To the Editor and the Referee,

We are writing to resubmit our manuscript to SciPost Physics. We thank the Referee for the pointing out the issues and have made changes to address them. We reply to Referee's comments, one by one below.

> **Referee's comment-** The authors do not provide an adequate review of where this work fits into the literature. Overall, the parts that I understand in this paper appear correct. The paper is unfortunately not clearly written. The explanations presented are often rushed, especially so for the most technically difficult arguments. Not enough trouble has been taken to put the work in context within the field. The authors cite no other works in the introduction. This must be fixed, and the relation of the manuscript to existing work clarified.

Authors' reply- We thank the referee for this general comment. Our changes are highlighted in red in a separately attached manuscript pdf. We discuss all the changes made to improve the clarity of the paper below. To begin, we have now added various references in the introduction and modified the following paragraph by adding the line in red.

> We further show in the paper that in 4+1d the electric, magnetic, and dyonic loop excitations can tunnel into each other across invertible domain walls in the bulk of the system. The domain walls, if interpreted as a time direction boundary, give the explicit unitary circuits that map the loop excitations into each other. This is similar to the $\mathbb{Z}_2$ electromagnetic duality symmetry in 2+1d $\mathbb{Z}_2$ gauge theory that exchanges the electric and magnetic particles (but unlike in our case, the symmetry leaves the dyon invariant), or the S_3 permutation symmetry in the three-fermion theory in 2+1d that permutes the three fermion particles [1].

> **Referee's comment-** Below Eq 23 the authors say: "The boundary is Z2 2-form gauge theory .... describes a Z2 topological order with confined charges" I'm confused: The 3+1 D toric code has **deconfined** charges! It's possible I've misunderstood the construction, but in any case the text is very confusing

> without further elaboration. Relatedly, just above eq 36 the authors say: "Similar to the e condensed and m condensed boundaries, there are two variants of the condensed boundary, corresponding to whether the particle in the boundary Z2 topological order is a boson or fermion" Again I'm confused: The 3D semion model has no deconfined particles, so it's meaningless to talk about exchange statistics of the particle. Again, the text is confusing.

Authors' reply- We thank the Referee for mentioning this confusion. The reason for using "confined" was that these charges are created at the boundary at the points where loop excitations intersect with the boundary. Due to the fact that the loop excitations have macroscopic energetic cost required to create them, these end-point excitations at the boundary are confined. Secondly, the 4+1D topological order is given a boundary by coupling the bulk to a 3+1d boundary topological order. For the e boundary, the boundary topological order we couple to can be a 3+1d toric code with an emergent boson or a 3+1d toric code with an emergent fermion. This charge (the emergent boson or fermion) becomes the end-point of the e-loop excitation coming from the bulk and the m-loop comes from the bulk m-loop. Hence, there are two variants. For the m boundary, we can couple to the two kinds of 3+1d toric codes in a similar way and hence there are two variants. Similarly, for the ψ boundary, the boundary topological order we couple to can be a 3+1d semion Walker-Wang model or 3+1d anti-semion Walker-Wang model. We have reworded our description of the boundary in the paper as follows.

> Generalizing the Beigi-Shor-Whalen (BSW) construction [2] to 4+1D and 2-form gauge fields, the e condensed gapped boundary of the 4+1D toric code is constructed by coupling to a 3+1D untwisted Z_2 topological order which can be the 3+1D toric code with an emergent boson or the 3+1D toric code with an emergent fermion. This charge (the emergent boson or fermion) becomes the end-point of the e-loop excitation coming from the bulk and the m-loop comes from the bulk m-loop.

and

> Similar to the e condensed and m condensed boundaries, there are two variants of the ψ condensed boundary, corresponding to whether the $\mathbb{Z}_2$ topological order

that is coupled to is the semion Walker-Wang model or the anti-semion Walker-Wang model.

Referee's comment-

Sec V and VI are very difficult to follow, and appear rather rushed.

To begin:

"As discussed for the Z_2 gauge theory, this symmetry is non-anomalous on orientable spacetime, where the integral of total derivative vanishes"

I've searched the text, and cannot find the discussion of anomalies in the Z_2 case prior to the quoted sentence.

