Dear Editor,

thanks for sending us the reports of two Referees. We are grateful to the Referees for their careful, positive and constructive reports that helped us improving our manuscript. We have indeed revised the manuscript complying with the requests of both Referees. Our detailed point-by-point response to the Referees has been uploaded.

In view of the already positive report of both Referees, and in view of the improvements that we have introduced in order to fully comply with the requests of both Referees, we are confident that this new version of our paper can be accepted for publication on SciPost Physics.

Best regards,

The authors

Reply to Referee #1.

We thank the Referee for her/his positive report. We also thank her/him for providing further useful comments and suggestions for improving it. We have introduced all the changes requested by the Referee. Therefore, we really hope that the new version of our paper can be accepted for publication on SciPost Physics.

Point 1. It is not clear to me why the chiral currents have to be even larger in the insulating phase than in the metallic phase. Can the authors comment more on this point?

Response to Point 1. We thank the Referee for the important question, which gives us the opportunity of adding more details about our understanding of this result.

In the original version of the manuscript we discussed some aspects of this point for the non-interacting system in the first part of Sec. 3. In this case, owing to the exact knowledge of the single-particle states and of their chiral character, we could associate the increase of the chiral current as a function of Ω in the metallic state, as well as its decrease in the insulator, to the change in the nature and in the occupation of the different bands as a function of Ω. We expanded and improved that section, emphasizing that the origin of the cusp-like maximum is the result of two competing effects which take place as Ω/t is increased. The first effect of Ω/t, which leads to an increase of the chiral current, is the splitting between energy bands with different chirality. For small splittings, the electrons occupy different bands with different chirality and therefore the global chiral current is small. Increasing the splitting, the lowest-energy band is occupied leading to an increase of the chiral current. The maximum is obtained at the metal-insulator transition, when the electrons occupy only one band. For larger values of Ω/t the chiral current decreases because the chiral character of each band is decreased. This mechanism is discussed in more detail in the revised and improved first part of Section 3 (see, in particular, the paragraph below Fig. 2, beginning with "*This peculiar behaviour...*").

When we turn to the interacting system, the states of the system are in principle many-body states and we can not use simple arguments to discuss their evolution and their contribution to the chiral current. On the other hand, our discussion of Sec. 4 shows that, also in the presence of intermediate and large interactions, the transition between the metal and the insulator shares many similarities with the non-interacting picture. In particular, to a large extent, it can be described in terms of a static mean-field theory, which leads to a renormalized band transition which is basically the same as that of the non-interacting system.

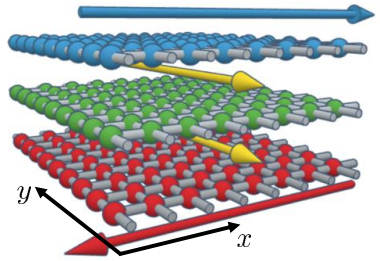
Therefore, our intuitive picture remains the one discussed in Sec. 3 also for the transition in the presence of interactions. We have emphasized also this point in the revised version of the manuscript, in particular, we have expanded the discussion concerning the role of interactions in Sec. IV.

Point 2. The current pattern is studied, via exact diagonalization, only in one dimension. Can the authors describe almost qualitatively how they are expected to behave in two dimensions? Related to this point, I do not understand much the reference to the Meissner effect in superconductors when we are in the insulating phase. Which should be the relation between the presented model and a superconductor?

Response to Point 2. We thank the Referee for prompting us to further comment this important aspect.

In the main text, we assumed, without loss of generality, that $\boldsymbol{\gamma} = \gamma\,\mathbf{u}_x$, i.e. that the Raman phase depends on the x-component of the site position. This implies that hopping processes along the x direction are marked by a flavor-dependent *complex* hopping amplitude $-te^{im\gamma(j_x-i_x)}$ (see

Hamiltonian (3) of the main text), where j_x and i_x are the x-components of the vector lattice sites $\mathbf{j} = (j_x, j_y)$ and $\mathbf{i} = (i_x, i_y)$ respectively.



Conversely, hopping processes along the y direction feature a *real* hopping amplitude $-t$, which is also flavor-independent. Therefore we expect no currents along "bonds" in the y direction and that the current patterns along the planes parallel to the xm-plane to resemble the ones illustrated in Fig. 6 (now Fig. 10). Of course, this is no longer the case if the vector $\boldsymbol{\gamma}$ is rotated and gets a non zero component along y.

As regards the second part of the Referee's question, we use the term "Meissner phase" to connect with previous literature in two-leg bosonic and fermionic ladders where this name has been used (see, e.g., Refs. [1, 2] and references therein). The reason to use the name "Meissner" for the current pattern visible, e.g., in the right panel of Fig. 6 (now Fig. 10), is that the current is indeed expelled from the bulk of the system, i.e. from the inner rungs and from the central leg of the ladder, and it only flows along its edges. This circumstance is visually similar to (but conceptually very different from) the expulsion of magnetic fields from the bulk of superconductors, the well-known Meissner effect.

We have made the following statement at the end of Appendix C.2 to explain the meaning of the term "Meissner": *This result, [the absence of inter-flavor current everywhere but at the edges] together with the continuity equations (31), supports the fact that, in the insulating state, currents are expelled from all inner bonds and can circulate only along the outer boundary of the synthetic two-dimensional system, a circumstance which is reminiscent of the Meissner effect in superconductors (see also Sec. 5 for an effective magnetic model accounting for this phenomenon)..*

Point 3. It is interesting to see how the insulating phase mixes band and Mott nature in it. Still, it is unclear to me why the limit of large U should not be dominated only by the Coulomb repulsive effects. I would expect that when U is the dominating energy scale, I should see only the Mott physics. Can the authors comment more on this?

