Dear Editor,

We wish to thank Dr. Mikael Fremling for a thorough review and for recommending our paper for publication in SciPost. We found his report to be illuminating and useful. Below, we address all of the comments made in Dr. Fremling's report and the associated improvements that we consequently made to the manuscript.

Sincerely,

Mostafa Tanhayi Ahari, Sumanta Bandyopadhyay, Zohar Nussinov, Alexander Seidel, and Gerardo Ortiz

Detailed Reply to the Referee:

* The abstract begins with: "We present microscopic, multiple Landau level, (frustration-free and positive semi-definite) parent Hamiltonians whose ground states, realizing different quantum Hall fluids, are parton-like and whose excitations display either Abelian or non-Abelian braiding statistics." From reading the main text, I understand that you argue for the EPP forms, which *can* give zero-energy solutions for frustration-free parent Hamiltonians. I could, however, not find a discussion or reference to how these Hamiltonians would look like, nor a recipe to construct them. If it is in the paper, could this then be pointed out more clearly? If it is not, perhaps the abstract should be changed slightly. I note that the authors on and off work with the TK Hamiltonian, but I understand this construction to be more general than that.

Response:
Positive semi-definite frustration-free Hamiltonians can enforce a wide range of clustering conditions, which become even richer when wave-functions with both homolorphic and anti-holomorphic degrees of freedom are concerned. This is so not only because of the added degree of of freedom, but also because of the new constraints (maximum LL index) that are enforced for the anti-holomorphic part only. There is then a general procedure to go from such Hamiltonians to necessary constraints on zero modes that manifest themselves in the form of an EPP. We try to better emphasize the generality of this in the new manuscript, but, as the referee observed, illustrate this in detail in the context of 4 LLs and an $M = 3$ clustering condition, where the TK-type Hamiltonian is appropriate. This is partially done for concreteness, and partially because this is the simplest parton state with Fibonacci anyons. The referee is also right in observing that an EPP is only derived as a necessary conditions. Additional logic is generally needed to establish 1-1 correspondance between "EPP satisfying structures" (root states) and zero modes. This, too, we exercise in detail in the context of the $n = 4, M = 3$ example. We have clarified these issues in the introduction.

* A general question on the importance of the LL structure: In the later section, the authors focus on deriving results for the $N_L = 4$ lowest LL:s. How important is the LL structure itself to these calculations? Can one replace the ladder operator algebra that moves particles up and down between LLs with, say, two internal spin-1/2 degrees of freedom and the corresponding $S^{+/-}$ operators? This would be relevant for layered materials where each layer is in the LLL.

Response:
Our formalism makes no use of the LL structure once a 2nd quantized projection operator Hamiltonian of the general form discussed is given. It will yield interesting results only when such a Hamiltonian is frustration free, but this can certainly be the case in a number of different contexts, for example, as the referee points out, when there are spin degrees of freedom as opposed to LL indices. We focus on the case where the additional degrees of freedom are associated to higher LLs, where traditional first-quantized wave functions are much harder to study algebraically for given clustering conditions, as already pointed out above.

* Between eqn. (8) and eqn. (9): The authors write that the eigenvalues of $n^b_{c,r}$ are $2j - m$ and m respectively. Could some text be added here to point out that $2j = n^b_c + n^b_r$ or similar, and that j could be a half-integer? When reading the text as it stands, I was very confused as to where j came from.

Response: Following the referee's suggestion, in the revised manuscript, we clarified the $2j = n^b_c + n^b_r$ is an integer (with being j a half-integer). We thank the referee for making this suggestion.

* Around eqn. (12) and eqn. (13): In equation (13), the authors introduce the determinant D...(1,2) , which they claim is being used in equations (12). However, as it is currently written since this is highly confusing since (12) is an equation for "kets" and (13) is a fave function. I can see that if one applied the "bra" $< 1, 2|$ from the left in (12), then the object D in (13) would appear, but it is not there at the moment.

Response: Heeding the referee request, in the revised manuscript, we underscore that the "bra" notation $\langle 1,2|$ furnishes a cogent link between Eq. (12) and Eq. (13).

Further, I find the notation with α_1 and α_2 quite inadequate since they are only indirectly defined. By comparing eqns. (12) and (13) I come to the conclusion that $\alpha_1 = n_1, j-k$ and $\alpha_2 = n_2, j+k$, but this is never written out explicitly.

Response: Prompted by the referee's suggestion, in the revised manuscript, we now explicitly write that $s_1 = j-k$ and $s_2 = j+k$.

* Equation (15): This equation is suggested as a natural map for $D(\alpha_1, \alpha_2)$. However, since the D in equation (13) never appears in any other equation except for (15), it is unclear how to interpret the suggested map.

Response: To elucidate our results, we expressed these in both the first and second quantization languages. In the revised manuscript, to avoid any confusion, we further expounded on the relationship between the second quantized (Eq. (12)) basis and the first quantized (position) basis (Eq. (13)).