Authors' reply- We thank the referee for the comment. We have removed the line about the Z_2 case and to address the comment on the anomaly, we added the following sentence in the main text

The $\mathbb{Z}_N$ two-form symmetry generated by the dyon surface operator is non-anomalous on orientable spacetime. To see this, we can turn on the background B_3 for the 2-form symmetry:

$$\frac{N}{2\pi}\int\left(b^1 db^2+(b^1+b^2)B_3\right)\ . \tag{1}$$

Under a two-form background gauge transformation $b^1\to b^1-\lambda, b^2\to b^2+\lambda, B_3\to B_3-d\lambda$ for two-form λ, the action changes by a total derivative and thus it is invariant on orientable spacetime.

Referee's comment- Continuing, I'm confused as to exactly what role time reversal symmetry is playing in the remainder of this paper. Does the H^4 obstruction only exist only when time reversal is imposed? I strongly suspect not, but the text above (63) suggests otherwise (or is a distracting aside whose relevance should be clarified).

Authors' reply- The 2-form symmetry is anomalous only when there is time-reversal symmetry. Or there is a mixed anomaly between the time-reversal symmetry and 2-form symmetry. We stress that the H^4 obstruction that we discuss means that the symmetry participates in a three-group, and it is different from the 't Hooft anomaly. We have added the following paragraphs in Sec VI to make it more clear:

> The defining feature of the H^3 obstruction in symmetry-enriched topological phase in 2+1d is that fusing the domain wall defects that generate the 0-form symmetry can produce additional Abelian anyon, which generates the one-form symmetry [1, 3, 4]. Such fusion algebra holds true even when the domain wall is supported on closed submanifolds. Equivalently, performing a 0-form symmetry transformation in the presence of background gauge field, which is specified by the configuration of 0-form symmetry domain wall defects, produces one-form symmetry background gauge field [4]. This is the global symmetry analogue of the Green-Schwarz mechanism in string theory. As a consequence, it is not possible to gauge the 0-form symmetry, where we sum over all possible insertions of the 0-form symmetry defects, without also summing over the insertions of one-form symmetry defect, since the fusion of the former produces the latter. The 0-form symmetry and one-form symmetry combines into a two-group symmetry, and the 0-form symmetry is not a "subgroup": we cannot gauge the 0-form symmetry alone in a two-group symmetry. This is familiar in group extension theory: consider $\mathbb{Z}_4 = \{1, \omega, \omega^2, \omega^3\}$ extension of $\mathbb{Z}_2$ by the $\mathbb{Z}_2' = \{1, \omega^2\}$ subgroup, with $\omega = i$. The element ω fuses into an element of $\mathbb{Z}_2'$, and thus we cannot gauge the "$\mathbb{Z}_2$ symmetry" generated by ω without also gauging $\mathbb{Z}_2'$: we need to gauge the entire extension $\mathbb{Z}_4$.
>
> In the following we will show that there is a similar obstruction for the $\mathbb{Z}_N$ loop permutation symmetry $T : (q_e, q_m) \to (q_e, q_m + q_e)$ in the $\mathbb{Z}_N$ two-form gauge theory in 4+1d. We will show that fusing the domain wall defects that generate the 0-form symmetry produces two-form symmetry defect. This means the 0-form symmetry has an "H^4 obstruction", or equivalently it participates in a three-group symmetry together with the two-form symmetry. This means we cannot gauge only the 0-form symmetry but not the two-form symmetry, since summing over the 0-form symmetry defect insertions necessarily contains sum over the one-form symmetry defect insertions as produced by the fusion of the

0-form symmetry defects.

Referee's comment- Focussing on Sec VI: My understanding of the obstruction is that it captures the fact you cannot proliferate D^N defects in the N even case. Correct? I'm less clear as to why you cannot proliferate D^N defects/gauge $m \leftrightarrow \psi$ symmetry.

Authors' reply- This will be clear below in a longer reply to a particular question by the referee on Eqns. (77)-(79). That question starts as "I think 77-79 form the proof that you cannot gauge $m \leftrightarrow \psi$ symmetry."

Referee's comment- Where does the final equality in (76) come from? Citation needed.

Authors' reply- We have added the explanation as follows

We can also use the mathematical identity

$$\frac{N^2}{4\pi}\int b^{(1)}b^{(1)} = \pi\int \frac{Nb^{(1)}}{2\pi}\frac{Nb^{(1)}}{2\pi} = \pi\int \frac{Nb^{(1)}}{2\pi}w_2(TM)| \tag{2}$$

where $w_2(TM)|$ is the restriction of the second Stiefel-Whitney class of the tangent bundle. We used the Wu formula $x_2 \cup x_2 = x_2 \cup (w_2(TM) + w_1(TM)^2)$ for $\mathbb{Z}_2$ two-cocycle x_2 [5], and we assumed the ambient spacetime is orientable.