Response to Point 3 This is indeed a central point of our manuscript, and we thank the Referee for giving us the opportunity to further clarify it. We emphasize that the mixed Mott-band character of the insulator is characteristic of systems where the $SU(N)$ symmetry is broken. In this reply we do not use the language of a synthetic dimension, but we consider the N species as internal labels of a site. From an intuitive point of view, the key point is that there is no competition between the states selected by the Hubbard U (any state with one fermion per site) and those selected by the symmetry-breaking field, which selects one single-fermion state or a linear combination on any site.

The simplest case of $SU(N)$ symmetry breaking is the half-filled $SU(2)$ Hubbard model in a magnetic field, which is the counterpart of the Raman fields. The Hubbard U selects configurations with one fermion per site which, in the absence of a magnetic field, can have either up or down spin. If however a magnetic field $-h\sum_i(n_{i\uparrow} - n_{i\downarrow})$ is added, the up spin configuration is favored and all the spins would align, leading to the simplest band insulator, where the spin-up band is completely filled and the spin-down is empty. Yet, this state is favoured by the Hubbard U as it can easily be seen considering a very small value of h. If h is much smaller than the bandwidth and in the absence of U the system remains metallic. On the other hand, if U is large enough, it selects singly occupied configurations which are then aligned by the small h. This implies, in particular, that the band-Mott insulator can not be obtained without including the Hubbard U. The results of our manuscript for larger number of components N and complex Raman fields are conceptually similar.

Another instructive argument can be obtained in the atomic limit $t = 0$. For the SU(2)-Hubbard model at half-filling, in the atomic limit the self-energy is given by $\Sigma(i\omega_n) = \left(\frac{U}{2}\right)^2 \frac{1}{i\omega_n}$, that has a pole at zero frequency. This is a direct consequence of the degeneracy of the spectrum of the atomic limit in the symmetric case. However, when such a degeneracy is lifted by applying a magnetic field $-h(n_\uparrow - n_\downarrow)$ the spin-dependent self-energy reads $\Sigma_{\uparrow(\downarrow)} = \pm\frac{U}{2}$. Despite we are by construction in a large-U regime, the self-energy is not singular and it is a constant, just like in a static mean-field calculations. The Mott insulator is therefore completely equivalent to a band insulator.

To comply with the Referee's request, we clarified this point by adding these considerations in Sec. IV of the main text. In particular, this concept is emphasized in this new paragraph: "*At the same time, the double occupations decrease due to their increased energetic cost. For a standard Hubbard model without symmetry breaking, this behavior is extended all the way to the Mott transition, where the quasiparticle weight vanishes continuously. In our model we find, instead, a distinct scenario, where the quasiparticle weight remains well different from zero for any value of U and it jumps to a large value close to 1 when the insulating state is reached. This reflects the fact that the self-energy becomes frequency-independent at low frequency and it is again completely different from the typical behavior of a Mott insulator, where the self-energy diverges as $1/i\omega_n$ and the quasiparticle weight vanishes.*"

Point 4. In Sec. 7, the fact that the chiral current for $N = 3$ is formally equal to the one for $N = 2$ is not obvious to me. Can the authors include more steps in the derivation of Eq. (23)?

Response to Point 4. We are happy to provide more information concerning the lengthy derivation of Eq. (23) [now Eq. (22)] and thus explicitly show that, both in the case $N = 2$ and in the case $N = 3$, the chiral current is proportional to the z-component of the *external product* between spin operators of adjacent sites.

In the case of $N = 3$ flavors, we can represent the local Hilbert space as $\mathbb{C}^3$ by identifying the basis of the Fock states with the canonical basis vectors of $\mathbb{C}^3$:

$$c_r^\dagger|0\rangle = |r\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad c_g^\dagger|0\rangle = |g\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad c_b^\dagger|0\rangle = |b\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where we identify the three flavors with colors rather than numbers for simplicity: [red (r) instead of +1, green (g) instead of 0, blue (b) instead of −1]. Local operators can be expressed by means of 3×3 hermitian matrices acting on this Hilbert space, which can always be written in terms of the generators of the Lie algebra $su(3)$. There are two conventionally used set of generators: the ladder operators S_{mn} [sparse matrices with 1 in position (m, n)] and the Gell-Mann matrices λ_i, $i = 1, ..., 8$.

We can express Gell-Mann matrices in terms of ladder operators as:

$$\begin{aligned} \lambda_1 = S_{gr} + S_{rg}, \quad \lambda_2 = i(S_{gr} - S_{rg}), \\ \lambda_4 = S_{br} + S_{rb}, \quad \lambda_5 = i(S_{br} - S_{rb}), \\ \lambda_6 = S_{bg} + S_{gb}, \quad \lambda_7 = i(S_{bg} - S_{gb}), \\ \lambda_3 = S_{rr} - S_{gg}, \quad \lambda_8 = \frac{1}{\sqrt{3}}(S_{rr} + S_{gg}) - \frac{2}{\sqrt{3}}S_{bb} \end{aligned} \tag{1}$$

and vice-versa ladder operators in terms of Gell-Mann matrices as:

$$\begin{aligned}
S_{rg} &= \frac{1}{2}(\lambda_1 + i\lambda_2), \quad S_{gr} = S_{rg}^\dagger, \\
S_{rb} &= \frac{1}{2}(\lambda_4 + i\lambda_5), \quad S_{br} = S_{rb}^\dagger, \\
S_{gb} &= \frac{1}{2}(\lambda_6 + i\lambda_7), \quad S_{bg} = S_{gb}^\dagger, \\
n_r &= S_{rr} = \frac{1}{2}\left(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8 + \frac{2}{3}\mathbb{I}\right), \\
n_g &= S_{gg} = \frac{1}{2}\left(-\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8 + \frac{2}{3}\mathbb{I}\right), \\
n_b &= S_{bb} = -\frac{1}{\sqrt{3}}\lambda_8 + \frac{1}{3}\mathbb{I}.
\end{aligned} \tag{2}$$

