We thank the referee for their rigorous and very constructive feedback.

We agree that a comparison of the Floquet-assisted superradiant phase to other dynamical phases improves our manuscript, and provides important context.

As a first comparison, we consider the dynamical normal phase (D-NP) that the referee mentions. The D-NP emerges in the Dicke model under parametric driving of the coupling parameter, resulting in a subharmonic oscillation between the two superradiant states. The D-NP is captured by the spontaneous emergence of density wave patterns in cavity-BEC systems. Cosme et al. (Phys. Rev. Lett. 121, 153001, Phys. Rev. A 100, 053615) have identified these states as time-crystalline. Keßler et al. have conducted experiments on such driven-dissipative cavity-BEC systems (Phys. Rev. Lett. 127, 043602). In these works, the driving is performed on the coupling parameter of the effective Dicke model. This type of driving is fundamentally different from the driving term in our work. The underlying mechanisms and behavior of the resulting phenomena are interesting, but distinct from the FSP. We are confident that they are not manifestations of the FSP, but distinct dynamically induced phases of the Dicke model, because they differ in the regime that is required for their emergence, and in their properties.

As a second comparison, we consider nitrogen vacancy center spins in diamond, which act as effective two-level systems that can be driven and coupled to a cavity to create a superradiant maser operating at room temperature (Nature 555, 493–496 (2018)). This is captured by an effective incoherently pumped Dicke model through a process that involves optically driving three Zeeman split levels (Sci. China Phys. Mech. Astron. 65, 217311 (2022)). This is substantially different from our work, where the coherent driving induces Floquet states that are brought into resonance with the cavity and are essential to the FSP. We include this comparison in the revision as nitrogen vacancies in diamonds might still present a potential platform for realizing the FSP by utilizing strong coherent driving at the suitable frequency.

Additionally, there has been a realization of a Floquet maser (Jiang et al. Sci. Adv. 7, 8 abe0719 (2021)) utilizing magnetic feedback circuits and dynamical Zeeman splitting through periodic magnetic driving of a 129Xe/87Rb mixture. They observe transient population inversion after flipping spin polarizations. The resulting decay is accompanied by superradiant emission. By periodically flipping polarizations they maintain continuous maser dynamics. When the driving is expressed via a quantized field, the effective Hamiltonian is comparable to the undriven Dicke model, albeit with a different coupling operator. The comb-like spectrum of the maser signal shows finite amplitudes at Floquet state energy differences, which are given by the driving frequency in this setup. This setup is interesting in its own right, but ultimately very different from our work, as the Floquet maser does not operate in a coherently driven steady state with population inverted Floquet bands, but uses a magnetic feedback flipping scheme that results in continuous multimode operation.

We include these discussions and the appropriate references in the revision and conclude that, to our knowledge, the FSP has not been discussed in other works. We further note that the dissipative scheme in our work is distinct from the description found in these references, further distinguishing our results from previously discussed dynamically induced phases.

We further answer the specific comments and questions by the referee:

(1) To directly create the FSP in a two-band solid, assuming a gap that is much larger than the bandwidth constitutes a good approximation of the model discussed in our manuscript. In that case the individual momentum modes will have similar dynamics due to their similar level spacing, thus remaining close to our case of equal level spacings for the two-level systems. This situation is not given in monolayer graphene. Rather, we provide here a minimal model that conceptualizes key features of a driven solid. A specific feature of optically driven solids is the form of the dissipative model. Our model describes the dissipative dynamics of two-band solids well - as has been discussed and shown in previous works (Phys. Rev. Research 2, 043408 (2020)) - and can be applied to two-band solids regardless of the dispersion relation, resulting in a more intricate system. In the minimal model we present here, we demonstrate that this dissipative model indeed supports the FSP. This is a non-trivial point to make, because the inversion that this generates on the Floquet states is influenced by the form and magnitude of the dissipation. The specific example of monolayer graphene is motivated by the population inversion demonstrated under strong circularly polarized driving in

previous work (Communications Physics 4, 248 (2021)). The resulting phase diagram in |alpha| will generally look different to our results, as different modes will contribute differently to the collective processes. However, the system is still susceptible to Floquet states that are brought into resonance with the cavity while experiencing population inversion. The exact shape of the FSP will depend on the individual Floquet bands and some band structures may be better suited for hosting a FSP than others, but the general mechanism may be present regardless. The exact study of what band structures lead to the FSP in what manner is subtle and will be explored elsewhere. We have emphasized these points more clearly in the revision.