* Above Figure 1: The authors write: "For finite size systems, the number of angular momentum orbitals in each LL is restricted by the number L of available distinct single particle angular momentum modes." Could a clarification be made here that L is refereeing to the number of angular momentum modes in the highest Landau level and not the lowest Landau level? As a consequence, is it true that $L >= NL$?

Response: Below Figure 1, we have clarified that the highest LL has L orbitals. There is no direct relation between N_L and L. Our focus, however, is on the (bulk ground state) situation in which $L \gg N_L$.

* First paragraph section III: in the text: "As we will explain, one may indeed precisely find these zero-energy states for any given number of particles N at filling fractions $\nu = (N-1)/(L-1)$." I find this statement weird as it would imply that zero energy states exist at any density. This seems to be contradictory to Figure 2, which makes it clear that there exists an N for which the energy is always strictly positive. Also, i'm confused as to why there is a -1 shift in N and L.

Response: We thank the referee for pointing this earlier confusing text. In the revised manuscript, we removed the word "any" and added further clarifications. The shift by (-1) has been adopted from earlier convention in the literature (see, e.g., Ref. [24]). In the LLL wavefunction, this shift yields the correct filling fractions for the disk and cylindrical geometries.

*Equation (41): Could the authors clarify what assumptions go into equation (41)? For instance, is there an assumption that L is kept constant as N is changed? Keeping L fixed seems important e.g., to be able to draw the diagram in Fig 2 and be able to define the densest zero energy state.

Response: We thank the referee for this comment. The new manuscript has been revised accordingly. In the subsection titled "Monotonicity of the ground state energy", L is indeed kept constant while allowing number of particles to vary. This point is now better clarified in the manuscript. We explicitly note that L may be constant while N may vary.

*Figure 3 and text below eqn (49): 1) In the figure, can the arrows pointing from one color to the other be made larger (they are hardly visible)?

Response: Adhering to the referee's suggestion, we modified the figure accordingly.

* The text under Eq. (49) claims that the "blue state" is non-expandable. Is this, however, true? Can i not obtain this state $(1_3, 0, 2_{1,3}, 0, 3_{0,2,3})$ from an inward squeeze of $(1_3, 1_1, 0, 1_3, 3_{0,2,3})$. If this is not the case, some extra text is needed explaining why.

Response: For the example provided, we now remark that $(1_3, 1_1, 0, 1_3, 3_{0,2,3})$ is not a possibility because a 1_1 state at angular momentum -2 does not exist. The figure, however, is primarily intended for explaining the squeezing process not the idea of non-expandable states. We have remove the word "non-expandable state" from the figure to make the discussion clearer. We further would like to point out that even if the configuration existed, it would not necessarily render the "blue state" non-expandable. This is so because the definition of "non-expandable" is relative to a given zero mode, and requires the coefficient of the configuration the referee envisions to be non-zero in that zero mode. We have further clarified this point in the manuscript.

Why are the L=2 orbitals included in the figure (and the state description?). If they are available, the blue state should also be reachable from an inward squeeze of a state that has electrons in the L=2 orbitals. To avoid confusion, I suggest that the L=2 orbitals are removed from the figure and the state description.

Response: To avoid confusion, we removed the $L = 2$ states from the new manuscript. However, as remarked in the preceding reply, the mere existence of $L = 2$ orbitals does not necessarily render the configuration non-expandable.

* Between equations (52) and (53): Underneath equation (52) there is the sentence: "The corresponding pseudofermion basis states are $zb_{ni}z_{j+ni}$ ". This sentence seems detached from the rest of the paragraph, as a) it is not clear why the functional form of the basis states is relevant, and b) z=x+iy is not used at all in this entire section (apart from this sentence). Can it be removed?

Response: The referee is absolutely correct in remarking that the functional form of the pseudofermions are not essential for understanding that section. We have removed the said sentence. We wanted to alert the readers that the functional form the pseudofermions is different than that of the fermions. In the revised manuscript, this information is now parenthetically provided.

* Below eqn (53): a) The text says: "There are 12 coefficients up to an overall normalization to satisfy 12 linear constraints from eq. (30)." I here wonder how the counting is made, are n and n' considered independent variables, or not. Naively I would have thought that $n < n'$, which should give 6 states. Alternatively I could let n and n' be independent which should give 4*4=16 states. How do I get 12?

Response: Due to the fermionic nature of the particles, once we fill one orbital with one particle, there are only 4-1=3 possible orbitals left for the second particle. As such, there is a total of $4 \times 3 = 12$ possibilities. Previously, in Eq. (53), omitting anti-symmetry we ended up with 12 coefficients. Following the comment by the referee, this has now been corrected.

b)The text further reads: "For a single angular momentum site constraints $\xi = 9, 12$ are not independent." This sentence would have been fine, except that the ξ-numbering is not clear. Looking at eqn (25) and (27), I see ξ appearing, but it is unclear which values it takes and if these are the same as the $\xi = 1, ..., 12$ that is implicitly used in the text under equation (53). Perhaps they can be listed in the appendix?