If one wants to describe the physics of the system in terms of an effective spin, one needs to extract a $su(2)$ subalgebra generated by three operators Σ^x, Σ^y, Σ^z from the $su(3)$ algebra mentioned above (because spin operators intrinsically satisfy the fundamental relation of $su(2)$, i.e. $[\Sigma^a, \Sigma^b] = i\varepsilon_{abc}\Sigma^c$). However, since this $su(2)$ algebra is acting on a three dimensional Hilbert space, it is describing spin-1 operators (we recall that the dimension of the representation N is related to the spin of the representation s by the relation $N = 2s + 1$). We can build this subalgebra by defining the three generators:

$$\Sigma^x = \frac{1}{\sqrt{2}}\left(\lambda_1 + \lambda_6\right), \qquad \Sigma^y = \frac{1}{\sqrt{2}}\left(\lambda_2 + \lambda_7\right), \qquad \Sigma^z = \frac{1}{2}\left(\lambda_3 + \sqrt{3}\lambda_8\right)$$

that indeed satisfy the proper commutation relation

$$[\Sigma^a, \Sigma^b] = i\varepsilon_{abc}\Sigma^c.$$

Interestingly, the Gell-Mann matrices can be rewritten in terms of linear and quadratic combinations of the spin-1 operators as

$$\begin{aligned}
\lambda_1 &= \frac{1}{\sqrt{2}}\left(\Sigma^x + \{\Sigma^x, \Sigma^z\}\right); \\
\lambda_2 &= \frac{1}{\sqrt{2}}\left(\Sigma^y + \{\Sigma^y, \Sigma^z\}\right); \\
\lambda_3 &= \frac{3}{2}\Sigma^z\Sigma^z + \frac{1}{2}\Sigma^z - \mathbb{I}; \\
\lambda_4 &= \Sigma^x\Sigma^x - \Sigma^y\Sigma^y; \\
\lambda_5 &= \{\Sigma^x, \Sigma^y\}; \\
\lambda_6 &= \frac{1}{\sqrt{2}}\left(\Sigma^x - \{\Sigma^x, \Sigma^z\}\right); \\
\lambda_7 &= \frac{1}{\sqrt{2}}\left(\Sigma^y - \{\Sigma^y, \Sigma^z\}\right); \\
\lambda_8 &= \frac{\sqrt{3}}{2}\left(\Sigma^z - \Sigma^z\Sigma^z + \frac{2}{3}\mathbb{I}\right);
\end{aligned} \tag{3}$$

At this point we can find a standard recipe to rewrite any local two-fermions operator in terms of spin operators: first of all we write it in terms of ladder operators, then we represent it by means of Gell-Mann matrices via (2) and finally we switch to the spin representation via (3). For instance, we can use this recipe to understand why the Hamiltonian in Eq. (12) [now Eq. (11)] of the manuscript, which is quadratic in the ladder operators (and hence quadratic also in terms of Gell-Mann matrices), is in fact *quartic* in terms of spin operators [see Eq. (21) of the previous version, or Eq. (20) of the new version]: precisely because Gell-Mann matrices are quadratic combinations of the spin operators.

Coming to the starting question, we can follow the recipe described above to rewrite the chiral current operator. The latter is still quadratic in the ladder operators (just like the Hamiltonian), but surprisingly and non trivially, it turns out that it can be written as a quadratic (not quartic) combination of spin operators.

The chiral current defined in Eqs. (14)-(16) [now Eqs. (13)-(15)] of the manuscript is already written in terms of ladder operators, so we can pass to the Gell-Mann matrices representation via (2):

$$\mathbf{I}_{\text{chir}} = -\frac{t^2}{U}\sum_{\mathbf{j}\,a}\left[2(\lambda_{\mathbf{j},4}\lambda_{\mathbf{j}+\mathbf{u}_a,5}-\lambda_{\mathbf{j},5}\lambda_{\mathbf{j}+\mathbf{u}_a,4})+(\lambda_{\mathbf{j},6}\lambda_{\mathbf{j}+\mathbf{u}_a,7}-\lambda_{\mathbf{j},7}\lambda_{\mathbf{j}+\mathbf{u}_a,6})+(\lambda_{\mathbf{j},1}\lambda_{\mathbf{j}+\mathbf{u}_a,2}-\lambda_{\mathbf{j},2}\lambda_{\mathbf{j}+\mathbf{u}_a,1})\right]\mathbf{u}_a. \tag{4}$$

Finally moving to the spin representation via (3), one obtains, after lengthy but straightforward algebraic calculations:

$$\mathbf{I}_{\text{chir}} = \frac{t^2}{U}\sum_{a}\sum_{\mathbf{j}}\left(\vec{\Sigma}_{\mathbf{j}}\times\vec{\Sigma}_{\mathbf{j}+\mathbf{u}_a}\right)_z\mathbf{u}_a, \tag{5}$$

This equation [which corresponds to the new Eq. (22)] is formally equivalent to Eq. (18) of the main text, the only difference being that $\{\vec{S}_j\}$ are spin-1/2 operators (as they are a representation of $su(2)$ in a 2-dimensional local Hilbert space), while $\{\vec{\Sigma}_j\}$ are spin-1 operators as stated above.

