1 Reply to Referee 1

Comment. *In the paper, Zhou et al. studied the ground state of the triangular-lattice quantum Ising model with spatial anisotropy and NNN interactions J', focusing on the quantum tricriticality induced by effective interactions between quantum strings. The quantum Ising model, or transverse-field Ising model, has been extensively studied for many years, but recently it attracts lots of attention in relation to the development of artificial quantum simulators with trapped ions, Rydberg atom arrays, etc. Based on the numerical QMC calculations, the authors investigated here the J' dependence of the energy of strings, assuming that it is given in the form with two different algebraic-decaying terms. As a results, it is concluded that "the tricriticality of the incommensurate phase is caused by the competing of effective long-range inter-string interactions with different power exponents."*

The paper is scientifically sounds and well written. However, I would recommend the manuscript for publication only after the points listed below are properly addressed.

Reply. We thank the referee for his positive evaluation as well as the critical comments that have helped us improve the manuscript. In the following we give a point-by-point reply to the comments.

Comment 1.1. *Page 4: In the first sentence, the authors state that all the bonds connecting parallel spins are aligned along the x-direction, forming stripe phase, for $J_x < J$. However, the actual ground state can be incommensurate phase when J_x/J is larger than a certain critical value between 0 and 1. This is confusing.*

Reply 1.1. In the second paragraph of §2, we start by considering the classical Ising limit $h = 0$, *i.e.*, the vertical axis of Fig. 4(c). Whereas the incommensurate phase only emerges at finite transverse field, at $h = 0$ all the $J_x < J$ region belongs to the stripe phase by optimising the energy classically. We start considering the quantum case only from the third paragraph, which is after the position the referee mentions. In the revised manuscript, we make clarification by replacing '*in the anisotropic case $J_x < J$*' by '*in the classical anisotropic case $J_x < J$ and $h = 0$*'.

Comment 1.2. *Page 10: The authors state that the leading order approximation of B and C (namely, constant and linear term, respectively) is good enough. However, according to the inset of Fig. 6(c), the liner term should also be considered for B, like $B \sim B_0 + B_1 J'$.*

Reply 1.2. The referee is correct that for an accurate description, it is better to include a linear term in B. However, approximating B to a constant suffices to reproduce the qualitative result. On one hand, unlike C which changes sign with J', the J'-dependence of B is not other drastic. On the other hand, the B-term interaction originates from the hinder of motion when strings are nearby; this mechanism does not depend on the presence of J'; by contrast, the C-term interaction directly comes from the NNN coupling. By approximating $B = B_0$ and $C = C_1 J'$, one can find that the stripe-incommensurate phase transition is first-order when $J' > 0$ and continuous when $J' < 0$, and there is therefore a multicritical point at $J' = 0$.

To clarify, in the revised manuscript, we add some related discussion: '*We also note that for a more accurate description, $B(J')$ should be approximated as $B(J') \approx B_0 + B_1 J'$ to take into account*

the fact that the NNN coupling modifies the vibration of string, but this modification does not change the scenario qualitatively and can be therefore considered as a higher-order term.'

Comment 1.3. *Although I understood that the energy can be well approximated in the form of Eq. (10) (except for the above-mentioned point), I could not get the idea of why it yielded the conclusion that the effective inter-string interaction can be written in the form $B/r^\alpha - C/r^\gamma$ (what is r?).*

Reply 1.3. Firstly, r is the distance between two nearby strings. To calculate the average energy of quantum string, we replace r by its average value $\bar{r} = L\rho_{\rm QS}$.

Regarding the interaction energy ansatz $V(r) = B/r^\alpha - C/r^\gamma$, we here give a more detailed explanation. There are two mechanisms of string interaction: the first, denoted $V_h(r)$, is a repulsion from the hinder of motion when strings are nearby; the second, denoted $V_{J'}(r)$, comes from the fact that the insertion of single string produces energy cost $3J'/2$ per string length, while two adjacent strings cost energy $2J'$ per string length, which is different than two individual strings; therefore we can write $V(r) = V_h(r) + V_{J'}(r)$. Originally, these two mechanisms all act in short-range; the vibration of strings then turns these short-range interactions into long-range ones. The exact determination of the form of $V_{J'}(r)$ and $V_h(r)$ is a difficult question. In a former work [3], the repulsion $V_h(r)$ from motion hinder has been determined to follow a power-law behaviour in the incommensurate phase a similar model (hardcore Bose-Hubbard model) by fitting the jumping points of the plateaux. As an extension, it is most natural and simplest to assume that $V_{J'}$ also follow a power law. We therefore write down the ansatz $V(r) = B/r^\alpha - C/r^\gamma$. As an evidence, this ansatz fits well with our numerical result (Fig. 6(c)), as explained in the main text.

To clarify, we have also added the above paragraph to the main text, added the sentence '*where r is the distance between two nearby strings*' to the Conclusion section, and deleted the exact expression from the abstract to avoid confusion.

Comment 1.4. *In the sketch of the phase diagram in Fig. 1, the tricritical point is located just on the vertical axis. However, it is not the case, is it? This is somewhat misleading. Also, using the effective energy functional Eq. (10) with numerically fitted values of coefficients, I think that the authors can estimate the location of the tricritical point (J_x and J') in a more quantitative way. It should help to improve the quality and completeness of the paper.*

Reply 1.4. The tricritical point in fact locates on the vertical axis $J' = 0$ exactly. The numerical evidence is provided in Fig. 5(b) inset. We determine the tricritical point at finite size as the J' value where the first plateau disappears, i.e., $E(\rho_{\rm QS} = 0) = E(\rho_{\rm QS} = 2/L) = E(\rho_{\rm QS} = 4/L)$, and performed a finite size scaling. The result shows that the tricritical point should lie on the vertical axis $J'_c = 0$.

Intuitively, the tricriticality between first order and continuous transition line is driven by the string interaction mediated by J' (C-term) changing from attractive to repulsive. As C-term is mediated by the NNN coupling J', it is natural that the tricritical point locates at the zero-point of this coupling.

In the manuscript, it is also mentioned that '*Moving towards the thermodynamic limit, this point is found to gradually approach $J' = 0$ in Fig. 5(b) inset*'.

We hope that with the improvements made, the Referee can agree that the manuscript is suitable for further consideration in SciPost Phys.