Response: We took the referee's remark to heart and expended considerable effort to make this clear. In the revised manuscript, we now very explicitly provide the matrix connecting the T^- to $\mathcal{T}^-$ bases along with the other associated details that address this question very perspicuously (see Appendix C, including Table IX therein).

*Equation (67): Can it be written out that the orbitals $\bar{\phi}$ in equation (67) are defined on a cylinder?

Response: In the revised manuscript, below Eq. 67, we state that $\bar{\phi}$ orbitals are defined in cylinder.

*Above (68): The authors introduce a gauge that is not the symmetric gauge nor the Landau gauge. In the literature, this gauge is often called the "tau-gauge" as it is perpendicular to the "tau"-vector. * Page 14 bottom: "Moreover, for $L_\Delta = 0$, there are anti-unitary operators that implement the combination of a mirror symmetry (in x or y) with time-reversal symmetry." For completeness, this is also present for the hexagon, with $L_\Delta = L/2$ and $|\tau = 1$.

Response: We thank the referee for bringing this up to our attention. We note that tau-gauge mentioned in the literature is indeed dual to our gauge.

* Page 16 left column: There are two "equations" with string sequences 2002002002002002002002002002002002002002 2001101100200200110110020020011011002002 and 200200200200200200200 11011011011011011011011 11010110101101101011010 which are long and hard to read. Can some tricks with color coding or other visual aids be inserted to highlight where the domain walls are forming?

Response: We are grateful to the referee for the suggestion to make these strings more readable. In the revised manuscript, we have inserted extra spaces to better mark the domain wall locations (instead of colors). A similar notation has already been adopted in Tables III and IV.

* Page 21, middle of the page: This is not a complete sentence: "From the completeness of our coherent states, given that the operation of inversion as described will produce a zero mode in the same topological sector with a quasihole localized at $2h_I - h$."

Response: We thank the referee for this comment. In the revised manuscript, we modified the said sentence to read: "From the completeness of our coherent states, the operation of inversion as described will produce a zero mode in the same topological sector with a quasihole localized at $2h_I - h$."

* Around eqn. (127):
a) The text above equation (127) refers back to the basis mentioned with regards to the pseudofermions below eqn (52). However, since the form in (127) was never used there (and I still recommend removing it), this reference becomes dangling. If the authors choose to keep this reference as is, then it should at least explicitly refer to eqn (52) to allow the reader to find the pseudofermions.

Response: Following the recommendation of the referee, in the revised manuscript, we inserted (i) a reference to Eq. (52) (around Eq. (127)) and (ii) further highlight (around Eq. (52)) the difference between the two orbital types.

b) If the reference is kept, it is worthwhile to remind the reader that $s_i = j + n_i$.

Response: We added the definition of s_i in Eq. (127).

c) a technical remark may be warranted here: That is, equation (127) (when the gaussian factor is restored) is not the same as the LL basis (since different ϕ_α are not orthogonal).

Response: We added a clarifying statement underscoring that these basis states are (by contrast to the LL orbitals) not orthogonal.

* Around eqn. (129): It may be advisable to mention that these calculations are specific for the symmetric gauge, and e.g. eqn (129) is only valid in that (symmetric) gauge.

Response: We incorporated this suggestion of the referee in the revised manuscript.

* eqn. (148): I never understood where equation 148 came from. For instance, for $N_L = 2$ and M=3, why am I not allowed to multiply three $\nu = 2$ wave functions? These states would also be a product of closed-shell wave functions, right?

Response: The condition of Eq. (148) ensures the overall parton state (a product of M Slater determinants) remains with N_L LLs. A product of three $\nu = 2$ Slater determinants would result in monomials containing $\bar{z}^3$ associated with $N_L = 4$ LLs.

Some minor language issues:

* Above equation (19): The text "the average distance between electrons is" should be changed to "the average distance between neighboring electrons is"

Response: We modified this sentence and used "correlation length".

*Page 8, Top first column: The text reads: "Since the size of LL orbitals is directly associated with the magnitude of its angular momentum....". Can a different word than "size" be used here? All the LL orbitals are equally large (cover the same area), but they have their density peaks at different radii.

Response: We replaced "size" with "spatial extent".

* Conclusions, second line: Inconsistent spelling of ansatz.

Response: We thank the referee for catching this mistake. This is corrected in the revised manuscript.

* Conclusions, middle of page 43:: Should there be an extra "and" in "Laughlin's wave function can be expressed as the product of the lowest LL (LLL) wave function with M Slater determinants of holomorphic (or anti-holomorphic) wave functions several filling fractions of the form 1/M (odd M), which are experimentally observed." such that it reads "Laughlin's wave function can be expressed as the product of the lowest LL (LLL) wave function with M Slater determinants of holomorphic (or anti-holomorphic) wave functions *and* several filling fractions of the form 1/M (odd M), which are experimentally observed."?

Response: We modified this sentence in the manuscript.