We have added some comments in Sec. 7 and Sec. 7.2 (now Sec. 5 and Sec. 5.2) of the manuscript that hopefully clarify a little bit more this subtle point.

Point 5. The yellow lines in Fig. 6 are not always well visible. If possible, I would suggest the authors to improve the presentation.

Response to Point 5. We have replaced the yellow color with the black color. We have also modified the text accordingly. Notice that, to comply with the request of the other Referee, we also changed the aspect ratio of the plaquettes.

Point 6. At the end of Sec. 6, the authors refer to Sec. 2, but I think that they refer to Sec. 3. Is this correct?

Response to Point 6. We thank the Referee for spotting this wrong cross reference. We have corrected it.

Reply to Referee #2.

We thank the Referee for his careful, constructive and positive report. We also thank him for providing further useful comments and suggestions for improving it. We reply point-by-point in the following.

Main Report

The authors study the chiral edge currents of quasi-3D bilayer systems subject to local interactions along the synthetic dimension.

I personally find the subject very interesting and timely, and their findings are presented in an appealing way. Although the counter-intuitive or "anomalous" behavior of currents upon repulsive interactions and their associated susceptibilities have already been addressed in the literature of (quasi) 1d ladder systems, their higher-dimensional analogs are far from being fully understood.

To my knowledge, this is the first study that presents and compares numerical and analytical results on the chiral currents of various ground states in these quasi-2d systems. The authors here use a mean-field approach, predictions from a strong coupling limit, and numerical DMFT simulations which all agree qualitatively (and quantitatively only in some limits). Therefore, their presented numerical results are thoroughly proof-checked and trustworthy.

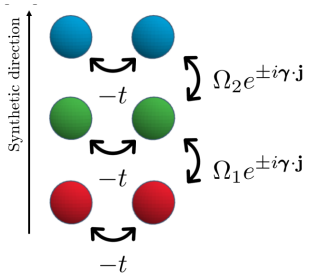
The authors further argue by ED results that 1d ladder systems can be used to understand the phases qualitatively. I find this connection very intriguing, but similarly, I have to say that this is perhaps the biggest weak point of this work. In my perspective, it is misleading to analyze the physics of 1d models and, based on macroscopic observables (the average chiral current), suggest that microscopic details (nodal current flows) are similar in a 2d lattice without an attempt to study small 2d clusters, or present the same quantities using a full-fledged 2d method capable of doing so. I suggest two possibilities to improve the manuscript in this respect. The authors could extend this section by including a study of small 2d clusters with an appropriate technique of their choice (e.g. density group renormalization group or auxiliary field quantum Monte-Carlo frameworks) which would not only bridge the gap between 1d and 2d, but significantly strengthen the concluding arguments as well. As an alternative, the authors could remove the misleading implications, combine the short sections 4 and 5 and move them to a dedicated appendix.

Reply to Main Report We thank the Referee for the positive and constructive report. We acknowledge that the novelty of our manuscript lies mainly in the results for the (2+1)d system, while the behavior of the chiral currents for the (1+1)d system has already been reported, as we already acknowledged in the original manuscript. Even if we can not fully agree, we accept the point of view of the Referee that the idea to interpret some aspects of the 2d system in light of results for the 1d system is not fully grounded on calculations, even if the overall agreement between the behavior of the chiral current is tempting.

While we certainly agree that including calculations for the 2d system with open boundary conditions able to show the current patterns in 2d would settle the question about the similarity between one and two dimensions, we are not in the position to include DMRG or auxiliary-field QMC results to the present manuscript. None of us has an expertise with these methods. The application of DMRG to two-dimensional systems is not very straightforward and the solution of the 2d Hubbard model with these methods (even in the standard case of $SU(2)$ symmetry without Raman fields) requires a specific expertise and a careful analysis of the way the methods work for the present models. For these reasons we postpone this investigation to future work and we resort to the second suggestion by the Referee, namely to eliminate the misleading statements and to move former Sec. 4 and 5 to an appendix (see Appendix C). We have modified the manuscript accordingly.

Point 1. Fig 1. (c), add missing hopping arrows

Response to Point 1. We have added the missing hopping arrows in Fig 1. (c). The new panel now reads:



Point 2. First term of Eq. (1) has a missing 1/2 in front (bonds are accounted for twice in the sum).

Response to Point 2. We thank the Referee for pointing out this misleading notation. In order to avoid the possible introduction of typos in the following equations, we address this point by giving an explicit declaration of how the sum $\sum_{\langle \mathbf{ij}\rangle}$ should be interpreted, rather than introducing the extra factor 1/2. In particular, we have added the following sentence: "*The symbol* $\langle \mathbf{ij}\rangle$ *here means that the sum runs over all the possible bonds connecting two nearest neighbor sites labeled* $\mathbf{i}$ *and* $\mathbf{j}$*, where each bond is only counted once.* ".

Point 3. Perhaps mention $I_{\rm chir} = \langle \frac{\partial H}{\partial \gamma} \rangle$ around Eq. (6), or is this equation only true in 1d?