Referee's comment-"Thus we find that the domain wall D^N is completely equivalent to insertion of the electric surface operator at a suitable locus in any correlation function." Why is this important? Does this prove there's an obstruction? Why?

Authors' reply- We have added the explanation as follows

Thus we find that the domain wall D^N is non-trivial (and thus there is an obstruction to gauging the $\mathbb{Z}_N$ symmetry). But it is completely equivalent to a higher-codimension operator: it is the same as insertion of the electric surface operator, that generates $\mathbb{Z}_N$ 2-form symmetry, at a suitable locus in any correlation function. Thus the fusion of 0-form symmetry defects produces a 2-form symmetry defect, indicating the 0-form symmetry mixes with the 2-form symmetry and it participates in a three-group. In particular, we cannot gauge the 0-form symmetry alone.

Referee's comment- I think 77-79 form the proof that you cannot gauge $m \leftrightarrow \psi$ symmetry. But I do not understand the construction in (77). Where does this come from? The authors need to spend more time explaining where (77) comes from, and why it corresponds to an attempt to gauge $m \leftrightarrow \psi$ symmetry.

Authors' reply- We have added the explanation as follows

We now repeat the discussion using background gauge fields instead of the symmetry defects. The symmetry defects and the corresponding background gauge fields are related by the Poincaré duality. Let us turn on the background gauge field B_1 for the $\mathbb{Z}_N$ 0-form symmetry and the background gauge field B_3 for the $\mathbb{Z}_N$ 2-form symmetry. The action is modified by the coupling

$$\int \frac{N}{4\pi} b^{(1)} b^{(1)} B_1 + \int b^{(1)} B_3 \ , \tag{3}$$

where we normalized $\oint B_1, \oint B_3 = 0, 1, \cdots, N-1$ mod N. We will investigate the relation between the background gauge fields, which manifest in the fusion algebra, by demanding the above coupling to be consistent. In particular, if we extend the fields to a 5+1d bulk, the dynamical field $b^{(1)}$ should be independent of the bulk extension. The bulk dependence is given by

$$\begin{aligned} &\int_{6d} \frac{N}{2\pi} b^{(1)} db^{(1)} B_1 + \pi \frac{Nb^{(1)}}{2\pi}\frac{Nb^{(1)}}{2\pi}\frac{dB_1}{N} + db^{(1)} B_3 + b^{(1)} dB_3 \\ &= \pi \int_{6d} \frac{Nb^{(1)}}{2\pi}\frac{Nb^{(1)}}{2\pi}\frac{dB_1}{N} + b^{(1)} dB_3 \ , \end{aligned} \tag{4}$$

where we simplified the first line using the property that $\frac{db^{(1)}}{2\pi}, \frac{N}{2\pi} b^{(1)}$ have integer periods. We can further simplify the equation using the identity

$$\begin{aligned} &\pi \int \frac{Nb^{(1)}}{2\pi}\frac{Nb^{(1)}}{2\pi}\frac{dB_1}{N} = \pi \int Sq^2\left(\frac{Nb^{(1)}}{2\pi}\right)\frac{dB_1}{N} \\ &= \pi \int Sq^2\left(\frac{Nb^{(1)}}{2\pi}\frac{dB_1}{N}\right) + \left(Sq^1 \frac{Nb^{(1)}}{2\pi}\right)\left(Sq^1 \frac{dB_1}{N}\right) + \frac{Nb^{(1)}}{2\pi} Sq^2\left(\frac{dB_1}{N}\right) \\ &= \pi \int w_2(TM)\left(\frac{Nb^{(1)}}{2\pi}\frac{dB_1}{N}\right) + \frac{Nb^{(1)}}{2\pi}\left(\frac{dB_1}{N}\frac{dB_1}{N}\right) \ , \end{aligned} \tag{5}$$

where we used the property $Sq^2(z)=z^2$ for $\mathbb{Z}_2$ two-form z, the Cartan formula $Sq^2(xy)=Sq^2xy+Sq^1xSq^1y+xSq^2y$ with $Sq^1\frac{dB_1}{N}=0$, and the Wu formula $Sq^2x_4=w_2(TM)x_4$ for $\mathbb{Z}_2$ 4-cocycle $x_4=\frac{Nb^{(1)}}{2\pi}\frac{dB_1}{N}$ mod 2. Thus for the bulk term to be independent of the dynamical field $b^{(1)}$, these backgrounds should satisfy

$$dB_3=\frac{N}{2}\left(w_2(TM)\frac{dB_1}{N}+\frac{dB_1}{N}\frac{dB_1}{N}\right)\text{ mod }N\equiv\Theta_4\text{ .,}\qquad(6)$$

