2 Reply to Referee 2

Comment. *The authors have studied the ground state phases of the transverse field Ising model with nearest and next nearest neighbor interactions on an anisotropic triangular model at weak transverse fields. They develop a description of the ground state order in terms of strings separating domains of columnar AFM order. Each string behaves as a $S = 1/2$ XY chain with long-range interaction. The central result of the work is the detection of an incommensurate phase and the appearance of a tricritical point when NNN interactions are introduced.*

The problem is interesting and the results are intriguing. However, in the opinion of the present referee, the manuscript can be improved before publication. In particular, can the authors address the following?

Reply. We thank the referee for his positive evaluation as well as the critical comments that have helped us improve the manuscript. In the following we give a point-by-point reply to the comments.

Comment 2.1. *How is the sign problem avoided in SSE? Previous QMC studies [13, 33, Nat. Commun. 11, 1111 (2020)] all use special trick with path integral / world line QMC to avoid the sign problem where the coupling in the imaginary time direction is shown to be ferromagnetic. Sandvik's work [Phys. Rev. E 68, 056701 (2003)] using SSE on TFIM with long range interactions use a bipartite lattice. Since the authors are probably the first to use SSE for TFIM on a triangular lattice, a discussion on the method would greatly add to the manuscript.*

Reply 2.1. The transverse field Ising model does not have sign problem regardless of the lattice, as one can always add a constant to eliminate the negative matrix elements. E.g., the AFM TFIM

$$H = J \sum_{\langle ij \rangle} S_i^z S_j^z - \Gamma \sum_i S_i^x \tag{R1}$$

can be rewritten in the following by adding a constant energy shift

$$H = J \sum_{\langle ij \rangle} (S_i^z S_j^z - 1/4) - \Gamma \sum_i (S_i^x + 1/2) \tag{R2}$$

$$= - \sum_{\langle ij \rangle} H_{ij}^d - \sum_i H_i^o, \tag{R3}$$

where the local operators and their matrix elements are

$$\begin{aligned} &H_{ij}^d = J(-S_i^z S_j^z + 1/4) \quad \langle \uparrow\downarrow | H_{ij}^d | \uparrow\downarrow \rangle = \langle \downarrow\uparrow | H_{ij}^d | \downarrow\uparrow \rangle = J/2, \quad \text{(others)} = 0 \\ &H_i^o = \Gamma(S_i^x + 1/2) \qquad \langle \uparrow | H_i^o | \uparrow \rangle = \langle \uparrow | H_i^o | \downarrow \rangle = \langle \downarrow | H_i^o | \uparrow \rangle = \langle \downarrow | H_i^o | \downarrow \rangle = \Gamma/2 \end{aligned} \tag{R4}$$

All the matrix elements are non-negative; there is therefore no sign problem. In addition, some reference [Phys. Rev. B 98, 174433 (2018)] also applies SSE to non-bipartite lattices; the original reference (Ref. [48]) does not restrict the simulation to bipartite lattice.

Comment 2.2. *How do the authors conclude that $V(r)$ decays faster than linear? Doesn't the hard core constraint generalize to fast decaying interaction in the presence of quantum fluctuation ($h > 0$)?*

We provide a detailed description on how we determine the form of the interaction between strings in the Reply 5. The power of the interaction is determined by fitting. In the presence of quantum fluctuation, the hardcore constraint generates a power-law dynamic interaction.

Comment 2.3. *In the phase diagram 4(c), can the authors discuss the stripe phase? It appears to be deep inside the incommensurate phase.*

Reply 2.3. The stripe phase is actually not incommensurate, as its Bragg peak situates at the commensurate M point in the momentum space (Fig. 3(a)i). The real-space spin configuration in the stripe phase has parallel spins in the same row and alternating spin-ups and spin-downs in adjacent rows (Fig. 2(a)). Unlike the incommensurate phase which only appears in the presence of quantum term h, the stripe phase optimises the energy in the classical limit $h = 0$ at $J_x < J$. The reason we choose the stripe state as the reference state is that any local operation acting upon this state violates the local constraint, hence there are no low-energy excitations within the stripe bulk.