Response to Point 3. We thank the referee for the interesting observation. The result is not only true in 1d, and it can easily be generalized to 2d. The subtle point here is that the current is now a vector, so we should define it as the *gradient* of the Hamiltonian with respect to the vector $\boldsymbol{\gamma}$ that defines the flux magnitude and direction. Furthermore, to make the definition consistent with Eqs. (4), (5), (7) of (the new version of) the manuscript one also needs a prefactor $-2L^{-d}/(N-1)$. The resulting expression is

$$\mathbf{I}_{\rm chir} = -\frac{2L^{-d}}{N-1}\langle \nabla_{\boldsymbol{\gamma}} \mathcal{H}\rangle.$$

We have added this equation [now Eq. (8)], together with a brief comment just after Eq. (7), as suggested by the Referee.

Point 4. Below Fig. 2: "... the flavor polarization ... changes" better: decreases?

Response to Point 4. We agree withe the Referee observation, and we have corrected the text accordingly.

Point 5. Above Fig. 3: please mention existing literature in 1d concerning anomalous Drude weight effects.

Response to Point 5. We thank the Referee for pointing out the very relevant existing literature about anomalous Drude weight in 1D systems. The references which we added are Refs. [3, 4, 5] of the present document (see Refs. [17, 18, 19] of the new version of the manuscript).

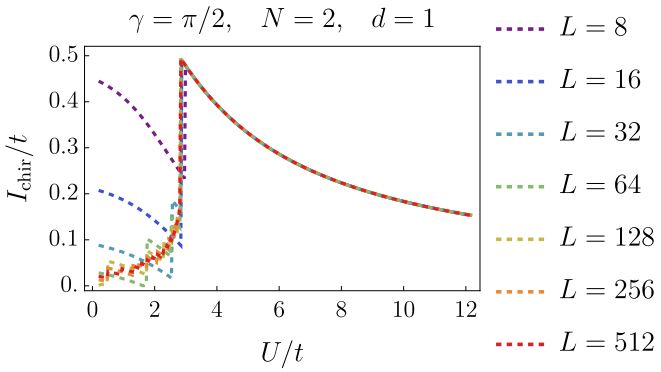


Figure 1: Finite size scaling for the chiral current obtained with $N = 2$, $\Omega = 0.4t$, $\gamma = \pi/2$ with the Hartree-Fock method. The spiky behavior of the current in the metallic phase is essentially a finite-size effect, which slowly disappears in the thermodynamic limit as L is sufficiently large. On the other hand, the hyperbolic behavior on the insulating phase and the presence of a peak at the phase transition are robust features that are not affected by finite-size effects.

Point 6. Below Fig. 4: "The T=0 results clearly show a qualitative agreement." Please comment on the strong v-shaped kink at U=0, which is clearly absent in your 2d estimates.

Response to Point 6. We thank the Referee for the suggestion. We believe that the detailed behavior of the current in the metallic phase, as obtained studying clusters of small size, is not representative of the thermodynamic limit, and is affected by finite-size effects. To motivate this statement, we can take advantage of the Hartree-Fock method, which is a flexible tool for making a proper finite-size scaling, due to its scalability to large system sizes.

The results are shown in Fig. 1 of this document, where one can appreciate how taking large systems regularizes the chiral current in the metallic phase, making the V-shaped behavior slowly disappear. On the other hand, it is also worth noticing how the main features of the chiral current (hyperbolic decrease in the insulator and presence of a sharp peak at the transition), that represent one of the main conclusions of our work, are not affected by finite size effects and are already quantitatively accurate in small clusters. We have added this figure to the manuscript (see new Fig. 6) together with a suitable comment at the end of Appendix A.

We believe that the reason why the current is so high in the metallic phase of small clusters with respect to its thermodynamic limit is the fact that a metallic phase is not easily distinguishable from an insulating phase when the system is small. Therefore, the disruptive interference mechanism responsible for the chiral current drop in the metal (described in Sec. 3) is not very effective when there are only few quasiparticle states available.

We have commented on this important point raised by the Referee in Sec. 4 of the original manuscript (new Appendix C.1) by adding the following sentence: "*The main difference emerging from these two methods is the quantitative value of the chiral current on the metallic phase, which is larger in systems of small size. This is a finite-size effect due to the presence of few quasiparticle states, that make a metal barely distinguishable from an insulator in a small cluster. The disruptive interference mechanism responsible for the current drop in the metal, described in Sec. 3, is thus not very effective in small systems, resulting in a large current. Nevertheless, it is worth observing that the chiral current behavior in the insulator, as well as the peak at the transition point is already captured in systems of small size.*"

Point 7. Section 5: see main report. Is it possible to artificially double the impurity unit cell in order to compute current correlations with DMFT? An alternative to bridge the gap between 1d and 2d would be the utilization of DMRG frameworks such as iTensors or TenPy, or auxiliary field quantum Monte-Carlo frameworks such als ALF, which have similar models already implemented.

Response to Point 7. As we mentioned in the reply to the main report, we believe that the implementation of DMRG, or other tensor-network methods as well as auxiliary-field QMC is far from trivial and beyond our current expertise despite the availability of libraries.

As far as the suggestion to double the unit cell in DMFT, this is in principle feasible and it amounts to use one of the cluster extensions of DMFT [cellular DMFT, Dynamical Cluster Approximation (DCA)]. However, the growth of the cluster significantly increases the computational cost of the calculation using our implementation (as a matter of fact the number of published results for finite clusters and $SU(N)$ with $N > 2$ is incredibly small). Moreover, small clusters can induce significant size effects which can be particularly significant for real-space patterns such as those of the currents we are inerested in. For these reasons we believe that all of these investigations have to be postponed to future work.

Point 8. Fig. 6: perhaps use the same aspect ratio between x and y links, otherwise it is impossible to guess by eye that the Kirchhhoff law is satisfied.