Let us relate the above equation to the fusion algebra of the 0-form symmetry. The relation between the background gauge fields implies which has the background gauge transformation (for simplicity, we omit the transformation of $w_2(TM)$)

$$\begin{aligned}&B_1\to B_1+NC_1, w_2(TM)\to w_2(TM)+d\lambda_1,\\&B_3\to B_3+\frac{N}{2}\left(w_2(TM)C_1+C_1dC_1\right)\lambda_1\frac{d(B_1+NC_1)}{2}\text{ mod }N\text{ ,.}\end{aligned}\qquad(7)$$

where we take a lift of the $\mathbb{Z}_N$ background gauge field B_1 in $\mathbb{Z}$, and change we find that changing the lift by NC_1, which is a 0-form background gauge transformation, produces additional background for the two-form symmetry. This agrees with the previous finding that fusing N of the $\mathbb{Z}_N$ 0-form symmetry domain wall defects gives a two-form symmetry defect.

The second term on the right hand side (6) can be explained as follows. Consider the junction of five domain walls that generate the $\mathbb{Z}_N$ 0-form symmetry. We first fuse two defects to form a codimension-two junction of three domain walls, then we add another domain wall to form a codimension-three junction of four domain walls, and adding another domain wall gives codimension-four junction of five domain walls. The term implies that the codimenion-four locus, *i.e.* one-dimensional, emits an electric loop. If the domain walls are $g_1,g_2,g_3,g_4,-(g_1+g_2+g_3+g_4)\in\mathbb{Z}_N$. then the electric loop is $\frac{N}{2}\left((g_1+g_2-[g_1+g_2])/N\right)\left((g_3+g_4-[g_3+g_4])/N\right)$, where we used the addition as the group multiplication in $\mathbb{Z}_N$, and $[x]=x\text{ mod }N$.

The first term on the right hand side (6) also has the following interpretation. If the theory is consistent as a bosonic theory and the symmetries are internal symmetries, then the fermion parity symmetry should act trivially. Suppose

$w_2(TM) = d\rho$, ρ is a $\mathbb{Z}_2$ one-form and it can be viewed as the background gauge field for the fermion parity symmetry. Then we find that the first term is equivalent to the shift $B_3 \to B_3 + \frac{N}{2}\rho\frac{dB_1}{N}$. If $B_1 = 0$, then the fermion parity acts trivially, as ρ is decoupled from the rest of the theory. If there is non-zero B_1, the fermion parity couples as follows. Consider a junction of three domain walls that generate the $\mathbb{Z}_N$ 0-form symmetry, labelled by $g_1, g_2, -(g_1+g_2)$, and the junction has codimension two. Then, we intersect the junction with the domain wall of the fermion parity symmetry, and obtain a new junction of codimension-three, $i.e.$ two-dimensional in space time. The junction contain the electric membrane operator labelled by $\frac{N}{2}(g_1 + g_2 - [g_1 + g_2])/N$. Now, if we braid the electric membrane supported at the junction with the magnetic loop labelled by q_m, we find the sign $(-1)^{q_m(g_1+g_2-[g_1+g_2])/N}$. If the theory is bosonic and the symmetries are internal, then the fermion parity domain wall should act trivially on the theory; here we find it is not the case. Concretely,if we take $g_1 = N-1, g_2 = 1$, then $(g_1 + g_2 - [g_1 + g_2])/N = 1$. Thus piercing the magnetic loop q_m with the domain wall labelled by N (more precisely, the domain wall obtained by fusing the $N-1$ and 1 domain wall) gives a point-like excitation with self-statistics πq_m, and when q_m is odd the point-like excitation is a fermion.

Referee's comment- 3. Below (69) "am" -> "an" 4. Repeated reference [20] and [36]. 5. Below (73) aciton-> action

Authors' reply- We thank the Referee for pointing out these typos and reference issues and we have now fixed them.

We hope that with these changes and additions, the manuscript is clear and suitable for publication.

Sincerely,

Authors

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