Response to the comments of the Second referee

We thank the referee for finding our analytical results new and interesting, and for thinking that our article warrants publication in SciPost Physics. Below we present our response to the specific comments of the second referee.

1. Finite dimensional MPS representations for steady states of stochastic processes were to the best of my knowledge first considered in

F.H.L. Essler and V. Rittenberg 1996 J. Phys. A: Math. Gen. 29 3375 K Mallick and S Sandow 1997 J. Phys. A: Math. Gen. 30 4513 .

We thank the referee for pointing out the above mentioned references. These are now cited as new references [23] and [24], in the second paragraph of the introduction in context of finite dimensional matrices for quadratic algebra.

2. As the authors explain very clearly, the process is not ergodic and certain "motifs" are conserved under the dynamics. Does this imply that the Liouvillian is block-diagonal with respect to some underlying symmetry, and the number of blocks scales exponentially with the system size? If this is the case the situation would be somewhat similar to what has been recently observed in certain stochastic processes (both classical and quantum)

K. Klobas et al J. Stat. Mech. (2018) 123202
F.H.L. Essler and L. Piroli Phys. Rev. E 102, 062210 (2020).

If there is indeed such a symmetry structure I think it would be very nice to construct it explicitly.

We thank the referee for raising this interesting question. Indeed, the non-ergodicity of our model is reflected through the block-diagonal structure of the transition rate matrix (dictating the evolution of the probabilities of configurations in the Master equation, Eq. (3) in main text). According to this important suggestion, we have added one paragraph in the end of Sec. 4 and Appendix E.
To illustrate the block-diagonal structure with an example, we consider a small system of size $L = 4$ where the number of impurity and vacancy are given by $N_+ = 1$ and $N_0 = 1$, respectively, and the

total number of species 1 and species 2 particles is $N_1 + N_2 = 2$. Total number of configurations in the configuration space, in this case, is 48. However, since there is no spatial disorder in the transition rates, we take into account the translational invariance of the model on a periodic lattice. Consequently, there are 12 independent configurations of the system, which we denote as follows (depending on the sequence of species 1 and 2)

$$
\begin{array}{lll}
11+0 \equiv I_1, & 110+ \equiv I_2, & 101+ \equiv I_3, \\
12+0 \equiv II_1, & 120+ \equiv II_2, & 102+ \equiv II_3, \\
21+0 \equiv III_1, & 210+ \equiv III_2, & 201+ \equiv III_3, \\
22+0 \equiv IV_1, & 220+ \equiv IV_2, & 202+ \equiv IV_3.
\end{array}
\tag{1}
$$

We have divided the 12 configurations in Eq. (1) into 4 sectors I, II, III, IV where the three configurations within a given sector are connected through the *drift dynamics*. To investigate the connectivity between these sectors through the *flip dynamics*, below we provide the full transition rate matrix for these 12 configurations (enumerated consecutively from I_1 to IV_3),

$$
\left(\begin{array}{cccccc|cccccc}
-\epsilon - w_{12} & 0 & p_1 & w_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & -p_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_1 & -p_1 - w_{12} & 0 & 0 & w_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
w_{12} & 0 & 0 & -\epsilon - w_{21} & 0 & p_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \epsilon & -p_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & w_{12} & 0 & p_2 & -p_1 - w_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -\epsilon - w_{12} & 0 & p_2 & w_{21} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \epsilon & -p_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & -p_2 - w_{12} & 0 & 0 & w_{21} \\
0 & 0 & 0 & 0 & 0 & 0 & w_{12} & 0 & 0 & -\epsilon - w_{21} & 0 & p_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & -p_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{12} & 0 & p_2 & -p_2 - w_{21}
\end{array}\right)
\tag{2}
$$

In Eq. (2) we clearly observe that the transition rate matrix in in block-diagonal form with *two* blocks. We observe that sector I is connected to sector II through flip dynamics, whereas sector III and IV are also connected to each other via flip dynamics. However, sectors (I, II) are disconnected from sectors (III, IV), thereby creating two separate blocks in the rate matrix.

Note that, in absence of the flip dynamics (i.e. $w_{12} = w_{21} = 0$), sectors I become disconnected from II, similarly III gets disconnected from IV, resulting in four blocks in the transition matrix. On the other hand, in the special limit when $N_1 + N_2 = 1$, we would have a single block with the system becoming ergodic.

Next we explore the variation in the number of blocks as the system size is increased. We keep $N_0 = 1$ throughout, because it appears that the number of blocks depends on the arrangements of $1, 2$ and $+$, but not on the location of vacancies. This might be better understood in a box-particle representation of the model where $1, 2, +$ denote boxes and 0-s are particles.

For $L = 5$, the special case $N_1+N_2 = 1$ ($N_+ = 3$) keeps the system ergodic with a single block only. But, as we increase $N_1 + N_2$, e.g. $N_+ = 2$ and $N_1 + N_2 = 2$, one can check that the rate matrix is block-diagonal with 3 blocks. With further increase in $N_1 + N_2(=3)$ which also corresponds to $N_+ = 1$, we have 4 blocks in the transition rate matrix. Below we present N_{blocks} in a tabular form, explicitly for a few sets of (L, N_+), with $N_0 = 1$ and $N_1 + N_2 = L - N_0 - N_+$,

L	N_+	N_{blocks}
4	1	2
4	2	1
5	1	4
5	2	3
5	3	1
6	1	8
6	2	6
6	3	3
6	4	1
7	1	16
7	2	15
8	1	32
8	2	32
9	1	64
9	2	74
10	1	128
10	2	160

In fact, for fixed system size L, with $N_0 = 1$, the general formulae for number of blocks N_{blocks} in the transition rate matrix, for cases $N_+ = 1$ and $N_+ = 2$ turn out to be

$$\begin{aligned}
N_+ = 1: \qquad & N_{blocks} = 2^{L-3}, \\
N_+ = 2: \qquad & N_{blocks} = 2^{L-6}L, \quad L \quad \text{even} \\
& \qquad\qquad = 2^{L-6}(L-1) + 2^{\frac{L-7}{2}}\left(2^{\frac{L-5}{2}} + 1\right), \quad L \quad \text{odd}, \\
N_+ = L-2: \qquad & N_{blocks} = 1.
\end{aligned} \tag{3}$$

It would be interesting to find out the analytical formula for the number of blocks in the transition rate matrix for any general N_+, which would contain the formulae in Eq. (3) as special cases. The articles mentioned in the referee's comment have been cited as new references [72] and [73] in the revised manuscript.