Response to Point 8. We agree with the Referee. Employing the same aspect ratio along the x-axis and the y-axis definitely improves the readability of the plot. We changed the figure accordingly. Notice that, to comply with the request of the other Referee, we also changed the color of the arrows.

Point 9. General comment on the mean-field approach: the currents in Fig. 6 left panel are clearly site-dependent. Provided currents in the 2d case behave similarly, why is this site dependence neglected on the mean-field (and DMFT) level?

Response to Point 9. This difference originates from the fact that in the mean field and DMFT approaches we have imposed periodic boundary conditions on the real directions, while in Fig. 6 (now Fig. 10) we show the site-resolved current pattern obtained using open boundary conditions. In the former case the system is invariant under lattice translations and every site and every bond are equivalent, while in the latter case we break translation invariance and we can detect spatial patterns of the current.

We have briefly clarified this point at the beginning of Sec. 5 of the original manuscript (now Appendix C.2), according to the Referee suggestion, by adding the following sentence: "*Once the qualitative agreement between DMFT and ED results has been established, one can use the latter to investigate the spatial configuration of observables. For this reason we consider the interacting* (1+1)*-dimensional system with open boundary conditions (OBC) along both the real and the synthetic directions, which introduces a difference between edge sites and bulk sites, and we investigate how the spatial current pattern is modified across the* U*-driven metal-insulator transition. We notice that DMFT can not be straightforwardly used with OBC because it requires translation invariance.*"

Point 10. Why "synthetic ladder" with quotation marks?

Response to Point 10. We have dropped the quotation marks in their only occurrence.

Point 11. Could the authors comment on my following question: Is the chiral current in these models related to any topological phase, or predecessors thereof, similarly to the scenario for quasi 1d ladders threaded with flux, which are related to quantum Hall phases via the coupled wire formalism?

Response to Point 11. We thank the Referee for driving our attention to the subtle topic of topology in fermionic ladder systems. To the best of our knowledge, we would argue that the chiral current flowing along the edge is *not* related to a topology-related edge mode of the system.

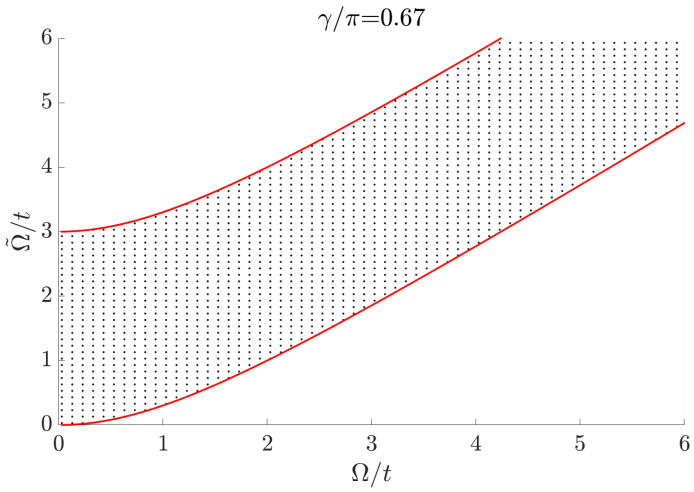


Figure 2: Plot of the Zak phase Z in the space of parameters $(\Omega/t, \tilde{\Omega}/t)$ of a $(1+1)d$ ladder Hamiltonian with an extra tunneling connecting the synthetic edges with amplitude $\tilde{\Omega}$. The stripe between the red lines corresponds to states with $Z = 1$ (hence topological), while the remaining regions correspond to states with $Z = 0$ (hence not topological). Here we consider $N = 3$, $\gamma/\pi = 2/3$, $U = 0$, $L = 1000$ sites. This plot is replicated from Fig 1.b of Ref. [6].

First of all, we would like to mention that a thorough investigation of the topology in ladder-like systems can be found in Ref. [6], where the authors identify the signature of topological order using the Zak phase. In Fig. 2 we show a similar topological phase diagram obtained replicating their calculation. The authors consider a $(1+1)d$ system similar to what we have presented in the manuscript, with an extra tunneling process in the synthetic direction connecting the synthetic edges with amplitude $\tilde{\Omega}$, and they argue that the Zak phase is always vanishing when $\tilde{\Omega} = 0$, as in our case. On the other hand, they find a topologically ordered phase in a range of parameters around the line $\tilde{\Omega} = \Omega$, which is however associated to the presence of edge modes localized at the *real* edges of the system when open boundary conditions along the *real* direction are taken into account. In addition, we would like to report a sentence from Ref. [7] which clearly confirms that the discussed chiral currents have *not* a topological origin: *"The edge currents studied here do not have a topological origin."*

Besides these considerations, supporting the idea that the chiral current in our model does not have topological origin, we can add some more arguments. First of all, we believe that the distinction between synthetic edge and synthetic bulk is very weak in systems where N is relatively small, as those considered in our work, thus making it difficult to disentangle bulk properties from edge properties of the model. Moreover, there is no intrinsic periodicity along the synthetic direction, because the hopping matrix element along each leg has a different phase, with no repeated patterns (in general). This means that, for an arbitrary flux γ, we can't identify a unit cell repeating itself along the synthetic direction, and we can not define a synthetic Brillouin zone. On the one hand, this supports the fact that there is no proper bulk and proper edge in the system, and on the other hand

this makes it difficult to define a suitable topological invariant, analogue to the aforementioned Zak phase, capable of accounting for the presence of toplogical edge modes along the synthetic direction.

