i + \frac{1}{2} = \frac{3}{2}, ..., L + \frac{1}{2}$. Let us begin by choosing one particular OBC such that the Hamiltonian is

$$H_{\text{SPT}}^{\text{OBC}} = -\sum_{i=1}^{L-1} \sigma_i^z \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z - \sum_{i=2}^{L} \tau_{i-\frac{1}{2}}^z \sigma_i^x \tau_{i+\frac{1}{2}}^z. \tag{A.1}$$

Suppose the boundary perturbation at the left end is supported on 2 sites, $1, \frac{3}{2}$. A generic symmetric perturbation takes the form

$$\Delta H_{\text{SPT}}^{\text{OBC}} = (\sigma_1^x)^{\beta_1} (\tau_{\frac{3}{2}}^x)^{\beta_{\frac{3}{2}}} \tag{A.2}$$

where $\beta_{1,\frac{3}{2}} \in \{0, 1\}$. [23] Let us find the local operators that commute with both $H_{\text{SPT}}^{\text{OBC}}$ and $\Delta H_{\text{SPT}}^{\text{OBC}}$. Any interaction commuting with $H_{\text{SPT}}^{\text{OBC}}$ are composed of the building blocks σ_1^z, $\sigma_1^x \tau_{\frac{3}{2}}^z, \tau_{L+\frac{1}{2}}^x \sigma_L^z$, $\tau_{L+\frac{1}{2}}^z$ and all the terms that already exist in (A.1). Using these building blocks, a generic term that might anticommute with the boundary perturbation takes the form

$$\mathcal{O}^{u_1 u_2 u_3 u_4} = (\sigma_1^z)^{u_1} (\sigma_1^x \tau_{\frac{3}{2}}^z)^{u_2} (\sigma_1^z \tau_{\frac{3}{2}}^x \sigma_2^z)^{u_3} (\tau_{\frac{3}{2}}^z \sigma_2^x \tau_{i+\frac{5}{2}}^z)^{u_4} \tag{A.3}$$

where $u_{1,2,3,4} \in \{0, 1\}$. Requiring $[\mathcal{O}, \Delta H_{\text{SPT}}^{\text{OBC}}] = 0$, we find that the coefficients need to satisfy the linear equations

$$\beta_1 (u_1 + u_3) + \beta_{\frac{3}{2}} (u_2 + u_4) = 0 \mod 2. \tag{A.4}$$

Note that $\beta_{1,\frac{3}{2}}$ are given, while u's are variables to be determined. There are 4 variables, and one equation, hence one is free to choose arbitrary value of u_1, u_2, such that u_i's for $i = 3, 4$ are constrained by the equation. One solution would be $u_3 = \beta_{\frac{3}{2}} - u_1, u_4 = \beta_1 - u_2$. On the other hand, the algebra between the operators $\{\mathcal{O}^{u_1 u_2 u_3 u_4}, U_A, U_B\}$ are

$$\begin{aligned} \mathcal{O}^{u_1 u_2 u_3 u_4} \mathcal{O}^{u_1' u_2' u_3' u_4'} &= (-1)^{u_1 u_2' + u_1' u_2} \mathcal{O}^{u_1' u_2' u_3' u_4'} \mathcal{O}^{u_1 u_2 u_3 u_4}, \\ U_A \mathcal{O}^{u_1 u_2 u_3 u_4} &= (-1)^{u_2} \mathcal{O}^{u_1 u_2 u_3 u_4} U_A, \\ U_G \mathcal{O}^{u_1 u_2 u_3 u_4} &= (-1)^{u_1} \mathcal{O}^{u_1 u_2 u_3 u_4} U_G. \end{aligned} \tag{A.5}$$

The commutation relations only depends on u_1, u_2! Hence we are free to choose two commuting independent operators $\mathcal{O}^{10 u_3 u_4}$ and $\mathcal{O}^{01 u_3' u_4'}$ whose common eigenvalues (a, b) label the ground

[23] For perturbations supported on 3 sites, one also allows $\sigma_1^z \sigma_2^z$. But for 2 site perturbation, Pauli Z operators are forbidden by the symmetries.

states $|(a,b)\rangle$, where $u_{3,4}$ and $u'_{3,4}$ are arbitrary solutions of (A.4). The four orthogonal ground states are thus given by

$$|(a,b)\rangle\,, \quad |(-a,b)\rangle = U_G\,|(a,b)\rangle\,, \quad |(a,-b)\rangle = U_A\,|(a,b)\rangle\,, \quad |(-a,-b)\rangle = U_A U_G\,|(a,b)\rangle\,. \tag{A.6}$$

The above discussion can easily be generalized to perturbation supported on arbitrary number sites. We thus conclude that, for the $\mathbb{Z}_2^A \times \mathbb{Z}_2^G$ gapped SPT, the exact four fold ground state degeneracy on an open chain is stable under boundary perturbation.

B Spectrum of Levin-Gu Model under Different Boundary Conditions

In this appendix, we show the energy spectrum of Levin-Gu model [35] under different boundary conditions analytically. The analytic results are confirmed by the numerical calculation.

B.1 Exact Solutions under PBC by Jordan-Wigner Transformation

The Hamiltonian of Levin-Gu model is

$$H_{\text{LG}} = -\sum_{i=1}^{L} \left(\sigma_i^x - \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z\right) \tag{B.1}$$

which respects the $\mathbb{Z}_2$ symmetry generated by

$$U_G = \prod_{i=1}^{L} \sigma_i^x \prod_{i=1}^{L} \exp\left(\frac{i\pi}{4}(1 - \sigma_i^z \sigma_{i+1}^z)\right). \tag{B.2}$$

We apply the Jordan-Wigner (JW) transformation which maps spin operator to fermion oeprator

$$\sigma_i^x = (-1)^{n_i} = 1 - 2f_i^\dagger f_i, \quad \sigma_i^z = \prod_{j=1}^{i-1} (-1)^{n_j} (f_i^\dagger + f_i) \tag{B.3}$$

where $n_i := f_i^\dagger f_i$ is fermion density operator. Note that when $i = 1$, we simply have $\sigma_1^z = f_1^\dagger + f_1$. We also assume PBC of the spins, i.e. $\sigma_i^a = \sigma_{i+L}^a$.

