

The paper is sound and well written. The main question addressed in the manuscript concerns the possibility of stabilising a linear dynamical system through suitable interactions even when a finite fraction of variables is not stable alone. I found the connection between stability and community interaction to be quite elegant. However, I have two main concerns about the manuscript in the current form. First, the novelty of the results needs to be better distinguished from similar results that have been obtained in close-related contexts using dynamical stability analysis and replica computations. The paper would also benefit from some discussion of how this theoretical approach might apply to real systems and experimental works. For the sake of clarity, I would then ask the authors to get back to a few listed points and better detail them.

Our Reply: We thank the Referee for writing a detailed report and providing us with comments to improve the paper.

The Referee raises two points. Regarding the point of novelty. We have added several references in the manuscript, including those mentioned by the two Referees, providing a more complete picture of the existing literature on random matrices with diagonal disorder. In addition to that, we have expanded the Introduction so that it explains more clearly the relation between the present paper and existing works.

We would like to emphasize that the formalism used, i.e., the cavity method for the spectral density of non-Hermitian matrices and the cavity method for eigenvalue outliers, are not novel, and we believe that the relevant papers have been cited. We are not aware of replica calculations for nonsymmetric matrices. However, to the best of our knowledge, the main result of the paper, namely that antagonistic interactions can stabilise systems with degrees of freedom that are unstable without interactions, and the fact that this is not possible with interactions that are uncorrelated, is novel. All the Figures 2, 3, 4, and 5 focus on this result. This is then also in our point of view the main interesting point of this paper and what distinguishes it from previous literature, notably Reference [29] which is the paper that is the most closely related to submitted manuscript.

Let us therefore discuss the differences between Reference [29] and the present paper. The Supplemental Material of Reference [29] contains analogous formulae for the leading eigenvalue when there are no eigenvalue outliers in the spectrum, while the formulae for eigenvalue outliers do not appear in [29]. However, more importantly, Reference [29] does not mention that antagonistic interactions can stabilise linear systems, contrarily to uncorrelated interactions. In fact, Reference [29] seems to claim the opposite, as in the discussion they state "First, we have developed an analytical method for predicting the stability properties of large ecological networks with high accuracy. Second, we have shown that local asymptotic stability cannot be achieved unless the majority of the diagonal entries in the community matrix are strongly negative." This may be because Reference [29] keeps the variance of the off diagonal matrix elements fixed, and thus does not investigate how the leading eigenvalues behave as a function of the interaction strength. Hence, the difference in behaviour of the leading eigenvalue as a function of σ depending on the sign of the Pearson correlation coefficient τ , is the main surprising result of our manuscript, which has not been stated before in the literature.

Regarding the application to real systems and experimental works. It should be noted that we studied a simple theoretical model in order to assess the stabilising feature of antagonistic interactions, which can serve as a null model in the analysis of experimental systems. To apply this result real systems, we would need the following:

- a system of many degrees of freedom that is in the vicinity of a stable fixed point;
- the Jacobian matrix describing the dynamics in the vicinity of the fixed point.

Given such a matrix, one could consider approximating it with an i.i.d. random matrix with nontrivial diagonal elements, and compare this with the theory in the present paper. However, in general, it is difficult to determine the interaction or Jacobian matrix of ecological systems, and most experimental data in ecology are qualitative, see Ref.[49]. We have clarified this in the Discussion as follows:

"The question of stability is also relevant for the study of experimental systems, see e.g. Refs.[46-48]], and the matrices studied in the present paper are null models for real-world world systems. However, as discussed in detail in Ref. [49], most ecological data on foodwebs is qualitative, and obtaining quantitative data in particular on the Jacobian, is challenging.

1. *After Eq.(1) the authors should contextualize better and possibly add specific references while stating "Differential equations of the form (1) appear in linear stability analyses of complex systems described by nonlinear differential equations of the form [...]"*.

Our Reply: Linear stability analysis is a common tool in the study of complex systems and we have pointed this out in the previous version of the manuscript with the sentence "In fact, linear stability analyses with random matrix theory have been used to study the onset of chaos in random neural networks [-], the stability of ecosystems [-], economies [-], or gene regulatory networks [-], and recently exact results for more realistic models based on complex networks have been derived [-]." All the references in this sentence use linear stability analysis to understand the stability of complex systems. Nevertheless, we agree that we could have contextualised earlier in the manuscript the use of Equation (1) for the study of complex systems. We have now modified the second paragraph as follows:

"Differential equations of the form Eq. (1) appear in linear stability analyses of complex systems described by nonlinear differential equations of the form $\partial_t \vec{y}(t) = \vec{f}(\vec{y}(t))$ where $\vec{y} = (y_1, y_2, \dots, y_n)$. For example, in theoretical ecology ecosystems are modelled with Lotka-Volterra equations, where the variable \vec{y} quantify the population abundances of the different species in the population [1]. Other examples are models for neural networks, for which \vec{y} represents the neuronal firing rate or the membrane potential [2-4], and models of economies [5], for which \vec{y} represents economic variables. If the differential system determined by \vec{f} admits a fixed point, defined as $\vec{f}(\vec{y}^*) = 0$, then the dynamics of $\vec{x} = \vec{y} - \vec{y}^*$ near the fixed point is given by Eq. (1), where \mathbf{A} is the Jacobian of \vec{f} . The linear stability of a complex system that settles in a fixed point state is thus determined by the real part of the leading eigenvalue λ_1 , which is defined as an eigenvalue of the Jacobian matrix \mathbf{A} that has the largest real part."

