We are thankful to the Referees for carefully reviewing the paper and providing useful comments and suggestions, which helped us to improve the manuscript.

Thanks to the comments of the Referees, we have made substantial improvement to the paper. The revisions made are listed below:

- We have improved the description for the construction of strings on Page 4 and Fig. 2;
- We have improved the description for the string energy in Eq. (2) on Page 4;
- We have added a footnote to justify the choice of ansatz on Page 9;
- We have added a more detailed description on whether $V_{J'}$ is attractive or repulsive on Page 9;
- We have added a footnote to discuss the different interaction power in hardcore boson model on Page 10;
- We have done a thorough proofreading to eliminate the errors.

In the following, we give a point-by-point response to the comments of all Referees. We believe that the changes made have improved our paper and hope that the current manuscript will be considered suitable for a further consideration in SciPost Physics.

2 Reply to Referee 2

Comment 2. *The authors consider the transverse field Ising model on a triangular lattice, which is an interesting model for rich critical behavior due to the interplay of frustration and quantum effects. By considering an anisotropic coupling it is possible to explain much of the behavior with an effective description in terms of interacting strings, which is supported by large scale numerical simulations.*

Apart from the necessary changes (below), I find the paper truly convincing. The work meets the acceptance criteria and should be published after those changes have been considered.

Reply 2.0. We thank the Referee for the recommendation for publication as well as the useful comments and suggestions, which helped us to improve the manuscript. In the following we give a point-by-point reply to these comments.

Comment 2.1. *In the description of the construction of strings on page 4, I could not understand the following sentence: "To avoid creation of triangle-rule-breaking defects (also known as spinon topological defects), each bisector within the string can only choose left-going or rightgoing directions" What is meant by "bisector"? Where do I see the directions in Fig 2? I recommend that the explanation is expanded in more detail.*

Reply 2.1. We thank the Referee for pointing out our ambiguous expression. In the revised manuscript, we have replaced the word 'bisector' by 'segment', and marked it in the Fig. 2 by the green and purple left and right-pointing arrows.

Comment 2.2. *In Eqs. (2) the energy of the quantum strings are defined. It should be explained if there is a kinetic energy as well or why it can be neglected.*

Reply 2.2. In fact, the vibration of the segment is described by an XY-chain and the kinetic energy corresponds to the energy of the XY-chain E_{XY} and is included in Eq. (2). In the revised manuscript, we have made that clear by adding a sentence 'where Δ is the energy gap and E_{XY} is the kinetic energy given by solving the effective spin-$1/2$ XY-chain'.

Comment 2.3. *In Eq. (5) the string density is defined, which appears to be quantized in the numerical simulations. Is it a conserved quantity or is there another explanation for this discrete behavior (finite size effect)? The change of the peak position is argued to become continuous in the thermodynamic limit, but does (density × length) remain quantized?*

Reply 2.3. The quantisation of string density is due to finite size effect. As the number of strings must be a even integer under periodic boundary condition, the string density must then be $2\mathbb{Z}/L_x$. In the limit of small quantum fluctuation where the triangle-rule cannot be violated, any local operation cannot change the number of quantum strings, so the string density is also conserved. In the thermodynamic limit, the quantisation step $2/L_y$ becomes infinitesimall, so the string density becomes continuous.

Comment 2.4. *The assumption of a power law interaction in Eq. (9) is not rigorously motivated as previous referees also commented. A discussion would be useful how important this assumed form is to the final outcome, or if other forms of two competing interactions have also been tried. The clear* *evidence of a long range attractive contribution to the interaction is surprising and interesting. What could be the mechanism? The newly inserted paragraph does not explain why one part is attractive.*

Reply 2.4. For the justification of the ansatz, there has in fact been a long debate between whether the interaction between strings should be exponential or power-law [J. Zaanen, Phys. Rev. B 40, 7391(R) (1989)]. We have tried both ansatz

$$V_1(r) = B(J')/r^{\alpha} - C(J')/r^{\gamma} \tag{R1}$$

$$V_2(r) = B(J')e^{-r/\xi_1} - C(J')e^{-r/\xi_2} \tag{R2}$$

for the second ansatz, the optimal parameters are calculated to be $\xi_1 = 0.19$ and $\xi_2 = 0.89$. The sum of residual squared of the power-law ansatz is 4.35×10^{-7} and for the exponential ansatz is 5.73×10^{-7}. We therefore adopted the power-law ansatz in the manuscript. The form of the ansatz does not affect our qualitative result as long as in $V(r) = V_h(r) + V_{J'}(r)$, $V_h(r)$ decays faster than $V_{J'}(r)$, and $V_{J'}(r)$ changes sign when $J' = 0$. We also note that the mechanism of the repulsion $V_h(r)$ is similar to the hard-core boson model, so there is no reason to expect that the form of interaction should be different. We have also added the discussion to a footnote in the main text.

For the reason of the attractive interaction, we added some further explanation to the newly added paragraph. At the presence of J', there is an additional mechanism of string interaction: apart from the repulsion from the hinder of motion when strings are nearby denoted $V_h(r)$, the second, denoted $V_{J'}(r)$, comes from the fact that the insertion of single string produces energy cost $3J'/2$ per string length, while two adjacent strings cost energy $2J'$ per string length, which is different than two individual strings [Fig. 6(a)]. Therefore, when two string segments are adjacent, there is an extra energy gain of J' when $J' > 0$, resulting in an attractive interaction, while when $J' < 0$, this becomes an energy cost of $|J'|$, resulting in a repulsive interaction.

Comment 2.5. *The relation to the hard-core boson model in Ref. [28] should be discussed in more detail, which seems to follow similar physics. What is different? Is the universal critical behavior the same?*

Reply 2.5. In the hardcore boson model, the interaction power is calculated to be $\alpha = 4.0(1)$, which is different from our result $\alpha = 7.5(1)$. The difference is because of the different manners of string vibration. In the hardcore boson model, the vibration of the string is described by an XY-chain with only next-nearest-neighbour interaction, which is different from our model where the vibration is described by an XY-chain with only nearest-neighbour interaction. As the interaction comes from adjacent strings hindering their motions, different manners of string vibration result in different interaction powers. We also added the discussion above to the revised manuscript. However, the discussion of universality is beyond the scope of this work.

Comment 2.6. *Editorial changes: Refs. [3] and [51] are identical. Please check for spelling mistakes ("incommensurte" on p. 2) and spurious articles (remove "the" in front of QMC simulations).*

Reply 2.6. We thank the Referee for the careful proofreading. In the revised manuscript, we have checked the text once again and corrected the mistakes.