Applying the JW transformation to the Levin-Gu model, we can rewrite (B.1) in terms of the fermions,

$$H_{\text{LG}} = -L + \sum_{i=1}^{L} \left(2f_i^\dagger f_i + (f_i^\dagger - f_i)(f_{i+2}^\dagger + f_{i+2})\right) \tag{B.4}$$

with boundary condition

$$f_{i+L} = -(-1)^F f_i, \quad F = \sum_{j=1}^{L} n_j. \tag{B.5}$$

After Fourier transformation and Bogoliubov transformation, this Hamiltonian is diagonal

$$H_{\text{LG}} = \sum_k \omega_k \left(c_k^\dagger c_k - \frac{1}{2} \right), \quad (-1)^{\sum_k c_k^\dagger c_k} = (-1)^F \tag{B.6}$$

where $\omega_k = 4|\cos k|$. There are zero modes if k can be either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, and whether they are realizable depends on the boundary condition. It turns out that depending on $L \in 4\mathbb{Z}, 4\mathbb{Z}+2$ or $2\mathbb{Z}+1$, the boundary condition behaves differently. We discuss them separately.

Case 1: $L \in 4\mathbb{Z}$

If $(-1)^F = -1$, the fermion chain has PBC. This means $k = \frac{2\pi j}{L}$ where $j = 0, \cdots, L-1$. Therefore, when $j = \frac{L}{4}$ and $j = \frac{3L}{4}$, we have two zero modes at $k = \frac{\pi}{2}$ and $k = \frac{3\pi}{2}$. Since $(-1)^F = -1$, the ground states are: $c_{\frac{\pi}{2}}^\dagger |\text{VAC}\rangle_{\text{PBC}}$ and $c_{\frac{3\pi}{2}}^\dagger |\text{VAC}\rangle_{\text{PBC}}$. The ground state energy is

$$E_{\text{GS}}^{\text{PBC}} = -2\sum_{j=0}^{L-1} |\cos(\frac{2\pi j}{L})| = -4\cot(\frac{\pi}{L}). \tag{B.7}$$

If $(-1)^F = 1$, the fermion chain has anti-periodic boundary condition (ABC) where $k = \frac{(2j+1)\pi}{L}$. Since $L \in 4\mathbb{Z}$, there is no zero mode. the ground state is $|\text{VAC}\rangle_{\text{ABC}}$ with ground state energy:

$$E_{\text{GS}}^{\text{ABC}} = -2\sum_{j=0}^{L-1} |\cos(\frac{(2j+1)\pi}{L})| = -\frac{4}{\sin(\frac{\pi}{L})}. \tag{B.8}$$

As $E_{\text{GS}}^{\text{ABC}} < E_{\text{GS}}^{\text{PBC}}$, the Levin-Gu model has an unique true ground state which is vacuum of ABC after Jordan-Wigner transformation.

Case 2: $L \in 4\mathbb{Z}+2$

If $(-1)^F = -1$, the fermion chain has PBC where $k = \frac{2\pi j}{L}$, $j = 0, \cdots, L-1$. Since $L = 4m+2 \in 4\mathbb{Z}+2$, there is no zero mode. The ground states are $c_{\frac{2m\pi}{4m+2}}^\dagger |\text{VAC}\rangle_{\text{PBC}}$, $c_{\frac{2\pi(m+1)}{4m+2}}^\dagger |\text{VAC}\rangle_{\text{PBC}}$, $c_{\frac{2\pi(3m+1)}{4m+2}}^\dagger |\text{VAC}\rangle_{\text{PBC}}$ and $c_{\frac{2\pi(3m+2)}{4m+2}}^\dagger |\text{VAC}\rangle_{\text{PBC}}$. The ground state energy is

$$E_{\text{GS}}^{\text{PBC}} = -2\sum_{j=0}^{L-1} |\cos(\frac{2\pi j}{L})| + 4\cos(\frac{m\pi}{2m+1}) = -\frac{4}{\sin(\frac{\pi}{L})} + 4\sin(\frac{\pi}{L}). \tag{B.9}$$

If $(-1)^F = 1$, the fermion chain has ABC where $k = \frac{(2j+1)\pi}{L}$. Since $L = 4m+2 \in 4\mathbb{Z}+2$, there are two zero modes at $j = m$ and $j = 3m+1$. The ground states are double degenerate $|\text{VAC}\rangle_{\text{ABC}}$ and $c_{\frac{\pi}{2}}^\dagger c_{\frac{3\pi}{2}}^\dagger |\text{VAC}\rangle_{\text{ABC}}$ with energy

$$E_{\text{GS}}^{\text{ABC}} = -2\sum_{j=0}^{L-1} |\cos(\frac{(2j+1)\pi}{L})| = -4\cot(\frac{\pi}{L}). \tag{B.10}$$

Since

$$E_{\text{GS}}^{\text{ABC}} - E_{\text{GS}}^{\text{PBC}} = -4\cot(\frac{\pi}{L}) + \frac{4}{\sin(\frac{\pi}{L})} - 4\sin(\frac{\pi}{L}) = -4\cot(\frac{\pi}{L})\Big(1-\cos(\frac{\pi}{L})\Big) < 0 \quad \text{(B.11)}$$

the Levin-Gu model has double degenerate ground states which is vacuum of ABC.