We have also added a brief discussion '*The ordering momentum is commensurate. The spin alignment alternates each row in the real space configuration [Fig. 2(a)]. This phase extends to* $J_x \to -\infty$ *and persists at the classical limit* $h = 0$.' to Section 3, Paragraph 2.

Comment 2.4. *Does "vibration" of the string refer to the process defined in Fig. 2(c)?*

Reply 2.4. Yes. In the revised manuscript, we have added the reference to Fig. 2(c) when mentioning the 'vibration'.

Comment 2.5. *It is not clear how the authors arrive at the functional dependence of the energy of the strings.*

As explained in §2, the energy of the string consists two parts. The first is the energy of single quantum string (Eq. (2)), which consists of the energy gap Δ and the kinetic energy obtained from mapping the string configuration to an effective XY chain; the second is the interaction between adjacent strings $V(\bar{r})$, where $V(r)$ is the interaction energy between adjacent strings, r is the distance between two nearby strings, and $\bar{r} = L\rho_{\mathrm{QS}}$ is its average value.

Regarding the interaction energy ansatz $V(r) = B/r^{\alpha} - C/r^{\gamma}$, we here give a more detailed explanation. There are two mechanisms of string interaction: the first, denoted $V_h(r)$, is a repulsion from the hinder of motion when strings are nearby; the second, denoted $V_{J'}(r)$, comes from the fact that the insertion of single string produces energy cost $3J'/2$ per string length, while two adjacent strings cost energy $2J'$ per string length, which is different than two individual strings; therefore we can write $V(r) = V_h(r) + V_{J'}(r)$. Originally, these two mechanisms all act in short-range; the vibration of strings then turns these short-range interactions into long-range ones. The exact determination of the form of $V_{J'}(r)$ and $V_h(r)$ is a difficult question. In a former work [3], the repulsion $V_h(r)$ from motion hinder has been determined to follow a power-law behaviour in the incommensurate phase a similar model (hardcore Bose-Hubbard model) by fitting the jumping points of the plateaux. As an extension, it is most natural and simplest to assume that $V_{J'}$ also follow a power law. We therefore write down the ansatz $V(r) = B/r^{\alpha} - C/r^{\gamma}$. As an evidence, this ansatz fits well with our numerical result (Fig. 6(c)), as explained in the main text.

To clarify, we have also added the above paragraph to the main text, added the sentence '*where r is the distance between two nearby strings*' to the Conclusion section, and deleted the exact expression from the abstract to avoid confusion.

Comment 2.6. *Should the width of the plateaus reduce to zero in the thermodynamic limit? What about the stripe phase? Does it have any finite-J_x extent in the thermodynamic limit?*

Reply 2.6. The width of the plateaux scales to zero in the thermodynamic limit. As an evidence, in the manuscript, we did the finite size scaling for a specific plateau ($\rho_{\mathrm{QS}} = 2/3$) and find that the ends of the plateau extrapolate to the same point [Fig. 4(b)]. The stripe phase will remain a finite region in the thermodynamic limit. As an intuitive argument, when the anisotropy is much stronger than the transverse field $J - J_x \gg h$, the stripe phase is always favoured. The estimation of the stripe-incommensurate critical point $J_{x,c} = J - 2h/\pi + \mathcal{O}(h^2)$ applies also to the thermodynamic limit.

To clarify, we have also added a sentence '*As demonstrated in finite-size scaling (Fig. 4(b)) of the plateau corresponding to the clock phase, the width of each plateaux scales to zero the string density ρ_{QS} should follow the continuous function (7) in the thermodynamic limit.*' to the related discussion.

We hope that with these changes made, the Referee could agree that the manuscript is suitable for further consideration for SciPost Phys.