We expect that inhomogeneous broadening would introduce dephasing that may be captured in an increase of the effective dissipation coefficients present in our model. Depending on the magnitude of the broadening, this might suppress the FSP. However, we find the FSP is more sensitive to the cavity loss rate and not so much to the two-level dissipation terms, such as the dephasing gamma_z. Hence, the effect of inhomogeneous broadening might not be large. Since the FSP originates from inverted Floquet states brought into resonance with the cavity such that detuned 2-level systems end up closer together than their original level spacings, we expect the effect on the linewidth to not be large. Particularly strongly detuned two-level systems might not participate in the FSP.

(2) The dissipative processes in the instantaneous eigenbasis of the two-level systems are motivated by modeling solid-state systems. This has been discussed in earlier works by Nuske et al. (Phys. Rev. Research 2, 043408 (2020)) and demonstrated by accurately describing anomalous Hall transport in light-driven graphene, as observed in experiment by McIver et al. (Nature Physics volume 16, pages 38–41 (2020)). Alternatively, as the referee suggests, we can ask the question if the FSP exists if the standard, stationary basis is used in the dissipative processes. Such a model might apply to cavity-BEC systems, or a large class of other cavity systems. The FSP does indeed require dissipation and is in particular sensitive to the cavity loss rate. This choice of dissipation does affect the results. We leave the details of the FSP in presence of dissipation in the stationary basis instead of solid-like dissipation to future research.

(3) We thank the referee for noticing this. Indeed, this factor of four is an oversight. We have corrected this mistake in the revision, as well as other notational inconsistencies and prefactor errors. This includes changes to the equations in the main text and we have redone all numerics accordingly. The results remain essentially unaltered and the conclusions in the manuscript do not change, however there are some quantitative changes in particular in Fig. 2. Note that the agreement between Fig. 2 (d) and Fig. 3 (c) has now improved further.

(4) Yes, this is correct. We agree that stating this more clearly is beneficial. We have done so in the revision.

(5) The specific reference to the cavity frequency is omitted in our phrasing. We agree that this can be confusing. What we are referring to as the good cavity regime is the case in which the cavity loss rate kappa is roughly equal to or smaller than the cavity frequency omega_c, where in the bad cavity regime kappa is much larger. In our case the ratio kappa over omega_c is not roughly equal to one, but rather on the order of 0.01, since the FSP is sensitive to the cavity loss rate. We have rephrased this to be more concise in the revision.

We think it is more instructive to think about the ratio of kappa and omega_c (or omega_d or omega_z), rather than bandwidth which does not need to be particularly large or larger than kappa by some margin. As discussed in question (1), the effects of particular band structures is intricate, but for a direct comparison the limit of bands that are flat on the order of their linewidth is desirable. In general, the conditions on the bandwidth compared to the cavity linewidth is unclear. Note that the FSP is more robust towards two-level dissipation than it is to cavity losses. In the revision we emphasize that in particular the small values of kappa are necessary.

(6) We thank the referee for this interesting question. In our numerics, the driving field is continuous and perfectly monochromatic. The emitted light expressed in the mean-field ansatz is equally as narrow in the steady state. As the referee points out, the width is limited by the integration time. To increase the visibility, we reworked Fig. 2 to now also show a smaller region of Fig. 2 (c). We expect that the emitted light in the FSP can overcome the linewidth of the driving field, resulting in line narrowing.

(7) Yes, that is the assumption we are referring to. We rephrase this in the revision to be more clear. It is true that the differences between those Floquet energies recover the approximate frequency in Fig. 2(c). There too is a slight deviation, due to the fact that this relies on the initial assumption that alpha oscillates with omega_c. An exact match between the Floquet energies and the maxima in n(omega) would be achieved when we introduce the driving term with the actual frequency of alpha in Fig. 2(c) instead of assuming that it is omega_c. The resulting Floquet energy difference would then give the frequency of alpha(omega) in a consistent way.

(8) We use the numerical solution of the magnitude of alpha for H1. This is the same solution of alpha that we show in Fig. 2. We initialize the numerics with alpha(t=0) equal to some small complex number, which can in Dicke models be seen as breaking the U(1) symmetry. More importantly however, the phase of the driving term breaks the symmetry and determines the complex phase of the FSP.