Point 12. If the conjecture that 1d and 2d insulating phases are qualitatively similar is correct, the chiral current is apparently carried by the exterior synthetic sheets. Could you confirm this by presenting/producing also DMFT results for $N = 4$?

Response to Point 12. This interesting comment by the Referee prompts us to make some remarks. When we conjecture that 1D and 2D insulating phases are qualitatively similar, we mean that:

1. There are no inter-flavor currents $I_{\mathbf{i};m,m+1}$ [see Eq. (14) of the new version] (except, at most, at the edges of the real direction associated to $\boldsymbol{\gamma}$).

2. There are no intra-flavor currents $I^{a}_{\mathbf{i},\mathbf{i}\pm\mathbf{u}_a;m}$ [see Eq. (13) of the new version] perpendicular to the real direction associated to $\boldsymbol{\gamma}$.

In other words, the current patterns illustrated in (the present version of) Fig. 10 are expected to be replicated along the second real dimension, as illustrated in Fig. 3 of the this document.

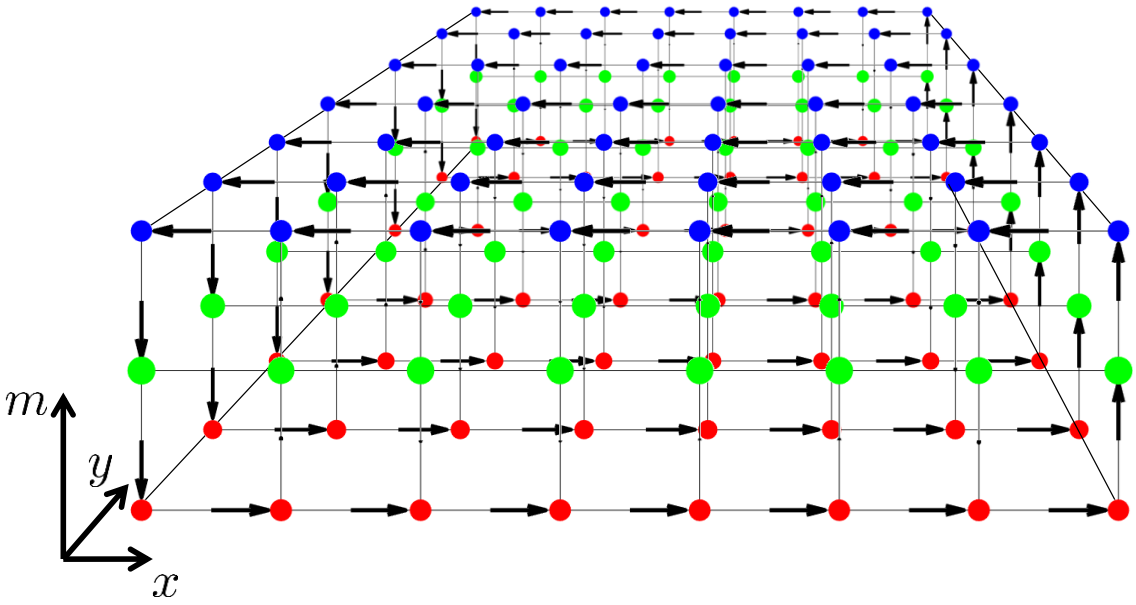


Figure 3: The current pattern characterizing the insulating phase of the ladder (in the plane xm) is expected to be replicated along the y-direction.

One can notice that, since $\boldsymbol{\gamma}$ is parallel to the x-axis, intra-flavor current flows parallel to the x-axis and has *zero* component along the y-axis. Similarly, interlayer currents (i.e. currents along the m-axis) are zero everywhere but at the two edges $x = 0$ and $x = 7$.

That being said, we can safely state that for a generic N-legs ladder (N-sheets heterostructure), the current is not necessarily localized along the external legs (sheets), even though we understand that the case $N = 3$ is possibly misleading in this sense, because there is only one inner leg (sheet).

We can also argue that, whenever N is odd and the Raman hopping amplitudes do not depend on the flavor index (i.e. $\Omega_m \equiv \Omega \ \forall m$), the current on the central flavor vanishes as a consequence of the point reflection symmetry of the Hamiltonian. Although we are going to characterize the physics of large-N synthetic ladders and heterostructures in a separate work, we anticipate that one should introduce *multiple* chiral-current operators, namely $N/2$ if N is even and $(N-1)/2$ if N is odd. For example, in the case $N = 4$ flavors, where the flavor index m takes the values $\{-3/2, -1/2, +1/2, +3/2\}$, one should introduce *two* different chiral-current operators: one involving the two *outer* layers $m = \pm 3/2$, another involving the two *inner* layers $m = \pm 1/2$. Figure 4 of the present manuscript schematically illustrates the basic Physics at stake. As requested by the Referee,

