In this paper, the authors study the stability of a system of randomly coupled linear equations

$$\partial_t x_j(t) = \sum_{i=1}^{n} A_{jk} x_k(t)$$

where matrix A is a large random non-hermitian matrix whose entries are given by

$$A_{jk} = \frac{J_{jk}}{\sqrt{n}}(1 - \delta_{j,k}) + D_j \delta_{j,k}$$

Here $J = (J_{jk})$ is an elliptic matrix whose off-diagonal entries J_{jk} and J_{kj} have a prescribed correlation $\tau \in [-1, 1]$ and $D = (D_j)$ is a random diagonal matrix.

For such a matrix A, it is theoretically possible to describe the limiting spectrum (by the cavity method for example), and thus to locate the dominant eigenvalue (eigenvalue with the largest real part) which drives the stability of the linear system: if the dominant eigenvalue has a negative real part, then the system is stable, and unstable otherwise.

The main contribution of the paper is to analyze the sensitivity of the dominant eigenvalue with respect to the correlation τ (and the standard deviation σ of the entries) of the interaction matrix J. Various distributions for matrix D are considered, some of them leading to rather explicit formulas. Among the interesting results provided in the paper: (1) the system might be stable even if the support of the distribution of D's entries intersects $\mathbb{R}^+$; (2) for a negative correlation τ, increasing the s-d. σ may stabilize the system. Overall the paper is interesting with non-trivial conclusions and nice simulations.

Our Reply: We thank the Referee for carefully reading the manuscript and providing us with comments to improve the paper.

The writing of Part 3 (exposition of the cavity method) may be improved for, as written, it does not completely unveil the mystery around the cavity method for newcomers. But maybe this mystery is inherent to the cavity method...

Our Reply: We thank the Referee for making us aware that the exposition of the cavity method was not clear enough. We have clarified some steps in Part 3, including the derivation of (19) by adding the steps (14)-(17), and the derivation of (21) by polishing and expanding the Appendix A. We hope the current exposition is clearer.

I am willing to support the publication of this paper if the writing is improved along the lines of this report.

Our Reply: We thank again the Referee for positively commenting on the paper, and we address each of the points below.

1. *The title and abstract are over precise. Please shorten the title and simplify the abstract.*

   Our Reply: We have shortened the title into:"*Antagonistic interactions can stabilise fixed points in heterogeneous linear dynamical systems*"; and we have shortened the abstract into: "*We analyse the stability of large, linear dynamical systems of degrees of freedoms with inhomogeneous growth rates that interact through a fully connected random matrix. We show that in the absence of correlations between the coupling strengths a system with interactions is always less stable than a system without interactions. Contrarily to the uncorrelated case, interactions that are antagonistic, i.e., characterised by negative correlations, can stabilise linear dynamical systems. In particular, when the strength of the interactions is not too strong, systems with antagonistic interactions are more stable than systems without interactions. These results are obtained with an exact theory for the spectral properties of fully connected random matrices with diagonal disorder.*"

2. *page 3, line 23: "in the model given by Eq. (2), ..." I do not understand this sentence: what does "in isolation" means?*

   Our Reply: In isolation means that we put all interactions J_{jk} to zero and thus the system is reduced to a set of uncoupled equations. We have changed the corresponding sentence as follows: "*In the model given by Eq.(2) it holds that in the absence of interactions ($J_{ij} = 0$) either all degrees of freedom are stable (when $d < 0$) or all degrees of freedom are unstable (when $d > 0$).*"

3. *p3, l8 from bottom: to weak → too weak*:

   Our Reply: The typo has been fixed.

4. *p3, l4 fb: "cavity method which is the mathematical method": I do not want to be fussy about terminology but it rather seems to me that cavity method is a highly efficient methods from theoretical physics to derive limiting spectral distributions for non-hermitian models, without bothering with the math details (such as the highly difficult control of the smallest sigular value, etc.). Anyway, this is just a remark.*

   Our Reply: We agree with the referee that the cavity method is a theoretical physics method that does not bother with rigorous proofs but instead focuses on the derivation of compact formulae for spectral observables given certain assumptions (such as the convergence of the spectral distribution in the limit of large matrix sizes). Therefore, we have changed the sentence into: *[...] we discuss the cavity method, which is a method from theoretical physics that we use to study the model in the limit of infinitely large random matrices.*

5. *p4, l9-13: it is a bit misleading to write that the leading eigenvalue only depends on the moments of (J_{ij}, J_{ji}), and then to write that it depends on the distribution pD. Please fix.*

   Our Reply: We thank the Referee for pointing out this misleading phrasing. We have changed the sentence into: *As will become clear later, in the limit of $n \gg 1$ the leading eigenvalue λ_1 is a deterministic variable that depends on the distribution p_{J_1,J_2} of (J_{ij}, J_{ji}) only through its first two moments given in Eq. (3), and hence we will not need to specify p_{J_1,J_2}.*