Case 3: $L \in 2\mathbb{Z}+1$

If $(-1)^F = 1$, the fermion chains has ABC where $k = \frac{(2j+1)\pi}{L}$ and where $j = 0, \cdots, L-1$. Now since $L = 2m+1 \in 2\mathbb{Z}+1$, there is no zero mode. The ground states is $|\text{VAC}\rangle_{\text{ABC}}$ with energy

$$E_{\text{GS}}^{\text{ABC}} = -2\sum_{j=0}^{2m} |\cos(\frac{(2j+1)\pi}{2m+1})| = -4\sum_{j=0}^{m-1} |\cos(\frac{(2j+1)\pi}{2m+1}) - 2. \quad \text{(B.12)}$$

If $(-1)^F = -1$, the fermion chain has PBC where $k = \frac{2\pi j}{L}$ where $j = 0, \cdots, L$. Now since $L = 2m+1 \in 2\mathbb{Z}+1$, there is also no zero mode. Here we note that the energy of $|\text{VAC}\rangle_{\text{PBC}}$ is the same as (B.12)

$$\begin{aligned} E_{\text{VAC}}^{\text{PBC}} &= -2\sum_{j=0}^{2m} |\cos(\frac{2\pi j}{2m+1})| = -4\sum_{j=1}^{m} |\cos(\frac{2\pi j}{2m+1})| - 2 \\ &= -4\sum_{j=1}^{m} |\cos(\frac{(2m-2j+1)\pi}{2m+1})| - 2 = -4\sum_{j=0}^{m-1} |\cos(\frac{(2j+1)\pi}{2m+1})| - 2. \end{aligned} \quad \text{(B.13)}$$

Since there is no zero mode, the ground state energy in $(-1)^F = -1$ sector must be higher than $E_{\text{VAC}}^{\text{PBC}}$ which coincides with the ground state energy (B.12) under the ABC and the unique true ground state is $|\text{VAC}\rangle_{\text{ABC}}$.

In summary, the ground state degeneracy of the Levin-Gu model under PBC is two if $L \in 4\mathbb{Z}+2$, and one otherwise. This proves (3.12).

B.2 Mapping to XX Chain and Charge of Ground State

When the system size is even ($L = 2m$), there is a unitary transformation [58]

$$U = \prod_{j=1}^{m} \exp(\frac{\pi i}{2}\sigma_{2j}^y) \prod_{j=1}^{m} i\frac{\sigma_{2j}^z + \sigma_{2j}^x}{\sqrt{2}} \prod_{j=1}^{m} \exp(\frac{\pi i(1-\sigma_{2j-1}^z)(1-\sigma_{2j}^z)}{4}) \quad \text{(B.14)}$$

which maps the Levin-Gu model to a XX chain with imaginary hopping constant.

$$\begin{aligned} UH_{\text{LG}}U^\dagger &= -\sum_{j=1}^{m}(\sigma_{2j-1}^z\sigma_{2j}^x - \sigma_{2j}^x\sigma_{2j+1}^x - \sigma_{2j-1}^x\sigma_{2j}^z + \sigma_{2j}^z\sigma_{2j+1}^x) \\ &= -\sum_{j=1}^{L} i\sigma_j^+\sigma_{j+1}^- + h.c. \end{aligned} \quad \text{(B.15)}$$

where $\sigma_j^+ = \sigma_j^z + i\sigma_j^x$. The imaginary hopping XX chain can be further mapped to a standard XX chain by a unitary transformation

$$U_1 = \prod_{j=1}^{L} \exp(\frac{\pi i}{2} j \sigma_j^y). \tag{B.16}$$

The resulting Hamiltonian is

$$U_1 U H_{\text{LG}} U^\dagger U_1^\dagger = -\sum_{j=1}^{L} (\sigma_j^z \sigma_{j+1}^z + \sigma_j^x \sigma_{j+1}^x) \tag{B.17}$$

with boundary condition

$$\sigma_{L+j}^z = i^L \sigma_j^z, \quad \sigma_{L+j}^x = i^L \sigma_j^x. \tag{B.18}$$

After taking the continuum limit [59, 60]

$$(\sigma^z + i\sigma^x) \propto e^{i\theta}, \quad \sigma^y \propto \frac{a}{2\pi} \partial_x \phi, \tag{B.19}$$

the low energy theory of standard XX chain is the free boson theory and the energy of eigenstate $|m, n\rangle$ is[24]

$$(E_{m,n} - E_{0,0}) \propto \frac{\pi}{2L} (m^2 + 4n^2) \tag{B.20}$$

where the integer pairs (m, n) are determined by the boundary conditions $\theta(x+L) = \theta(x) + 2\pi m$ and $\phi(x+L) = \phi(x) + 2\pi n$. By combining (B.17), (B.18) and (B.20), we conclude as follows.

1. When $L \in 4\mathbb{Z}$, the Levin-Gu model is equivalent to the XX chain with PBC where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Its energy minimizes at a unique value $(m, n) = (0, 0)$, and the unique ground state is $|0, 0\rangle$.

2. When $L \in 4\mathbb{Z} + 2$, the Levin-Gu model is equivalent to the XX chain with ABC where $m \in \mathbb{Z} + 1/2$ and $n \in \mathbb{Z}$. Its energy minimizes at two distinct values $(m, n) = (\pm\frac{1}{2}, 0)$, and there are two degenerate ground states $|\pm\frac{1}{2}, 0\rangle$.

This is consistent with the results from JW transformation in (B.1).

Moreover we can obtain the $\mathbb{Z}_2$ symmetry (B.2) after transformation

$$U_G' = U_1 U U_G U^\dagger U_1^\dagger = \prod_{j=1}^{L} \sigma_j^y \prod_{j=1}^{\frac{L}{2}} \exp\left(\frac{\pi i}{4} (2 + \sigma_{2j-1}^x \sigma_{2j}^z - \sigma_{2j}^z \sigma_{2j+1}^x) \right). \tag{B.21}$$

After taking the continuum limit (B.19), the $\mathbb{Z}_2$ symmetry operator in the low energy is given by

$$U_G' = i^{\frac{L}{2}} \exp\left(\frac{i}{2} \int \partial_x \phi dx - \frac{i}{2} \int \partial_x \theta dx \right). \tag{B.22}$$

[24] Since we are only interested in ground state degeneracy, we don't consider excitations of the oscillator modes.