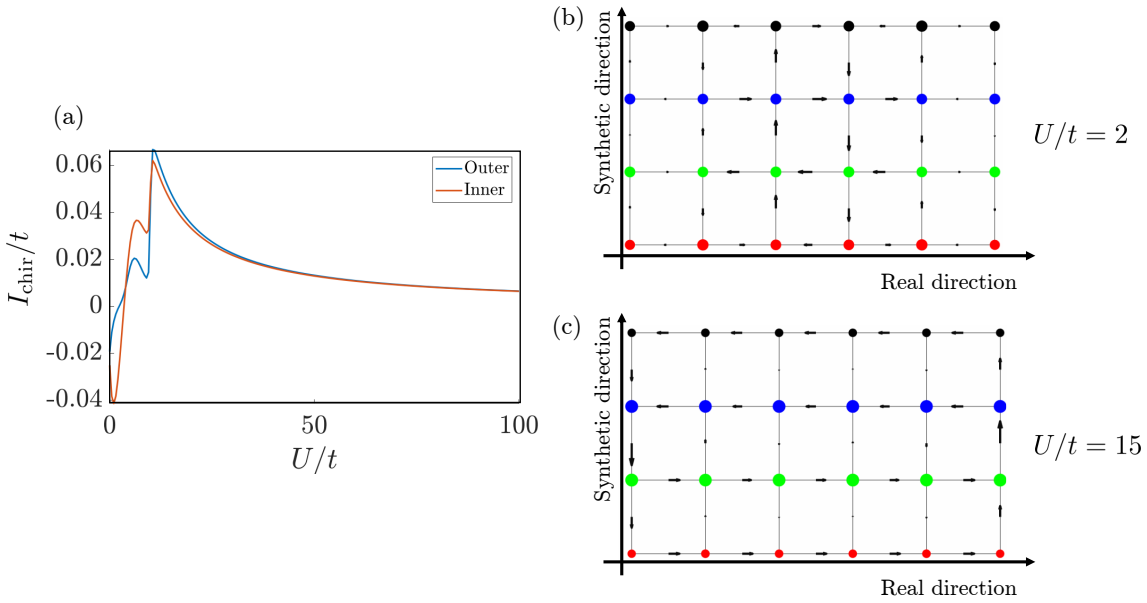


Figure 4: Panel (a): functional dependence of the inner and outer chiral currents on control parameter U/t. It is clear that, deep in the insulating regime, the chiral current along the two *inner* layers is far from being zero but, instead, it is almost equal to the the chiral current along the two *outer* layers. Panels (b) and (c): typical arrangement of current patterns in the metallic phase and in the insulating phase, respectively. In the latter phase, we remark that both the outer and the inner layers are flown by non-zero currents. Moreover, rung currents are zero everywhere but at the two (real) edges. We expect these properties not to be limited to synthetic ladders, but to extend also to synthetic heterostructures. Data obtained for $\Omega/t = 0.5$, $\gamma = \pi/5$.

we can also integrate the previous discussion providing early results obtained with DMFT for the $N = 4$ heterostructure. Looking at Fig. 5, we can see how in general the currents in every sheet are non vanishing. Moreover, the current behavior is quite intriguing and worthy of further investigation, as inner currents become larger than external currents in the insulating phase.

To stress the important role of symmetry, and to clarify possible sources of misunderstanding, we have added the following sentence at the end of Sec 2: "*Before moving to the discussion of the particular cases $N = 2$ and $N = 3$, it is worth mentioning that for an arbitrary N, Hamiltonian (3) has a point reflection symmetry, as long as the Raman tunneling amplitudes do not depend on m. This means that the Hamiltonian is invariant under a simultaneous reflection on the real and the*

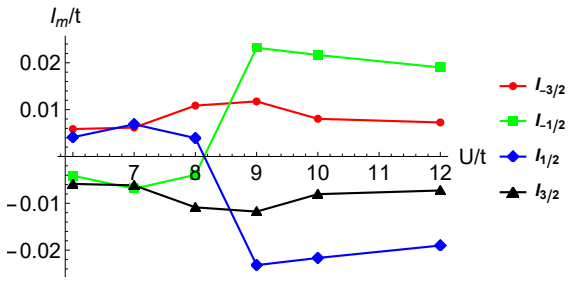


Figure 5: Sheet currents in a $N = 4$ heterostructure obtained via DMFT with $\Omega = 0.4t$, $\gamma = \pi/2$.

synthetic space: i.e. $c_{\mathbf{i},m} \to c_{-\mathbf{i},-m}$. *As a consequence of this symmetry, it is possible to prove that* $\langle I_{a,-m}\rangle = -\langle I_{a,m}\rangle$, *which in turn implies that, when* N *is odd and* $m = 0$ *is a possible flavor index, the current on the central leg vanishes:* $\langle I_{a,0}\rangle = 0$. *However we remark that in general the currents flowing on inner legs (i.e. legs with* $m \neq 0, \pm\mathcal{I}$*) are non-vanishing, so that when* $N > 3$ *one can in principle define* $N/2$ *chiral currents if* N *is even, and* $(N-1)/2$ *chiral currents if* N *is odd.*".

Point 13. Can you compute and present the current between the synthetic sheets with DMFT? This would further establish a similarity to Fig. 6.

Response to Point 13. Unfortunately we believe that it is not possible to establish a straightforward comparison between DMFT results and the panels of Fig. 6 (now Fig. 10). The reason is that the Vortex-Meissner transition, i.e. the presence of non trivial current patterns along the rungs of the structure, is peculiar of systems with open boundary conditions and cannot be reproduced by systems where periodic boundary conditions are implemented. In fact the "rung currents" obtained with DMFT, as well as with the mean field approach, should be vanishing for every U as a matter of principle to ensure the Kirchhoff current law on every node. This stems from considering every site indistinguishable from the others (only applicable to systems with periodic boundary conditions): which makes it impossible to have different currents in two rungs or in two bonds of the same leg. Therefore, the presence of a non-vanishing rung current on the rung $(\mathbf{i}, m) - (\mathbf{i}, m+1)$ is incompatible with having the same currents along the two bonds $(\mathbf{i}-1, m) - (\mathbf{i}, m)$ and $(\mathbf{i}, m) - (\mathbf{i}+1, m)$, implying that the former should be zero.

Nevertheless, we have computed such currents within DMFT as a useful consistency check of our algorithm and we have verified that they actually vanish.

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