6. *p4,l9 fb: "finite set" the uniform distribution is supported on a bounded or compact set. But the set $[d_-, d_+]$ is not finite.*

   Our Reply: We have replaced finite by bounded: *Since the uniform distribution is supported on a bounded set,[...]*

7. *p4, last sentence: "when p_D has finite support on positive values $d > 0$". Unclear, please fix.*

   Our reply: We thank the Referee letting us clarify this sentence. We have changed it into: *In other words, we ask whether it is possible to have $\mathrm{Re}[\lambda_1] < 0$ even when there exists a value $d > 0$ such that $p_D(d) > 0$.*

8. *p6, correct typos: correctiosn - the he - two typeS.*

   Our reply: Have all been corrected.

9. *p8, equation (22): I do not understand the status of $\delta(y)$ on the r.h.s. Also it will be nice to precise where in [27] we can find this formula.*

   Our Reply: The delta distribution is required because the distribution ρ is defined on the complex plane and since all eigenvalues of $\mathbf{A}$ are real we need a $\delta(y)$ on the right-hand side of this equation. This formula appears in equation (8.8) of chapter 8 in [27]. We have altered the text as follows: *we can use the Sokhotski-Plemelj inversion formula (see e.g. Chapter 8 of [27]) [...]* and we have added the sentence: *Note that the delta distribution $\delta(y)$ on the right hand-side of Eq.(22) specifies that the eigenvalues of $\mathbf{A}$ are real, and hence the distribution ρ defined on the complex plane equals zero for all values $y \neq 0$.*

10. *p10, l8 fb: the the boundary → the boundary. Please fix.*

    Our Reply: Fixed!

11. *p12, caption of Fig 4: Is really $\tau = 1$ in the lowest curve? It seems to me that it should be $\tau = -1$. In this case, the system is always (asymptotically) stabilized. It $\tau > -1$, then as some point $\sigma > \sigma_\star(D)$, the system becomes unstable. The limiting case $\tau = -1$ should probably correspond to a vertical spectrum centered on the mean of p_D ? Am I delusional? Anyhow, the third curve (diamonds) should be more commented.*

    Our Reply: Indeed it should be $\tau = -1$. This was an unfortunate typo.

    In general, the limiting case $\tau = -1$ does not correspond with a vertical spectrum. This is because the matrix is not anti-Hermitian as the diagonal elements are real-valued. A notable exception is when all diagonal elements are equal to each other, in which case $\mathbf{A}$ is an antiHermitian matrix plus a matrix proportional to the identity matrix. Also, when $\sigma \to \infty$, then the spectrum converges to a vertical spectrum centered around the mean of p_D.

We have clarified this with the following paragraph:

In Fig. 4 we compare Eq. (30) with numerical results of the leading eigenvalue obtained through the direct diagonalisation of matrices of finite size n. The numerics corroborate well the analytical results that are valid for infinitely large n. We make a few interesting observations from Fig. 4: (i) for $\tau = 0$, the leading eigenvalue is a monotonically increasing function of the interaction strength σ implying a continuous increase of the width of the spectrum as a function of σ; (ii) for $\tau = -0.8$, the leading eigenvalue is a nonmonotonic function of σ. Initially, for small values of σ, the width of the spectrum decreases as a function of σ, while for large enough values of σ the width of the spectrum increases as a function of σ; (iii) for $\tau = -1$, the leading eigenvalue is monotonically decreasing. In this case, the width of the spectrum decreases continuously as a function of σ and converges for large σ to a vertical spectrum centered on the mean value of p_D.

12. *Bibliography: you may want to cite*

    - *the paper "Properties of networks with partially structured and partially random connectivity" by Ahmadian, Fumarolla, Miller, Phys. Rev. E (2015), about non-hemitian matrix models for neuroscience applications,*
    - *the paper "Positive Solutions for Large Random Linear systems, Proc. A.M.S., 2021", by P. Bizeul, J. Najim, which describes feasibility and stability for large Lotka-Volterra systems.*

    Beside these papers devoted to applications, many papers on non-hermitian matrices (from a math perspective) may deserve to be cited:

    - *"Low rank perturbations of large elliptic random matrices" by Sean O'Rourke, David Renfrew , EJP 2014. on the elliptic law,*
    - *V. L. Girko, Elliptic law, Theory of Probability and Its Applications, Vol. 30, No. 4 (1985). landmark paper on elliptic law*
    - *"Non-Hermitian random matrices with a variance profile (I): deterministic equivalents and limiting ESDs" by Cook et al. Electron. J. Probab. 23: 1-61 (2018)*

Our Reply: We thank the Referee for the suggested papers and giving us the opportunity to make the bibliography more complete. We have cited those papers in the revised version, and in addition, we have also included a couple of other papers.