The charge of the state can be found by acting U'_G on $|m,n\rangle$,

$$U'_G\,|m,n\rangle = i^{\frac{L}{2}} e^{i\pi(n-m)}\,|m,n\rangle\,. \tag{B.23}$$

Therefore when $L \in 4\mathbb{Z}$, the charge of ground state $|0,0\rangle$ is $(-1)^{L/4}$. When $L \in 4\mathbb{Z}+2$ the charges of ground states $|\pm\frac{1}{2},0\rangle$ are $\pm(-1)^{\frac{L-2}{4}}$. This proves (3.14) for even L.

B.3 Spectrum under Open Boundary Condition

In this section, we use the transformations (B.14) and (B.16) to discuss spectrum of Levin-Gu model under OBC

$$H_{\text{LG}}^{\text{OBC}} = -\sum_{i=2}^{L-1}\left(\sigma_i^x - \sigma_{i-1}^z\sigma_i^x\sigma_{i+1}^z\right). \tag{B.24}$$

There are two boundary operators σ_1^z and σ_L^z commuting with Hamiltonian.

When $L \in 2\mathbb{Z}$, the Hamiltonian (B.24) and the boundary operators $\sigma_{1,L}^z$ after the transformation are given by

$$U_1 U H_{\text{LG}}^{OBC} U^\dagger U_1^\dagger = -\sum_{j=1}^{\frac{L}{2}-1}\left(\sigma_{2j-1}^x\sigma_{2j}^x + \sigma_{2j+1}^z\sigma_{2j+2}^z + \sigma_{2j}^x\sigma_{2j+1}^x + \sigma_{2j}^z\sigma_{2j+1}^z\right), \tag{B.25}$$

$$U_1 U\sigma_1^z U^\dagger U_1^\dagger = -\sigma_1^x, \quad U_1 U\sigma_L^z U^\dagger U_1^\dagger = (-1)^{\frac{L}{2}+1}\sigma_L^z. \tag{B.26}$$

After taking the continuum limit, the boundary operators are $-\sin\theta(x=0)$ and $(-1)^{\frac{L}{2}+1}\cos\theta(x=L)$. As the ground state should be the eigenstate of the boundary operators $-\sigma_1^x, (-1)^{\frac{L}{2}+1}\sigma_L^z$, $-\sin\theta(x=0) = \pm1, (-1)^{\frac{L}{2}+1}\cos\theta(x=L) = \pm1$. They determine the boundary conditions $\theta(x=0) = \pm\frac{\pi}{2}$ and $\theta(x=L) = 0$ or π. The ground state energy under these four boundary conditions are exactly the same.

When $L \in 2\mathbb{Z}+1$, we only do the transformation (B.14) for even number of sites, say, $i = 1, ..., L-1$. We still do $\pi/2$ rotation along y direction, i.e. U_1 in (B.16), on the L-th site. The Hamiltonian (B.24) and the boundary operators after the transformation are given by

$$U_1 U H_{\text{LG}}^{OBC} U^\dagger U_1^\dagger = -\sum_{j=1}^{\frac{L-1}{2}}\left(\sigma_{2j-1}^x\sigma_{2j}^x + \sigma_{2j}^x\sigma_{2j+1}^x\right) + \sum_{j=1}^{\frac{L-3}{2}}\left(\sigma_{2j+1}^z\sigma_{2j+2}^z + \sigma_{2j}^z\sigma_{2j+1}^z\right) \tag{B.27}$$

$$U_1 U\sigma_1^z U^\dagger U_1^\dagger = -\sigma_1^x, \quad U_1 U\sigma_L^z U^\dagger U_1^\dagger = \sigma_L^x. \tag{B.28}$$

After taking the continuum limit, the boundary operators are $-\sin\theta(x=0)$ and $\sin\theta(x=L)$ which implies boundary conditions are $\theta(x=0) = \pm\frac{\pi}{2}$ and $\theta(x=L) = \pm\frac{\pi}{2}$, and the signs are uncorrelated. Unlike even size, the states with different boundary conditions have different energies,

$$E_{(\mp\frac{\pi}{2},\pm\frac{\pi}{2})} - E_{(\pm\frac{\pi}{2},\pm\frac{\pi}{2})} \propto \frac{1}{L} \tag{B.29}$$

where the signs are correlated. Therefore the true ground states are double degenerate and are in the sector with boundary conditions $\theta(x=0) = \theta(x=L) = \pm\frac{\pi}{2}$.

C Equivalence Between Ground sector of $\mathbb{Z}_4$ SPT criticality and Levin-Gu model

In this section, we show the ground state of the pre-decorated model (3.8) of $\mathbb{Z}_4^\Gamma$ strong SPTC is the same as the Levin-Gu model (3.10) with $\tau_i^x = 1$.

Let us begin with the pre-decorated model (3.8) with PBC, which we reproduce here

$$U_{DW} H_{\text{SPTC}} U_{DW}^\dagger = -\sum_{i=1}^{L} \left(\sigma_i^x - \sigma_{i-1}^z \tau_{i-\frac{1}{2}}^x \sigma_i^x \tau_{i+\frac{1}{2}}^x \sigma_{i+1}^z + \tau_{i-\frac{1}{2}}^x \right). \tag{C.1}$$

Since the last term commutes with all other terms, the Hibert space can be divided into sectors with different τ^x configurations. In different sectors, the sign of term $\sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z$ is decided by $\tau_{i-\frac{1}{2}}^x \tau_{i+\frac{1}{2}}^x$. It is easy to see that the number of terms with $\tau_{i-\frac{1}{2}}^x \tau_{i+\frac{1}{2}}^x = -1$ must be even, since $\prod_{i=1}^L \tau_{i-\frac{1}{2}}^x \tau_{i+\frac{1}{2}}^x = 1$. We prove the splitting of ground state energy of first two terms in (C.1) with different τ configuration is order of $1/L$ or exactly zero. Therefore, when L is large enough, the state in the ground state sector of (C.1) satisfies $\tau_{i+\frac{1}{2}}^x = 1$ for each i.