2. *On page 3, the authors claim the importance of "focusing on the case of negative τ that is of particular interest for ecology" but without explaining the actual reason. I agree with this point as it allows to describe prey-predator interactions, however, I think the authors should clarify better and give more ecological motivations.*

Our Reply: We agree that the relevance of negative τ for ecology was not explained. Below equation (3), we have added the following explanation:

"The sign of the parameter τ is important in theoretical ecology as it determines the nature of the trophic interactions between two species. If the interactions are on average competitive ($J_{ij} < 0$ and $J_{ji} < 0$) or mutualistic ($J_{ij} > 0$ and $J_{ji} > 0$), then $\tau > 0$. On the other hand, if the interactions are on average antagonistic ($J_{ij} > 0$ and $J_{ji} < 0$ or $J_{ij} < 0$ and $J_{ji} > 0$), then $\tau < 0$ [9-11,18]. In theoretical ecology, antagonistic interactions are also called predator-prey interactions as they describe trophic interactions between two species for which one predate on the other."

3. *Concerning Sec. 2, "As will become clear later, in the limit of $n \gg 1$ the leading eigenvalue is a deterministic variable that only depends on the moments of the distribution given in Eq. (3)". For an experienced researcher in disordered systems, it is immediate to get the underlying reasoning. However, I do not think it is enough for a non previously aware reader to understand why the random average can be neglected and, hence, the leading eigenvalue becomes a purely deterministic quantity (as related to the properties of the Stieltjes transform specifically at $n \rightarrow \infty$).*

Our Reply: We know that the boundary of the spectrum of \mathbf{A} is deterministic when $p_D(x) = \delta(x - d)$. This is the so-called circular law. Hence, it is not a surprise that self-averaging also holds in the case of arbitrary diagonal distributions $p_D(x)$.

We have modified the sentence as follows:

As will become clear later, just as is the case for the circular law [30,31], in the limit of $n \gg 1$ the boundary of the spectrum of \mathbf{A} is a deterministic curve in the complex plane that depends on the distribution p_{J_1, J_2} of (J_{ij}, J_{ji}) only through its first two moments given in Eq. (3), and hence we will not need to specify p_{J_1, J_2} .

In addition, we have clarified some steps in the derivation of the main result presented in Appendix A. In particular, below equation (44), we comment on the use of the law of large numbers in the self-consistent equation for the resolvent, which gives an intuitive understanding of the universal behaviour of the spectrum.

4. *As for Eq. (20), i.e. the condition to find the boundary of the support, I would urge the authors to be more cautious as this condition has been already found in the literature using RMT techniques. Albeit in the symmetrical case, this stability bound has been precisely derived to point out marginal stability in large well-mixed ecosystems. See Biroli et al., New Journal of Physics (2018) and, more in detail, Altieri et al., SciPost Physics (2022) for a specific application of the Schur complement formula (Eq. (48) in the main text along with the derivation given in the Appendix). Same comment when introducing Eq. (21).*

Our Reply: We thank the Referee for providing interesting references.

In the previous manuscript we did not claim that the Schur complement formula has not been used before. In fact, the theory for the Schur complement formula has been reviewed in the paper [33] for the case of nonHermitian matrices, as we stated in the previous manuscript.

The references of the referee regard disordered Lotka-Volterra systems with symmetric couplings. The interesting result from those papers is that the spectrum of the Hessian found is consistent with a semi-circle law albeit discounted by the number of extinct species (see e.g. Biroli et al, New Journal of Physics (2018)), and hence random matrix arguments do apply in this case.

We have included these relevant references in several places in the manuscript.

5. *In the Conclusions, the authors claim that "A is the Jacobian matrix, which is in general different from the interaction matrix. Nevertheless, studies in, among others, ecology [10,11] and neuroscience [8,9], show that nonlinear systems do exhibit regimes with one unique stationary fixed point and random matrix theory can provide insights on system stability in this regime."*

I agree with the statement that, although the interaction matrix and the Jacobian (or "community matrix") are not the same matrix, their stability properties - as well as the feasibility of equilibria - are closely related, but I would remain cautious about possible generalizations.

This outcome has been stressed recently in a bunch of works (see for instance, Grilli et al. (2017), Stone (2018)) according to which May's argument still applies thanks to the relationship between the Jacobian and the interaction matrix. This reasoning is valid for linear systems or generalized Lotka-Volterra models, as presented in [10], but it is not necessarily true for more complex models, for instance with sub-linear growth. With sub-linear production the stability of the community matrix is no longer directly related to that of the interaction matrix and the elements of the Jacobian matrix would strongly depend on equilibrium abundances.

Our Reply: The papers of Grilli and Stone cited by the Referee study random matrices for which each row is multiplied by a certain positive random number representing the population abundance. The structure mimics Jacobian matrices in differential equations described by randomly coupled Lotka-Volterra equations. We believe that the main result of the paper, namely that antagonistic interactions can stabilise linear dynamical systems would still apply to those ensembles, as Eq.(51) still applies. Nevertheless, it would be interesting to study this more realistic ensemble for Lotka-Volterra systems, and we added the following paragraph in the discussion:

In an ecological setting, the Jacobian matrix of a set of randomly coupled Lotka-Volterra equations have a specific structure, viz., all elements in a row are multiplied by the population abundance so that

$$A_{ij} = D_i J_{ij}. \tag{1}$$

The spectra of such random matrix ensembles have been studied in Refs. [45,46], and it would be interesting to study the stabilising effect of antagonistic interactions in this setup.

Regarding the comment about sub-linear growth. It is not entirely clear to us what the Referee is referring to, but we do think we have been careful with all statements in the manuscript.