When $L \in 2\mathbb{Z}+1$, we can prove the first two terms in (C.1) with any τ configuration can be mapped to the standard Levin-Gu model by a unitary transformation.

This implies the ground state energy of any τ configuration is same as that of the standard Levin-Gu model. To see the unitary transformation, let us assume that the sign of two terms $\sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z$ and $\sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^z$ are both -1 where $1 \leq i < j \leq L$.[25] There is always a unitary transformation which can cancel these two -1 and preserve sign of other terms: If i,j are both odd (even), the unitary transformation is $\prod_{i<2k<j} \sigma_{2k}^x$ ($\prod_{i<2k+1<j} \sigma_{2k+1}^x$). If i is odd (even) and j is even (odd), the unitary transformation is $\prod_{i<2k<L} \sigma_{2k}^x \prod_{1\leq 2k+1<j} \sigma_{2k+1}^x$ ($\prod_{j<2k<L} \sigma_{2k}^x \prod_{1\leq 2k+1<i} \sigma_{2k+1}^x$) which can do the job only when $L \in 2\mathbb{Z}+1$. Since the number of terms with -1 sign is even, we can cancel these -1s step by step and obtain the standard Levin-Gu model at last.

When $L \in 2\mathbb{Z}$, we apply the unitary transformation (B.14) and (B.16) on the first two terms and then obtain XX chain with several minus coupling constants :

$$H_{\mu^1,\mu^2} = -\sum_{j=1}^{L} (\mu_{j,j+1}^1 \sigma_j^z \sigma_{j+1}^z + \mu_{j,j+1}^2 \sigma_j^x \sigma_{j+1}^x) \tag{C.2}$$

where μ^1 and μ^2 can be ± 1. They are decided by the configuration of τ^x but we don't need to know the exact relationship. We only use the fact that $l + l' \in 2\mathbb{Z}$ where l and l' are number of -1 in μ^1 and μ^2.[26]

We note that the spectrum of Hamiltonian (C.2) only depends on $l, l' \mod 2$, and is independent of the configuration of μ^1 and μ^2. The reason is as follows. The sites of -1 in μ^1 can be

[25] We only focus on the "fundamental domain" where $1 \leq i < j \leq L$ and do not use periodicity $i \sim i + L$ here.

[26] $l + l' \in 2\mathbb{Z}$ can be seen from the transformation (B.14) and (B.16), which maps $\sigma_{2j-1}^x \to \sigma_{2j-1}^z \sigma_{2j}^z$, $\sigma_{2j}^x \to \sigma_{2j-1}^x \sigma_{2j}^x$, $-\sigma_{2j-1}^z \sigma_{2j}^x \sigma_{2j+1}^z \to \sigma_{2j}^x \sigma_{2j+1}^x$ and $-\sigma_{2j}^z \sigma_{2j+1}^x \sigma_{2j+2}^z \to \sigma_{2j}^z \sigma_{2j+1}^z$.

labeled as $\mu^1_{j_1,j_1+1}, \mu^1_{j_2,j_2+1}, \cdots \mu^1_{j_l,j_l+1}$ where $j_1 < j_2 < \cdots < j_l$. After the unitary transformation $\prod_{k=j_i+1}^{j_{i+1}} \sigma^x_k$, μ_{j_i,j_i+1} , $\mu_{j_{i+1},j_{i+1}+1}$ will become 1 without changing spectrum. Similar for μ^2.

As $l + l'$ are even, there are only two equivalence classes for spectrum: $l = l' = 0$ and $l = l' = 1$. The first case is XX chain with PBC. In the second case, we can choose $\mu^1_{L,1} = \mu^2_{L,1} = -1$ without loss of generality. This is XX chain with the ABC. The splitting between ground state energy of these two boundary conditions is order of $1/L$ which completes our proof.

Besides, one can apply this argument to the $\mathbb{Z}_4$ SPTC with TBC and OBC as well. Generally, the ground state sector is Hilbert subspace which has eigenvalue 1 of the third term in the Hamiltonian (3.16), (3.18) and (3.21).

D Edge Degeneracy of SPTC

In section 2.3.3 and 3.2.3, we discussed the degeneracy of weak and strong SPTC under OBC by studying the dimension of irreducible representation of operators commuting with the Hamiltonian. In this appendix, we rederive the degeneracy under OBC in an alternative way. We first undecorate the domain wall which maps the SPTCs to the Ising and Levin-Gu models under OBC respectively, and then use the results in section B to rederive the degeneracy.

D.1 Edge Degeneracy of $\mathbb{Z}_2 \times \mathbb{Z}_2$ Weak SPTC

In section 2.3.3, we studied the $\mathbb{Z}_2 \times \mathbb{Z}_2$ weak SPTC under OBC, with the Hamiltonian (2.20),

$$H^{\text{OBC}}_{\text{SPTC}} = -\sum_{i=1}^{L-1} \left(\sigma^z_i \tau^x_{i+\frac{1}{2}} \sigma^z_{i+1} + \sigma^z_i \sigma^z_{i+1} \right) - \sum_{i=2}^{L} \tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}}. \tag{D.1}$$

After U_{DW} transformation, the Hamiltonian is given by

$$U_{DW} H^{\text{OBC}}_{\text{SPTC}} U^\dagger_{DW} = -\sum_{i=1}^{L-1} \left(\tau^x_{i+\frac{1}{2}} + \sigma^z_i \sigma^z_{i+1} \right) - \sum_{i=2}^{L} \sigma^x_i \tag{D.2}$$

$\tau_{L+\frac{1}{2}}$ decouples from the Hamiltonian which gives two ground state degeneracy. The σ^z_0 commutes with Hamiltonian which gives two fixed boundary conditions on the left end and the right end is free boundary condition. Therefore we have four exact ground states. But this is unstable under symmetric perturbations as noted in section 2.3.3. We can add the boundary term (2.23) which becomes

$$-\sigma^z_L \tau^x_{L+\frac{1}{2}} \tag{D.3}$$

after conjugated by U_{DW}, i.e. domain wall undecoration. Now $\tau^x_{L+\frac{1}{2}}$ no longer decouples, which lifts degeneracy due to free boundary condition on the right, and ground state degeneracy reduces to two.

D.2 Edge Degeneracy of $\mathbb{Z}_4$ Strong SPTC

In section 3.2.3, we studied the $\mathbb{Z}_4$ strong SPTC under OBC, with the Hamiltonian (3.21)

$$H_{\text{SPTC}}^{\text{OBC}} = -\sum_{i=2}^{L}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}}\right) - \sum_{i=1}^{L-1}\sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1}. \tag{D.4}$$

After undecorating the domain wall, we obtain the Levin-Gu model under OBC

$$U_{DW}H_{\text{SPTC}}^{\text{OBC}}U_{DW}^{\dagger} = -\sum_{i=1}^{L-1}\tau^x_{i+\frac{1}{2}} - \sum_{i=2}^{L-1}(\sigma^x_i - \sigma^z_{i-1}\tau^x_{i-\frac{1}{2}}\sigma^x_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1}) - (\sigma^x_L - \sigma^z_{L-1}\tau^x_{L-\frac{1}{2}}\sigma^x_L\tau^x_{L+\frac{1}{2}}). \tag{D.5}$$

The ground state should be the eigenstate of $\tau^x_{i-\frac{1}{2}}$ $(i < L+1)$ with eigenvalue 1. The low energy effective Hamiltonian is :

$$U_{DW}H_{\text{SPTC}}^{\text{OBC}}U_{DW}^{\dagger}|_{\text{low}} = -\sum_{i=2}^{L-1}(\sigma^x_i - \sigma^z_{i-1}\sigma^x_i\sigma^z_{i+1}) - (\sigma^x_L - \sigma^z_{L-1}\sigma^x_L\tau^x_{L+\frac{1}{2}}). \tag{D.6}$$

Since $\tau^x_{L+\frac{1}{2}}$ commute with effective Hamiltonian, we can redefine $\tau^x_{L+\frac{1}{2}}$ as σ^z_{L+1} and (D.6) becomes (B.24) with system size $L+1$. We thus conclude that when $L \in 2\mathbb{Z}+1$, the ground state degeneracy is four and when $L \in 2\mathbb{Z}$ the ground state degeneracy is two.

E $\mathbb{Z}_4^{\mathbb{T}} \times \mathbb{Z}_2$ Strong SPTC

In this section we discuss another example of strong SPTC which respects the $\mathbb{Z}_4^{\mathbb{T}} \times \mathbb{Z}_2$ symmetries. We will also discuss the PBC, TBC and OBC.

E.1 Lattice Hamiltonian

Let us assign three spin-$\frac{1}{2}$s τ, σ and μ per unit cell and the Hamiltonian is:

$$H_{\mathbb{Z}_4^{\mathbb{T}}\times\mathbb{Z}_2} = \sum_j\left(\mu^z_j\tau^x_{j+\frac{1}{2}}\mu^z_{j+1} + \sigma^z_j\mu^z_j\tau^x_{j+\frac{1}{2}}\mu^z_{j+1}\sigma^z_{j+1} + \sigma^x_j\mu^x_j + \sigma^x_j\right) - \sum_j \tau^z_{j-\frac{1}{2}}\mu^x_j\tau^z_{j+\frac{1}{2}}. \tag{E.1}$$

This Hamiltonian respects the following symmetry:

$$\mathbb{Z}_4^{\mathbb{T}} : U_{\mathbb{T}} \equiv \prod_j\left(\frac{1+\mu^x_j}{2}\sigma^x_j + \frac{1-\mu^x_j}{2}i\sigma^y_j\right)K, \quad U_{\mathbb{T}}^2 = \prod_j \mu^x_j \tag{E.2}$$

$$\mathbb{Z}_2^{\tau} : U_{\tau} \equiv \prod_j \tau^x_j. \tag{E.3}$$

where $\mathbb{T}$ stands for time reversal, and K is the complex conjugation.

To see that (E.1) is a $\mathbb{Z}_2 \times \mathbb{Z}_4^{\mathbb{T}}$ strong SPTC, we show that it can be obtained by starting with a $\mathbb{Z}_2^{\tau} \times \mathbb{Z}_2^{\mathbb{T}}$ anomalous critical theory, and decorating the $\mathbb{Z}_2^{\tau}$ domain wall by 1d $\mathbb{Z}_2^{\mu}$ gapped SPT, where $\mathbb{Z}_2^{\mu}$ is generated by $U_{\mathbb{T}}^2$. Let us apply U_{DW} of τ and μ on both the Hamiltonian (E.1) and the symmetry operators (E.2) and (E.3).

$$U_{DW} U_{\mathbb{T}} U_{DW}^{\dagger} = \prod_j \left(\frac{1 + \mu_j^x \tau_{j-\frac{1}{2}}^z \tau_{j+\frac{1}{2}}^z}{2} \sigma_j^x + \frac{1 - \mu_j^x \tau_{j-\frac{1}{2}}^z \tau_{j+\frac{1}{2}}^z}{2} i\sigma_j^y \right) K, \tag{E.4}$$

$$U_{DW} U_{\tau} U_{DW}^{\dagger} = U_{\tau}, \tag{E.5}$$

$$U_{DW} H_{\mathbb{Z}_4^{\mathbb{T}} \times \mathbb{Z}_2} U_{DW}^{\dagger} = \sum_j (\tau_{j+\frac{1}{2}}^x + \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z + \sigma_j^x + \tau_{j-\frac{1}{2}}^z \sigma_j^x \mu_j^x \tau_{j+\frac{1}{2}}^z) - \sum_j \mu_j^x. \tag{E.6}$$

In (E.6), since the last term commutes with all other terms, the energy eigenstates are eigenstates of μ_j^x. Similar to the proof in the $\mathbb{Z}_4$ strong SPTC, we can consider the spectrum of first four terms in the Hamiltonian (E.6) with different configurations of μ^x. These four terms can be mapped to an XX chain by applying the unitary transformations (B.14):

$$H(\{\mu_j^x\}) = \sum_{j=1}^{L} \sigma_j^z \tau_{j+\frac{1}{2}}^z + \tau_{j+\frac{1}{2}}^z \sigma_{j+1}^z + \sigma_j^x \tau_{j+\frac{1}{2}}^x + \tau_{j-\frac{1}{2}}^x \sigma_j^x \mu_j^x. \tag{E.7}$$

According to the proof in appendix C, we know the spectrum of the (E.7) is invariant if we flip even number of μ^x. Thus, the spectrum of first four terms in (E.6) is that of XX chain with boundary condition: $\sigma_{L+j}^x = \pm\sigma_j^x$ and $\sigma_{L+j}^z = \sigma_j^z$, where we take $\pm$ sign if there are even or odd number of $\mu^x = -1$ respectively. After taking the continuum limit (B.19), these two boundary conditions are PBC and ABC for θ respectively. The splitting between the corresponding ground state energy is also of order $1/L$. Thus in the low energy state sector, one can find that $\mu_j^x = 1$. The effective Hamiltonian and symmetry are those of the boundary model of 2+1d $\mathbb{Z}_2^{\mathbb{T}} \times \mathbb{Z}_2$ SPT [58]:

$$U_{DW} U_{\mathbb{T}} U_{DW}^{\dagger}|_{\text{low}} = \prod_j \left(\frac{1 + \tau_{j-\frac{1}{2}}^z \tau_{j+\frac{1}{2}}^z}{2} \sigma_j^x + \frac{1 - \tau_{j-\frac{1}{2}}^z \tau_{j+\frac{1}{2}}^z}{2} i\sigma_j^y \right) K, \tag{E.8}$$

$$U_{DW} H_{\mathbb{Z}_4^{\mathbb{T}} \times \mathbb{Z}_2} U_{DW}^{\dagger}|_{\text{low}} = \sum_j (\tau_{j+\frac{1}{2}}^x + \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z + \sigma_j^x + \tau_{j-\frac{1}{2}}^z \sigma_j^x \tau_{j+\frac{1}{2}}^z). \tag{E.9}$$

Moreover the proof on the equivalence between ground state sector and XX chain can be generalized to twisted boundary conditions and open boundary conditions. We conclude that the ground state sector of different boundary conditions is always Hibert subspace which has eigenvalue 1 of the last term in the Hamiltonian (E.10) and (E.14).

E.2 Charge of Twisted Boundary Condition

We show that the charge of the ground state under TBC is nontrivial, implying that (E.1) is a nontrivial SPTC. Let us start by twisting the boundary condition using the $\mathbb{Z}_2^{\tau}$ symmetry, which we denote as $\mathbb{Z}_2^\tau$-TBC. The Hamiltonian (E.1) becomes

$$H_{\mathbb{Z}_4^{\mathbb{T}}\times\mathbb{Z}_2}^{\mathbb{Z}_2^\tau} = \sum_{j=1}^{L}(\mu_j^z\tau_{j+\frac{1}{2}}^x\mu_{j+1}^z + \sigma_j^z\mu_j^z\tau_{j+\frac{1}{2}}^x\mu_{j+1}^z\sigma_{j+1}^z + \sigma_j^x\mu_j^x + \sigma_j^x) - (\sum_{j=1}^{L-1}\tau_{j-\frac{1}{2}}^z\mu_j^x\tau_{j+\frac{1}{2}}^z - \tau_{L-\frac{1}{2}}^z\mu_j^x\tau_{\frac{1}{2}}^z).$$

The ground state satisfies

$$\tau_{j-\frac{1}{2}}^z\mu_j^x\tau_{j\frac{1}{2}}^z = 1 \quad (0 < j < L); \qquad \tau_{L-\frac{1}{2}}^z\mu_j^x\tau_{\frac{1}{2}}^z = -1. \tag{E.10}$$

which implies that the ground state has a nontrivial $\mathbb{Z}_2^\mu$ charge

$$\prod_{j=1}^{L}\mu_j^x\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^\tau} = -\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^\tau}. \tag{E.11}$$

On the other hand, if we twist by $\mathbb{Z}_2^\mu$ symmetry, the SPT criticality Hamiltonian becomes

$$\begin{aligned} H_{\mathbb{Z}_4^{\mathbb{T}}\times\mathbb{Z}_2}^{\mathbb{Z}_2^\mu} = \quad & \sum_{j=1}^{L-1}(\mu_j^z\tau_{j+\frac{1}{2}}^x\mu_{j+1}^z + \sigma_j^z\mu_j^z\tau_{j+\frac{1}{2}}^x\mu_{j+1}^z\sigma_{j+1}^z) + \sum_{j=1}^{L}(\sigma_j^x\mu_j^x + \sigma_j^x - \tau_{j-\frac{1}{2}}^z\mu_j^x\tau_{j+\frac{1}{2}}^z) \\ & -\mu_L^z\tau_{\frac{1}{2}}^x\mu_1^z - \sigma_L^z\mu_L^z\tau_{\frac{1}{2}}^x\mu_1^z\sigma_1^z \\ & = \tau_{\frac{1}{2}}^z H_{\mathbb{Z}_4^{\mathbb{T}}\times\mathbb{Z}_2}\tau_{\frac{1}{2}}^z. \end{aligned} \tag{E.12}$$

It is straightforward to check that $|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^\mu}$ has $\mathbb{Z}_2^\tau$ charge 1:

$$U_\tau\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^\mu} = U_\tau\tau_{\frac{1}{2}}^z U_\tau^\dagger U_\tau\,|\text{GS}\rangle = -\tau_{\frac{1}{2}}^z\,|\text{GS}\rangle = -\,|\text{GS}\rangle_{\text{tw}}^{\mathbb{Z}_2^\mu}. \tag{E.13}$$

E.3 Open Boundary Condition

To consider OBC, we truncate the spin chain so that σ-spins and μ-spins live on $i = 1, ..., L$, and τ-spins live on $i = \frac{3}{2}, ..., L + \frac{1}{2}$. We only keep the terms in (E.1) that are fully supported on the spin chain. The Hamiltonian is

$$H_{\mathbb{Z}_4^{\mathbb{T}}\times\mathbb{Z}_2}^{\text{OBC}} = \sum_{j=1}^{L-1}\mu_j^z\tau_{j+\frac{1}{2}}^x\mu_{j+1}^z + \sigma_j^z\mu_j^z\tau_{j+\frac{1}{2}}^x\mu_{j+1}^z\sigma_{j+1}^z + \sum_{j=1}^{L}\sigma_j^x\mu_j^x + \sigma_j^x - \sum_{j=2}^{L}\tau_{j-\frac{1}{2}}^z\mu_j^x\tau_{j+\frac{1}{2}}^z. \tag{E.14}$$

There are two boundary operators $\mu_1^x\tau_{\frac{3}{2}}^z$ and $\tau_{L+\frac{1}{2}}^z$ commuting with Hamiltonian. Since both of them anticommute with U_τ, there must be at least two exactly degenerate ground states of (E.14).

The exact ground state degeneracy can be determined by undecorating the domain wall, by applying U_{DW} on (E.14):

$$U_{DW}H_{\mathbb{Z}_4^{\mathbb{T}}\times\mathbb{Z}_2}^{\text{OBC}}U_{DW}^\dagger = \sum_{j=1}^{L-1}\tau_{j+\frac{1}{2}}^x + \sigma_j^z\tau_{j+\frac{1}{2}}^x\sigma_{j+1}^z + \sum_{j=1}^{L}\sigma_j^x + \sum_{j=2}^{L}\tau_{j-\frac{1}{2}}^z\sigma_j^x\mu_j^x\tau_{j+\frac{1}{2}}^z + \sigma_1^x\mu_1^x\tau_{\frac{3}{2}}^z - \sum_{j=2}^{L}\mu_j^x$$

and the two boundary operators becomes μ_1^x and $\tau^z_{L+\frac{1}{2}}$. In the ground state sector $\mu_j^x = 1$ for $2 \le j \le L$. The Hamiltonian in the low energy then simplifies to

$$U_{DW} H^{\mathrm{OBC}}_{\mathbb{Z}_4^{\mathbb{T}} \times \mathbb{Z}_2} U^\dagger_{DW}|_{\mathrm{low}} = \sum_{j=1}^{L-1} (\tau^x_{j+\frac{1}{2}} + \sigma^z_j \tau^x_{j+\frac{1}{2}} \sigma^z_{j+1}) + \sum_{j=1}^{L} \sigma^x_j + \sum_{j=2}^{L} \tau^z_{j-\frac{1}{2}} \sigma^x_j \tau^z_{j+\frac{1}{2}} + \sigma^x_1 \mu^x_1 \tau^z_{\frac{3}{2}}. \tag{E.15}$$

Under the unitary transformation (B.14), this Hamiltonian is mapped to

$$U \left(U_{DW} H^{\mathrm{OBC}}_{\mathbb{Z}_4^{\mathbb{T}} \times \mathbb{Z}_2} U^\dagger_{DW}|_{\mathrm{low}} \right) U^\dagger = \sum_{j=1}^{L-1} \sigma^z_j \tau^z_{j+\frac{1}{2}} + \tau^z_{j+\frac{1}{2}} \sigma^z_{j+1} + \sum_{j=1}^{L} \sigma^x_j \tau^x_{j+\frac{1}{2}} + \sum_{j=2}^{L} \tau^x_{j-\frac{1}{2}} \sigma^x_j + \sigma^x_1 \mu^x_1 \tag{E.16}$$

and the two boundary operators become μ^x_1 and $\tau^x_{L+\frac{1}{2}}$. The Hamiltonian (E.16) can be understood as an XX chain on an open chain with size $2L$ and one spin-$\frac{1}{2}$ per unit cell.

Similar to the $\mathbb{Z}_4$ strong SPTC, we can redefine μ^x_1 as $\tau^x_{\frac{1}{2}}$. After taking the continuum limit (B.19), σ^x and τ^x are mapped to $\sin\theta$. Thus $\mu^x_1 = \pm 1$ and $\tau^x_{L+\frac{1}{2}} = \pm 1$ correspond to the boundary conditions $\sin\theta(x = 0/L) = \pm 1$ which implies $\theta(x = 0/L) = \pm\frac{\pi}{2}$. There is an energy splitting between the ground states of two boundary conditions

$$E_{(\frac{\pi}{2}, -\frac{\pi}{2})/(-\frac{\pi}{2}, \frac{\pi}{2})} - E_{(\frac{\pi}{2}, \frac{\pi}{2})/(-\frac{\pi}{2}, -\frac{\pi}{2})} \propto \frac{1}{L}. \tag{E.17}$$

In summary, the ground state degeneracy under OBC is two.

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